Homological dimensions of crossed products

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- $K^-({}_RP)$, $K^b({}_RP)$: (right) bounded homotopy category of chain complexes of finitely generated left projective R-modules.

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- (Aljadeff) A skew group ring $A\#G$ with commutative A and finite group G has finite global dimension if and only so does A and the trivial representation A is projective.
- (Li) If a finite dimensional k-algebra A has a complete set of primitive idempotents closed under the action of G, $A\#G$ has finite global dimension if and only if so dose A and a Sylow p-group $S \leqslant G$ acts freely on this set, where p is the characteristic of k.

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- Q2: Consider crossed products which include skew group rings as special examples.

Definition of Crossed products

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- $\alpha(x, 1) = \alpha(1, y) = 1$.
- The crossed product is defined to be $A_\alpha^\sigma G = \bigoplus_{x \in G} A\sigma_x$, a direct sum of free modules with multiplication $(a\sigma_x) * (b\sigma_y) = a\sigma_x(b)\alpha(x, y)\sigma_{xy}$.

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- Note that A is a G-module via σ if and only if $\alpha(x, y)$ is in the center of A for all $x, y \in G$.
- If σ is trivial, this gives *twisted group rings*; if α is trivial, this gives *skew group rings*.
- A is an $A_\alpha^\sigma G$-module if and only if α is trivial. This is called the *trivial representation* of a skew group ring.

Homological dimensions

Given a left Noetherian ring R, a complex $X^\bullet \in K^-({}_RP)$ has *amplitude* $amp(X^\bullet)$

$$\sup\{i \in \mathbb{Z} \mid X^i \neq 0\} - \inf\{i \in \mathbb{Z} \mid X^i \neq 0\}$$

and *length* $l(X^\bullet)$

$$\inf\{amp(Y^\bullet) \mid X^\bullet \text{ is quasi-isomorphic to } Y^\bullet\}.$$

Homological dimensions

- $\text{gl.dim}\, R = \sup\{l(P^\bullet) \mid P^\bullet \in K^-({}_R P) \text{ and } H^i(P^\bullet) \neq 0 \text{ for at most one } i\}$;

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- sgl.dim $R = \sup\{l(P^\bullet) \mid P^\bullet \in K^b({}_R P)$ is indecomposable$\}$.

Induction and Restriction on module categories

Let G be a finite group, and $H \leqslant G$ be a subgroup.

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- Since ${}_{A_{\alpha}^{\sigma}H}A_{\alpha}^{\sigma}G_{A_{\alpha}^{\sigma}H} = A_{\alpha}^{\sigma}H \oplus B$, the induction and restriction functors are exact and preserve projective modules.

Chain maps

- For $M \in A_\alpha^\sigma H$-mod, define:

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- They are $A^\sigma_\alpha H$-module homomorphisms and commute with differentials, and $\eta \circ \delta$ is the identity.

Chain maps - Continuation

- Suppose that $|G : H|$ is invertible in A. For $N \in A_{\alpha}^{\sigma}G$-mod, define $\varphi : N \to N\downarrow_{H}^{G}\uparrow_{H}^{G}$:

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Theorem: Let $H \leqslant G$ be a subgroup.

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- and $K^b_0({}_{A^\sigma_\alpha H}P), K^b_0({}_{A^\sigma_\alpha G}P)$.

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- Note that $T \cong A$ if α is trivial. Otherwise, it is not true since $\mathfrak{I}$ contains all $\alpha(x, y) - 1$ for $x, y \in G$.
- For $M \in A_\alpha^\sigma G$-mod, there is a natural isomorphism $M^G \cong \mathrm{Hom}_{A_\alpha^\sigma H}(T, M)$ of A^G-modules, where $M^G = \{v \in M \mid \sigma_x v = v, \forall x \in G\}$.

Trivial representations - continue

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- One direction is trivial. For the other direction, note that $\mathrm{Hom}_{A^{\sigma}_{\alpha}G}(M, -) \cong \mathrm{Hom}_A(M, -)^G$. But both $\mathrm{Hom}_A(M, -)$ and $-^G \cong \mathrm{Hom}_{A^{\sigma}_{\alpha}G}(T, -)$ are exact.

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- Proposition: If T is projective, then $A_\alpha^\sigma G$ and A have the same global dimension and finitistic dimension.

Global dimension of crossed products

- Proposition: For $M \in A^{\sigma}_{\alpha}G$-mod, $\mathrm{pd}_{A^{\sigma}_{\alpha}G} M$ is either infinity or equal to $\mathrm{pd}_A M$. Consequently, $\mathrm{gl.dim}\, A^{\sigma}_{\alpha}G$ is either infinity or equal to $\mathrm{gl.dim}\, A$.

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- *Proof.* Using the following isomorphisms:
  $A_\alpha^\sigma G \otimes_A - \cong \mathrm{Hom}_A(A_\alpha^\sigma G, -)$;
  $\mathrm{Ext}^i_{A_\alpha^\sigma G}(M, A_\alpha^\sigma G \otimes_A -) \cong \mathrm{Ext}^i_A(M, -)$.

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- The global dimension of $A^{\sigma}G$ is finite if and only if so is $\mathrm{gl.dim}\, A$ and the trivial representation A is projective.

Questions

- If $\mathrm{gl.dim}\, A_\alpha^\sigma G < \infty$, then $\mathrm{gl.dim}\, A < \infty$ and $\mathrm{pd}_{A_\alpha^\sigma G}\, T < \infty$. Is the converse true? (This holds for skew group rings.)

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- If $\mathrm{pd}_{A_\alpha^\sigma G}\, T = 0$, what can we say about $\mathrm{sgl.dim}\, A_\alpha^\sigma G$ and $\mathrm{sgl.dim}\, A$?
- What can we say about the homological dimensions of A^G?