Arc-transitive cycle decompositions of tetravalent graphs

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Abstract

A cycle decomposition of a graph $\Gamma$ is a set $C$ of cycles of $\Gamma$ such that every edge of $\Gamma$ belongs to exactly one cycle in $C$. Such a decomposition is called arc-transitive if the group of automorphisms of $\Gamma$ that preserve $C$ setwise acts transitively on the arcs of $\Gamma$. In this paper, we study arc-transitive cycle decompositions of tetravalent graphs. In particular, we are interested in determining and enumerating arc-transitive cycle decompositions admitted by a given arc-transitive tetravalent graph. Among other results we show that a connected tetravalent arc-transitive graph is either 2-arc-transitive, or is isomorphic to the medial graph of a reflexible map, or admits exactly one cycle structure.

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1. Introduction and definitions

By a cycle in a graph we mean a connected regular subgraph of valence 2. If $\Gamma$ is a graph and $C$ is a set of cycles in $\Gamma$ such that each edge of $\Gamma$ belongs to exactly one cycle in $C$, then the pair $(\Gamma, C)$ is a cycle decomposition of $\Gamma$. 
Cycle decompositions of various types of regularity have been extensively studied. For example, the question for which values of $k$ a cycle decomposition of the complete graph $K_n$ into cycles of fixed length $k$ exist was long an open problem, settled only recently by Alspach and Gavlas [1] and Šajna [15]; see also [2]. Cycle decompositions which are invariant under certain cyclic groups of automorphisms attract considerable attention (see [14,18]). One of the aims of this paper is to initiate the study of cycle decompositions which are invariant under an arc-transitive group of automorphisms. Let us start with precise definitions and statements.

An isomorphism between cycle decompositions $(\Gamma_1, C_1)$ and $(\Gamma_2, C_2)$ is a graph isomorphism $\Gamma_1 \rightarrow \Gamma_2$ which maps every cycle from $C_1$ to a cycle in $C_2$. A symmetry or automorphism of a cycle decomposition $(\Gamma, C)$ is an isomorphism between $(\Gamma, C)$ and itself. The group of all automorphisms of $(\Gamma, C)$ will be denoted by $\text{Aut}(\Gamma, C)$. If $G \leq \text{Aut}(\Gamma, C)$, then we will say that $(\Gamma, C)$ is a $G$-invariant cycle decomposition.

If $\text{Aut}(\Gamma, C)$ acts transitively on vertices, arcs, or edges, then $(\Gamma, C)$ is said to be vertex-transitive, arc-transitive, or edge-transitive, respectively. Similarly, if $G$ is a subgroup of $\text{Aut}(\Gamma, C)$ which acts transitively on vertices, arcs, or edges, then $(\Gamma, C)$ is $G$-vertex-transitive, $G$-arc-transitive, or $G$-edge-transitive, respectively. We will often refer to an arc-transitive cycle decomposition of a tetravalent graph as a cycle structure.

Consider the octahedron $O$ as shown in Fig. 1.

![Octahedron](image)

Fig. 1. The octahedron.

The set $C_1 = \{(1, 6, 12, 9), (2, 5, 10, 8), (3, 4, 11, 7)\}$ of three four-cycles is a cycle decomposition of $O$. Moreover, the full symmetry group $\text{Aut}(O)$ preserves $C_1$ and is transitive on arcs of $O$. Thus $C_1$ is a cycle structure on $O$. While there are no other cycle decompositions of $O$ invariant under $\text{Aut}(O)$, there are two which are invariant under a smaller arc-transitive subgroup. One of these is $C_2 = \{(1, 2, 3), (4, 9, 10), (5, 6, 11), (7, 8, 12)\}$, and the other is $C_3 = \{(1, 4, 5), (2, 6, 7), (3, 8, 9), (10, 11, 12)\}$. Thus we see that a tetravalent arc-transitive graph can admit more than one cycle structure. Moreover, since $\text{Aut}(O, C_2) = \text{Aut}(O, C_3)$, we see that a fixed group of a graph can admit more than one cycle structure.

The results of this paper are a part of a larger project investigating edge-transitive tetravalent graphs. Our initial motivation for the present work was a certain construction [13] which produces, from a cycle structure, a tetravalent semisymmetric graph. For applying this construction it is desirable to be able to find all cycle structures in a given graph $\Gamma$. One approach to this task is in two steps: find all arc-transitive groups of symmetries of $\Gamma$, and then, for each such group $G$ find all cycle decompositions of $\Gamma$ which are invariant under $G$. A way to tackle the second step is given in Theorem 4.2, where we show that a given group may admit at most three different cycle structures. On the other hand, while the first step is computationally feasible for graphs $\Gamma$ whose symmetry group $\text{Aut}(\Gamma)$ is relatively small, it will fail when $\text{Aut}(\Gamma)$ is large and contains...
many arc-transitive subgroups. For such graphs, alternative approaches will be discussed in the paper.

In particular, in Section 4.1 we prove that a connected tetravalent arc-transitive but not 2-arc-transitive graph \( \Gamma \) either admits exactly one cycle structure, or it is isomorphic to the medial graph of a reflexible map. In the latter case, \( \Gamma \) admits at least three (not necessarily non-isomorphic) cycle structures (see Theorem 4.5).

Perhaps more crucially, we also see cycle structures as an important tool for investigating tetravalent graphs which are arc-transitive but not 2-arc-transitive. As we show in Theorem 4.2, every such graph admits at least one cycle structure, and by analyzing how cycles in a cycle structure interlace one can hope to gain some information about the graph itself. We discuss this approach in more detail in Section 5.

We will prove in Section 4 results describing the occurrence of cycle structures in terms of “local action” of a group acting on a graph. We will also examine in Section 5 cycle structures in graphs which are 1-arc- but not 2-arc-transitive.

2. Consistent cycles

We summarize here the theory of consistent cycles, which is relevant to cycle structures. For a cycle \( C \) in \( \Gamma \) and a group \( G \leq \text{Aut}(\Gamma) \), consider the setwise stabilizer \( G_C \leq G \) of \( C \). If the permutation group \( G^C_C \), induced by \( G_C \) on \( C \), contains the cyclic group of order the length of \( C \), then, following [3], we say that \( C \) is \( G \)-consistent. An element of \( G_C \) which induces a one-step rotation of \( C \) is then called a shunt for \( C \). If, in addition, \( G^C_C \) is isomorphic to the full symmetry group of \( C \) (that is, if \( G^C_C \) is dihedral of order twice the length of \( C \)), then we say that \( C \) is \( G \)-symmetric; and if \( G^C_C \) is cyclic, then we say that \( C \) is \( G \)-chiral. Clearly, if \( C \) is \( G \)-consistent, then \( C^g \) is also \( G \)-consistent for every \( g \in G \). Hence the set of all \( G \)-consistent cycles can be partitioned into \( G \)-orbits.

Observe that if \( (\Gamma, \mathcal{C}) \) is a \( G \)-invariant cycle decomposition for some arc-transitive group \( G \leq \text{Aut}(\Gamma) \), then \( \mathcal{C} \) is a \( G \)-orbit of symmetric \( G \)-consistent cycles. Moreover, the self-overlap of \( \mathcal{C} \) is 0, where by a self-overlap of a set of cycles \( \mathcal{C} \) in \( \Gamma \) we mean the largest number of consecutive edges that are shared by two distinct cycles in \( \mathcal{C} \). (More on the overlaps of consistent cycles can be found in [9].) It is easy to see that the converse holds as well, yielding the following simple but useful observation.

**Lemma 2.1.** Let \( \Gamma \) be a \( G \)-arc-transitive graph and let \( \mathcal{C} \) be a set of cycles in \( \Gamma \). Then the following are equivalent:

1. \( (\Gamma, \mathcal{C}) \) is a cycle decomposition and \( G \leq \text{Aut}(\Gamma, \mathcal{C}) \);
2. \( \mathcal{C} \) is a \( G \)-orbit of symmetric \( G \)-consistent cycles with self-overlap 0.

A long-forgotten result of Conway and Biggs [3] easily implies the following theorem, whose complete proof was first given in [8].

**Theorem 2.2.** (See [8, Corollary 4.2].) Let \( \Gamma \) be a \( G \)-arc-transitive graph of valence \( k \), and let \( s \) and \( c \) denote the numbers of \( G \)-orbits of \( G \)-symmetric and \( G \)-chiral cycles in \( \Gamma \), respectively. Then \( s + 2c = k - 1 \). In particular, if \( k = 4 \), then either \( s = 3 \) and \( c = 0 \), or \( s = 1 \) and \( c = 1 \).
3. Examples

3.1. Wreath graphs

The wreath graph, $W(n,k)$, $n \geq 3$, has $nk$ vertices in $n$ sets $S_i$, $i \in \mathbb{Z}_n$, each of size $k$, where each vertex in $S_i$ is adjacent to all vertices in $S_{i-1}$ and $S_{i+1}$. In short, $W(n,k)$ is the lexicographic product of a cycle of order $n$ and the edgeless graph of order $k$. Observe that $W(n,k)$ is tetravalent if and only if $k = 2$. In this case we let $S_i = \{a_i, b_i\}$, $i \in \mathbb{Z}_n$, and we can picture the graph as in Fig. 2. The graph $W(4,2)$ is isomorphic to $K_{4,4}$ and needs to be treated separately; we will do this at the end of this section. In all other cases, we can describe all cycle structures on $W(n,2)$, up to isomorphism, as follows.

Let $G = \text{Aut}(W(n,2)) \cong \mathbb{Z}_2^n \rtimes D_n$. Let $C_1$ be the set of all 4-cycles consisting of the edges between $S_i$ and $S_{i+1}$ for $i \in \mathbb{Z}_n$, let $C_2$ be the set of all $n$-cycles traversing the sets $S_i$ in the natural order once, and let $C_3$ be the set of all $2n$-cycles traversing the sets $S_i$ in the natural order twice. Clearly each of the sets $C_1$, $C_2$ and $C_3$ is a $G$-orbit of $G$-consistent cycles, and by Theorem 2.2 there are no other $G$-consistent cycles.

The set $C_1$ itself forms a cycle structure; every edge belongs to exactly one cycle in $C_1$. In the case $n = 6$, this is shown in Fig. 2(a).

Consider one cycle from $C_3$, of length $2n$. Its complement also contains $2n$ edges. If $n$ is odd, these fall into two $n$-cycles, but if $n$ is even, they form another $2n$-cycle, which must also be in $C_3$. Together they form a cycle structure, with a group of order $8n$, as in Fig. 2(b).
If there is a cycle structure $C$ consisting of $n$-cycles from $C_2$, it must contain four of them. Since we are only trying to find cycle structures up to isomorphism, we may assume that $C$ contains the cycle $C_1 = (a_0, a_1, \ldots, a_n)$. Then there are two possibilities:

1. One of the other three $n$-cycles is $C_2 = (b_0, b_1, \ldots, b_{n-1})$. Then the complement of $C_1$ and $C_2$ contains $2n$ edges, and they fall into two $n$-cycles exactly when $n$ is even. In this case the other two cycles are $(a_0, b_1, a_2, b_3, \ldots, a_{n-2}, b_{n-1})$ and $(b_0, a_1, b_2, a_3, \ldots, b_{n-2}, a_{n-1})$, and they are clearly in $C_2$. These four cycles then form a cycle structure. An example with $n = 6$ is in Fig. 2(c).

2. If $C_2 \notin C$, then the rotation about $C_1$ permutes the three cycles in $C \setminus \{C_1\}$ cyclically. But this is possible if and only if $n$ is divisible by 3. The other three cycles are then $(a_0, b_1, a_2, b_3, a_4, b_5, \ldots, a_{n-3}, b_{n-2}, b_{n-1})$, $(b_0, a_1, b_2, a_3, b_4, a_5, \ldots, b_{n-3}, a_{n-2}, a_{n-1})$ and $(b_0, b_1, a_2, b_3, b_4, a_5, \ldots, b_{n-3}, b_{n-2}, a_{n-1})$, and they are clearly in $C_2$. These three cycles together with $C_1$ then form a cycle structure. An example with $n = 6$ is in Fig. 2(d).

In the special case of $W(4, 2)$, the consistent cycles have lengths 4, 6, 8. Because 6 does not divide 16, the number of edges, there cannot be a cycle structure of 6-cycles. The set $C_1$ is a cycle structure of 4-cycles, and is isomorphic to the structure of 4-cycles from $C_2$ mentioned in case (1) above. Also, any 8-cycle and its complement form a cycle structure on $W(4, 2)$. These two are the only isomorphism classes of cycle structures in this graph.

Thus, a wreath graph may have 1, 2, 3 or 4 isomorphism classes of cycle structures. We know of no other graph or family of graphs which admits more than three.

3.2. Medial graphs of reflexible maps

Let $\mathcal{M}$ be a reflexible map. (A map is called reflexible if its symmetry group acts transitively on the flags of the map; see, for example, [17] for details.) The medial graph of $\mathcal{M}$, denoted by $\text{MG}(\mathcal{M})$, is a graph having one vertex for each edge of $\mathcal{M}$, with two of its vertices adjacent when the corresponding edges belong to the same face and share an endpoint. If $\Lambda = \text{MG}(\mathcal{M})$, then every symmetry of $\mathcal{M}$ acts as a symmetry of $\Lambda$. Because the map is reflexible, this group acts transitively on the arcs of $\Lambda$. Let $C$ be the collection of cycles of $\Lambda$ corresponding to the faces of $\mathcal{M}$. Then $C$ is invariant under Aut$(\mathcal{M})$, showing that $(\Lambda, C)$ is a cycle structure, which will be denoted by $\text{CS}(\mathcal{M})$.

Besides the map $\mathcal{M}$, there are five more reflexible maps sharing the same medial graph $\Lambda$. These are $\text{D}(\mathcal{M})$, $\text{P}(\mathcal{M})$, $\text{DP}(\mathcal{M})$, $\text{PD}(\mathcal{M})$ and $\text{PDP}(\mathcal{M})$, where the dual and the Petrie operators D and P are as used in [17]. By starting with one of these five maps in place of $\mathcal{M}$ we would, in principle, get five more cycle structures on $\Lambda$. However, since $\mathcal{M}$ and $\text{PDP}(\mathcal{M})$ have the same faces, the corresponding cycle structures are identical. The same is true for the pairs $\text{D}(\mathcal{M})$, $\text{DP}(\mathcal{M})$, and $\text{P}(\mathcal{M})$, $\text{PD}(\mathcal{M})$. We have thus seen that every medial graph of a reflexible map $\mathcal{M}$ admits three cycle structures, namely $\text{CS}(\mathcal{M})$, $\text{CS}(\text{D}(\mathcal{M}))$, and $\text{CS}(\text{P}(\mathcal{M}))$.

To illustrate this phenomenon recall the example of the octahedron from Section 1. The graph $\mathcal{O}$ is the medial graph of the tetrahedron $T$. The cycle structures $C_2$ and $C_3$ correspond to $\text{CS}(T)$ and $\text{CS}(\text{D}(T))$, while $C_1$ corresponds to $\text{CS}(\text{P}(T))$. Note that the self-duality of $T$ causes $C_2$ and $C_3$ to be isomorphic (though still distinct).
3.3. Line graphs of cubic graphs

Closely related to the idea of the medial of a map is the line graph of a graph. If \( \Lambda \) is any graph, the **line graph** of \( \Lambda \), \( L(\Lambda) \), is the graph whose vertices are (or correspond to) edges of \( \Lambda \), two being adjacent when they have a vertex in common. If \( \Lambda \) is regular of valence \( k \), then \( L(\Lambda) \) is regular of valence \( 2(k-1) \). In particular, \( L(\Lambda) \) is tetravalent when \( \Lambda \) is cubic.

In this case the vertices of \( L(\Lambda) \) which correspond to edges in \( \Lambda \) incident to one vertex induce a 3-cycle in \( L(\Lambda) \); call it a **vertex-cycle**. Assume now that \( \Lambda \) is 2-arc-transitive. Then \( L(\Lambda) \) is arc-transitive. Moreover, if \( \Lambda \) is not \( K_4 \), then the only 3-cycles in \( L(\Lambda) \) are, in fact, the vertex-cycles, and hence they form a cycle structure whose group is \( \text{Aut}(\Lambda) \).

4. Cycle structures and the local action

Let \( \Gamma \) be a connected tetravalent graph, let \( v \in V(\Gamma) \), and let \( G \) be an arc-transitive subgroup of \( \text{Aut}(\Gamma) \). Consider the vertex stabilizer \( G_v \) and the permutation group \( G_v^\Gamma(v) \), induced by \( G_v \) on the neighborhood \( \Gamma(v) \). The group \( G_v^\Gamma(v) \) will sometimes be referred to as the **local action** of \( G \). By arc-transitivity, \( G_v^\Gamma(v) \) is isomorphic to one of the following five transitive groups of degree 4: \( \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4, D_4, A_4 \) and \( S_4 \).

If \( G_v^\Gamma(v) \) is isomorphic to one of the 2-transitive groups \( A_4, S_4 \), then \( G \) acts transitively on the set of 2-arcs of \( \Gamma \), and \( \Gamma \) is called \( (G,2)-\text{arc-transitive} \).

If \( G_v^\Gamma(v) \) is isomorphic to one of the regular groups \( \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \), then \( G \) acts regularly on the set of arcs of \( \Gamma \), and \( \Gamma \) is called \( G-\text{arc-regular} \).

Finally, if \( G_v^\Gamma(v) \cong D_4 \), then \( \Gamma \) is called \( (G,1\frac{1}{2})-\text{arc-transitive} \). These graphs will be discussed in detail in Section 5.

In this section, we will be interested in the problem of determining how many \( G \)-invariant cycle structures \( \Gamma \) admits. The main result of the section is Theorem 4.2, where the answer is given in terms of the local action of \( G \). We first prove the following auxiliary result.

**Lemma 4.1.** Let \((\Gamma,\mathcal{C})\) and \((\Gamma,\mathcal{D})\) be distinct cycle decompositions of a connected tetravalent graph \( \Gamma \). If \( G \) is a subgroup of \( \text{Aut}(\Gamma,\mathcal{C}) \cap \text{Aut}(\Gamma,\mathcal{D}) \) acting transitively on the vertices of \( \Gamma \), then \( G_{uv} = 1 \) for every edge \( \{u,v\} \) of \( \Gamma \).

**Proof.** Let us first prove that two distinct cycles \( C_0 \in \mathcal{C} \) and \( D_0 \in \mathcal{D} \) cannot share two consecutive edges. Suppose the contrary, and let \( \{u_0, v_0\} \) and \( \{v_0, u'_0\} \) be two edges that are consecutive on \( C_0 \) and on \( D_0 \). Let \( w_0 \) and \( w'_0 \) be the other two neighbors of \( v_0 \). Then the edges \( \{w_0, v_0\} \) and \( \{v_0, w'_0\} \) belong to the same cycle \( C'_0 \in \mathcal{C} \), and well as to the same cycle \( D'_0 \in \mathcal{D} \). By vertex-transitivity, the same situation occurs at every vertex \( v \), that is, the four cycles \( C, C' \in \mathcal{C} \) and \( D, D' \in \mathcal{D} \) containing \( v \) come in pairs \( \{C,D\} \) and \( \{C',D'\} \) such that \( C \) and \( D \) share two edges incident with \( v \), and \( C', D' \) share the other two edges incident with \( v \). On the other hand, this cannot be the case for the vertex \( v \) in which the cycles \( C_0 \) and \( D_0 \) split. This contradiction shows that whenever two cycles \( C \in \mathcal{C} \) and \( D \in \mathcal{D} \) share two consecutive edges, they coincide.

We now show that \( \mathcal{C} \cap \mathcal{D} = \emptyset \). Suppose, on the contrary, that \( C \in \mathcal{C} \cap \mathcal{D} \), and let \( v \) be a vertex of \( C \). Then the other two cycles \( C' \in \mathcal{C} \) and \( D' \in \mathcal{D} \) containing \( v \) share two consecutive edges (namely, the two edges incident with \( v \) that are not contained in \( C \)), and by what we have proved above, we see that \( C' = D' \). By vertex-transitivity, the same situation occurs at every vertex \( v \),
which implies that every cycle in $C$ coincides with some cycle in $D$. But then $C = D$, which contradicts the assumptions.

We have thus shown that a cycle in $C$ cannot share two consecutive edges with a cycle in $D$. Now let $(u, v)$ be an arbitrary arc of $\Gamma$, and let $C$ and $D$ be the cycles from $C$ and $D$, respectively, which contain $(u, v)$. By the above argument, the vertex following $(u, v)$ on $C$ is not the same as the vertex following $(u, v)$ on $D$. On the other hand, $G_{uv}$ fixes $C$ and $D$ pointwise, and so it fixes three neighbors of $v$. But then $G_{uv}$ fixes all four neighbors of $v$, and by arbitrariness of $(u, v)$ and connectivity of $\Gamma$, we conclude that $G_{uv}$ is trivial. $\square$

**Theorem 4.2.** Let $G$ be an arc-transitive group of automorphisms of a connected tetravalent graph $\Gamma$. Then the following holds:

(i) If $G$ is 2-arc-transitive (equivalently, $G^{\Gamma(v)}_v \cong A_4$ or $S_4$), then there are no $G$-invariant cycle decompositions of $\Gamma$;

(ii) If $G^{\Gamma(v)}_v \cong D_4$ or $Z_4$, then there is a unique $G$-invariant cycle decomposition of $\Gamma$;

(iii) If $G^{\Gamma(v)}_v \cong Z_2 \times Z_2$, then there are exactly 3 $G$-invariant cycle decompositions of $\Gamma$.

**Proof.** We begin with the following observation: If $(\Gamma, C)$ is a cycle decomposition such that $G \leq \text{Aut}(\Gamma, C)$, $v$ is a vertex of $\Gamma$ and $C \in \mathcal{C}$ such that $v \in C$, then the $C$-neighbors of $v$ form a block of imprimitivity of $G^{\Gamma(v)}_v$. This immediately implies statement (i).

To prove statement (ii), assume that $G^{\Gamma(v)}_v \cong D_4$ or $Z_4$. These groups (in their natural actions on four points) admit a unique system of imprimitivity. Two edges will be called *opposite*, if they share an endpoint, say $u$, and if the other two endpoints form a block of imprimitivity for $G^{\Gamma(u)}_u$. Note that each edge is opposite to exactly two other edges (one at each end). Hence equivalence classes of the transitive envelope of the relation “opposite” form a cycle decomposition $C$ of $\Gamma$. Since $G^{\Gamma(u)}_u$ admits a unique system of imprimitivity it is easy to see that this decomposition $C$ is $G$-invariant. Thus there is at least one $G$-invariant cycle decomposition of $\Gamma$. Since each local system of imprimitivity is unique, $C$ is unique. This completes the proof of statement (ii).

Let us now prove statement (iii). Note that $G^{\Gamma(v)}_v \cong Z_2 \times Z_2$ implies that $G$ acts regularly on the arcs of $\Gamma$. By Theorem 2.2 we know that there are three $G$-orbits of $G$-consistent cycles. In view of Lemma 2.1 it suffices to show that all these orbits are symmetric and have self-overlap 0. We first show that the orbits are symmetric.

Let $C$ be one of these three $G$-orbits, let $C \in \mathcal{C}$, let $v$ be a vertex of $C$, let $u, w$ be the $C$-neighbors of $v$, and let $x, y$ be the other two neighbors of $v$. Since $G$ acts regularly on the arcs of $\Gamma$ there exist unique involutions $\alpha, \beta \in G_v$ acting on $\Gamma(v)$ as permutations $(u, w)(x, y)$ and $(u, x)(w, y)$, respectively. It is well known that $G$ is generated by the stabilizer $G_v = \langle \alpha, \beta \rangle$ and the unique involution $\gamma$ which reverses the arc $(v, w)$. By the definition of consistent cycles, the setwise stabilizer $G_C$ of $C$ contains a cyclic subgroup $Z$ of order the length of $C$. The group $Z$ is generated by an automorphism which maps $(u, v)$ to $(v, w)$. However, $\alpha \gamma$ also maps $(u, v)$ to $(v, w)$, and by regularity of the action of $G$ on the arcs of $\Gamma$, it follows that $Z = \langle \alpha \gamma \rangle$. Since $(\alpha \gamma)^a = a a \alpha a \gamma \alpha = \gamma \alpha = (\alpha \gamma)^{-1}$, we see that $\alpha$ normalizes $Z$, and therefore preserves the set of orbits of $Z$. On the other hand, $C$ is an orbit of $Z$ (in its action on the edges of $\Gamma$), and $a$ maps the edge $\{u, v\}$ of $C$ to the edge $\{v, w\}$ of $C$. Therefore, $\alpha \in G_C \setminus Z$, showing that $G_C$ is dihedral and hence $C$ is symmetric.

Finally, since $G_v$ acts regularly on $\Gamma(v)$, it is clear that self-overlap of every symmetric $G$-orbit of $G$-consistent cycles is 0. $\square$
4.1. Relationship with maps

As we have already seen in Section 3.2, cycle structures arise naturally in the context of maps. As we will now show, every cycle structure which is invariant under an arc-regular group $G$ acting locally as $\mathbb{Z}_2 \times \mathbb{Z}_2$ is isomorphic to $\text{CS}(M)$ for some reflexible map $M$, where the CS operator is defined in Section 3.2. We start with a general lemma about how to obtain a map from a set of cycles in a tetravalent graph.

Lemma 4.3. Let $\Gamma$ be a connected tetravalent graph and let $C$ be a set of cycles of $\Gamma$ with self-overlap 1 and with the property that every edge of $\Gamma$ belongs to exactly two cycles in $C$. Then the topological space obtained from $(\Gamma, C)$ by spanning each cycle in $C$ by a disc is a closed surface.

Proof. By the definition of surface, we need to show that every point of the obtained topological space $X$ has a neighborhood homeomorphic to the unit disc in the Euclidean plane. Clearly, every point from the interior of any of the added discs satisfies this property; and since every edge of $\Gamma$ belongs to exactly two cycles in $C$, this also is true for every point lying in the interior of an edge.

Let us now consider a point of $X$ which corresponds to a vertex $v$ of $\Gamma$. Let $a \in \Gamma(v)$, and let $A, B \in C$ be the cycles of $\Gamma$ containing the edge $(a, v)$. Let $b \in \Gamma(v) \setminus \{a\}$ be such that the 2-arc $(a, v, b)$ lies on $A$, and let $c \in \Gamma(v) \setminus \{a\}$ be such that the 2-arc $(a, v, c)$ lies on $B$. Since the overlap of $A$ and $B$ is 1, we obtain $b \neq c$. Now let $d$ be the neighbor of $v$ other than $a, b$ and $c$. Let $C, D \in C$ be the cycles of $\Gamma$, containing the edge $(d, v)$. Since the edge $(a, v)$ is not contained in any of $C$ and $D$, and since the overlap of $C$ and $D$ is 1, we may assume that the 2-arc $(d, v, b)$ lies on $C$, and that the 2-arc $(d, v, c)$ lies on $D$, as in Fig. 3. It is now clear that $v$ has a neighborhood homeomorphic to the unit disc in the Euclidean plane. □

Lemma 4.4. Let $G$ be an arc-transitive group of automorphisms of a connected tetravalent graph $\Gamma$, and let $v \in V(\Gamma)$. Suppose that $G^{(v)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then for every $G$-invariant cycle structure $C$ on $\Gamma$ there exists a reflexible map $M$, such that $C$ is isomorphic to $\text{CS}(M)$.

Proof. By Theorem 4.2, there are three $G$-invariant cycle structures on $\Gamma$; let $D$ be any of them other than $C$. Since $C$ and $D$ each have self-overlap 0, and since we showed in the proof of Lemma 4.1 that cycles from the two structures have no two consecutive edges in common, we see that the collection $E = C \cup D$ satisfies the hypotheses of Lemma 4.3. Thus, there is a map on a surface whose faces are the cycles of $E$, and every edge joins a face from $C$ to a face from $D$.
We form a new map on this surface by choosing a new vertex in the center of each face from $D$; at each vertex $v$ of $\Gamma$, we then join the two faces corresponding to $D$-cycles containing $v$ by a new edge through $v$. For example, consider Fig. 3, and suppose that face $C$ is from $C$; then $B$ is also from $C$, while $A$ and $D$ are from $D$. The new edge corresponding to $v$ goes diagonally from the center of $A$ through $v$ to the center of $D$. The new vertices and new edges form a map $M$ on the surface, and it is clear that $\Gamma = MG(M)$.

To show that $M$ is reflexible, first observe that every element of $G$ acts as a symmetry of $M$. Since $G$ is transitive on vertices of $\Gamma$, it must be transitive on edges of $M$. Since each $G_v$ has order 4, each stabilizer of an edge of $M$ must have order 4, and so $M$ is reflexible, as required. 

We conclude the section with the main result of this paper, announced in the introduction.

**Theorem 4.5.** Let $\Gamma$ be a connected tetravalent arc-transitive but not 2-arc-transitive graph. Then either $\Gamma$ admits exactly one cycle structure, or $\Gamma$ is isomorphic to the medial graph of a reflexible map. In the latter case, $\Gamma$ admits at least three (not necessarily non-isomorphic) cycle structures.

**Proof.** If $Aut(\Gamma)$ contains an arc-transitive subgroup $G$ with $G_v^{\Gamma(v)} \cong \mathbb{Z}_2^2$, then, by Lemma 4.4, $\Gamma$ is isomorphic to the medial graph of a reflexible map, and the result holds. Assume henceforth that no such subgroup exists. In particular, $Aut(\Gamma)_v^{\Gamma(v)} \cong D_4$ or $\mathbb{Z}_4$. Then by Theorem 4.2, $Aut(\Gamma)$ admits a unique cycle structure, say $C$. Note that $C$ is $G$-invariant for any subgroup of $Aut(\Gamma)$. On the other hand, if $G \leq Aut(\Gamma)$ is arc-transitive, then by our assumption that $G_v^{\Gamma(v)} \not\cong \mathbb{Z}_2^2$ and Theorem 4.2, $G$ admits a unique cycle structure. Hence $C$ is the only cycle structure in $\Gamma$. 

5. Cycle structures and $1\frac{1}{2}$-arc-transitive graphs

Recall that every tetravalent $G$-arc-transitive graph falls into one of the following three families: $(G, 2)$-arc-transitive graphs (where $G_v^{\Gamma(v)}$ is isomorphic to $A_4$ or $S_4$), $(G, 1\frac{1}{2})$-arc-transitive graphs (where $G_v^{\Gamma(v)}$ is isomorphic to $D_4$), and $G$-arc-regular graphs (where $G_v^{\Gamma(v)}$ is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$).

The families of $(G, 2)$-arc-transitive and $G$-arc-regular graphs were extensively studied and the structure of each is fairly well understood. For example, it is known that if $G_v^{\Gamma(v)} \cong \mathbb{Z}_4$, then $\Gamma$ is the underlying graph of a tetravalent orientable map, which might be chiral or reflexible. Similarly, if $G_v^{\Gamma(v)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\Gamma$ is the medial graph of a reflexible map. The fact that the vertex-stabilizer $G_v$ is of order 4, makes the family of $G$-arc-regular graphs accessible to a computer based generation; see [4,10].

The $(G, 2)$-arc transitive graphs were studied by Gardiner and Weiss [5,6,16]. It follows from their work that the order of $G_v$ in such graphs divides $2^4 \cdot 3^6$. Similarly as in the $G$-arc-regular case, the fact that the order of $G_v$ is bounded by a constant enables one to produce a complete list of all such graphs on up to a given number of vertices. This has been done recently in [11] for the upper bound of 512 vertices.

In a $1\frac{1}{2}$-arc-transitive graph, on the other hand, vertex-stabilizers can be arbitrarily large. For example, the vertex-stabilizer in the wreath graph $W(n, 2)$ has order $2^n$. This makes the task of finding all $1\frac{1}{2}$-arc-transitive graphs on up to a given number of vertices difficult. Some progress
on such graphs of girth 4 was made in [12]. In what follows we will describe a possible approach to $1\frac{1}{2}$-arc-transitive graphs via studying their cycle structures.

Since graphs admitting an arc-regular group of automorphisms are isomorphic to underlying graphs of regular maps or their medials, we shall concentrate on those $1\frac{1}{2}$-arc-transitive graphs for which every arc-transitive group of automorphisms is also $1\frac{1}{2}$-arc-transitive. In view of Corollary 4.5 such graphs admit a unique cycle structure.

Let $(\Gamma, C)$ be a $G$-invariant cycle structure. The intersection of two intersecting cycles in $C$ is called an attachment set. By arc-transitivity of $G$ all attachment sets have the same size, called the attachment number of $(\Gamma, C)$, and denoted by $\text{Att}(\Gamma, C)$. Note that the attachment sets form a system of imprimitivity for the action of the group $G$ on the vertices of $\Gamma$.

In general, the attachment number can be arbitrarily large: consider the wreath graph $W(n, 2)$ together with the cycle structure consisting of two cycles of length $2n$, described in Section 3.1, where the attachment number equals $2n$. However, the situation changes if the cycle structure is invariant under a $1\frac{1}{2}$-arc-transitive group.

**Lemma 5.1.** Let $\Gamma$ be a connected tetravalent graph, and let $(\Gamma, C)$ be a cycle decomposition which is invariant under a $1\frac{1}{2}$-arc-transitive group $G$. Then $\text{Att}(\Gamma, C) \leq 2$. Moreover, if $\text{Att}(\Gamma, C) = 2$, then the length of cycles in $C$ is even, and the attachment sets correspond to pairs of antipodal vertices of cycles in $C$.

**Proof.** Let $v$ be a common vertex of two distinct cycles $C_1, C_2 \in C$. Since $G^{\Gamma(v)}_v \cong D_4$, there exist an element $g \in G_v$ fixing the $C_1$-neighbors of $v$ and swapping the $C_2$-neighbors of $v$. Such an element $g$ therefore fixes $C_1$ pointwise, and acts as a reflection on $C_2$. Therefore, $g$ fixes at most one more vertex of $C_2$, namely the vertex which is antipodal to $v$ in $C_2$. The claim of the lemma now follows. \qed

Cycle structures with attachment number 1 will be called loosely attached, and those with attachment number 2, antipodally attached.

Suppose now that $\Gamma$ is a $1\frac{1}{2}$-arc-transitive graph. Let $G = \text{Aut}(\Gamma)$ and let $(\Gamma, C)$ be the unique $G$-invariant cycle structure (see Theorem 4.2).

Suppose that the cycles in $C$ are antipodally attached. Then the length of the cycles in $C$ is even, say $2r$. If $r = 2$, then it is easy to see that $\Gamma$ is a wreath graph described in Section 3.1. We will assume henceforth that $r \geq 3$.

Let $B$ be the collection of attachment sets of $(\Gamma, C)$, and consider the quotient graph $\Gamma_B$ whose vertex set is $B$, and two elements of $B$ are adjacent whenever there is an edge in $\Gamma$ joining a vertex of one to a vertex of the other. Since $r \geq 3$, $\Gamma_B$ is a simple graph of valence 4. Moreover, the mapping $\varphi : \Gamma \to \Gamma_B$ which maps a vertex of $\Gamma$ to the attachment set containing it is a covering projection. (A graph morphism is a covering projection if it is surjective and maps the neighborhood of each vertex bijectively onto the neighborhood of its image. See, for example, [7] for a thorough treatment of covering projections of graphs.) Therefore, $\Gamma$ can be considered as a two-fold cover of $\Gamma_B$. Furthermore, the automorphism $\tau$ of $\Gamma$ which swaps the vertices in each attachment set preserves the decomposition $C$, and is therefore contained in $G$. This implies that $\varphi$ is a regular covering projection, and that $G$ is a lift of a $1\frac{1}{2}$-arc-transitive group of automorphisms of $\Gamma_B$, isomorphic to $G/\langle \tau \rangle$. Note that $\Gamma_B$ together with the projection of $C$ along $\varphi$ forms a loosely attached cycle decomposition which is invariant under $G/\langle \tau \rangle$. In this sense we can conclude that the antipodally attached cycle structure $(\Gamma, C)$ arises as a regular two-fold cover of a loosely attached cycle structure $(\Gamma', C')$, in such a way that $\text{Aut}(\Gamma, C)$ is a
lift of some $1\frac{1}{2}$-arc-transitive subgroup of $\text{Aut}(\Gamma', C')$. We see from this that the first step in the study of $1\frac{1}{2}$-arc-transitive graphs should be to study the case of loosely attached $1\frac{1}{2}$-arc-transitive cycle structures.

Loosely attached cycle structures constitute a topic which deserves a careful and detailed treatment. As a possible approach, we suggest analyzing the corresponding graph of cycles, whose vertices are the cycles in the cycle structure with two cycles being adjacent if and only if they intersect. If $(\Gamma, C)$ is a loosely attached $(G, 1\frac{1}{2})$-arc-transitive cycle structure and $\Lambda$ the corresponding graph of cycles, then $G$ acts transitively on the arcs of $\Lambda$. What is more, the stabilizer of a vertex of $\Lambda$ in $G$ acts on the neighborhood as a dihedral group. Note also the valence of $\Lambda$ equals to the length of the cycles in $C$. The central question here is: Which $G$-arc-transitive graphs $\Lambda$ (of arbitrary valence) where $G^A_v(v) \cong D_4$ arise as graphs of cycles, and how to reconstruct the cycle structures from their graphs of cycles. We leave the detailed analysis of these questions for the future projects.

We end this section with an example illustrating the above discussion. Let $\Gamma$ be the skeleton of the map $\{4, 4\}_{3, 3}$, shown in Fig. 4.

The labeling of the vertices shows the identifications of edges. Let $G$ be the automorphism group of the map. It is not difficult to see that the map $\{4, 4\}_{3, 3}$ is reflexible, and so its skeleton is $(G, 1\frac{1}{2})$-arc transitive. In Fig. 4, the unique cycle structure for $G$ consists of the horizontal lines, such as $(7, 8, 9, 10, 11, 12)$, and the vertical lines, such as $(5, 11, 17, 2, 8, 14)$. These two adjacent cycles meet at 11 and at 8, and so they are antipodally attached. Identifying antipodal vertices gives the graph underlying the regular map $\{4, 4\}_{3, 0}$, shown in Fig. 5. The underlying graph is isomorphic to the Cartesian square $C_3 \square C_3$. Again, the cycle structure consists of the vertical and horizontal lines, but now they are loosely attached.
This example will generalize in two ways: First, the map \( \{4, 4\}_{k,k} \) is always reflexible, and so its graph is \( 1\frac{1}{2} \)-arc transitive. Its horizontal and vertical lines form a cycle structure with cycles of length \( 2k \), and these are antipodally attached. Identification of antipodal vertices results in the factor map \( \{4, 4\}_{k,0} \), whose skeleton is loosely attached.

Secondly, let \((\Gamma, C)\) be any cycle structure in which the cycles have odd length \( k \). Let \( B(\Gamma) \) be the bipartite double cover (also called the canonical double) of \( \Gamma \). Each cycle in \( C \) will be covered by a single \( 2k \)-cycle in \( B(\Gamma) \). These \( 2k \)-cycles will form a cycle structure, and it will be antipodally attached. The two cycle structures will have the same local actions. More precisely, if \( G \) is any arc-transitive subgroup of \( \text{Aut}(\Gamma, C) \), then \( G \) lifts to a group of symmetries of the larger structure having the same local action.

Note that when \( k \) is odd, these two constructions coincide; i.e., \( B(\{4, 4\}_{k,0}) = \{4, 4\}_{k,k} \).

References

[18] S.-L. Wu, H.-L. Fu, Cyclic \( m \)-cycle systems with \( m \leq 32 \) or \( m = 2q \) with \( q \) a prime power, J. Combin. Des. 14 (2006) 66–81.