Unbiased estimation of Markov jump systems with distributed delays

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ABSTRACT

The unbiased $H_\infty$ filtering problem is considered for a class of Markov jump systems (MJSs) with distributed time-delays. Based on the selected Lyapunov–Krasovskii functional, it gives a sufficient condition for the existence of the mode-dependent unbiased $H_\infty$ filter such that the filtering error dynamic MJSs is stochastically stable and satisfies a prescribed level of $H_\infty$ disturbance attenuation in an infinite time-interval. The design criterions are presented in the form of linear matrix inequality techniques, and then are described as the optimization problems. At last, two numerical examples are employed to illustrate the effectiveness of the developed techniques.

1. Introduction

In the past several decades, dynamical systems with distributed time-delays have received considerable attention. Many efforts have been made to study the stability and control problems of this kind of systems. Among these results, an important and popular approach is to construct a Lyapunov–Krasovskii function for the distributed time-delays. Then by using the bonding techniques to the cross-term, delay-dependent or delay-independent criteria are obtained. Comparing with the delay-independent criteria, the delay-dependent ones are less conservative conditions for the stability analysis and controller synthesis. And the measuring index is the maximum allowable upper bound in the delays, for instance [1–6]. But for filtering schemes, the conservative conditions are seldom considered. The filtering problem is to estimate the unavailable state variables of a known system by using past measurement.

During the past several years, $H_\infty$ filtering problems have been investigated extensively. By comparison with the traditional Kalman filtering scheme [7], the noise sources in the $H_\infty$ setting are arbitrary signals with bounded energy or average power, and no exact statistical properties are required to be known. For more results on this topic, readers are referred to Refs. [8–12]. It should point out that the dimensions of the filtering error system, which is constructed by the dynamics and the filter, are twice that of the dynamics. This may bring the computational complexity, especially in practical engineering process where the number of states is numerous. For these, the so-called unbiased conditions [13] can provide a good help. Under this condition, the dimension of the unbiased filtering error dynamics equals to that of the dynamic system. Moreover, no restriction is imposed on the stability of original dynamics while taking into account the unbiased condition. So this filtering approach has extensive application in unstable systems. For more results on this topic, we refer readers to [14–19] and the references therein.

In general, the parameters of many dynamical systems are subject to random abrupt changes due to, for example, sudden environment changes, subsystem switching, system
noises, failures occurred in components or interconnections and executor faults, etc. Markov jump systems (MJSs), as a special kind of hybrid systems with two components, the mode and the state, can be employed to model the above phenomena. The dynamics of jump modes and continuous states in MJSs are respectively modeled by finite state Markov chains and differential equations. With two dynamical mechanisms, time-evolving and event-driven, the applications of MJSs are more comprehensive since the pioneering work of Krasovskii and Lidskii on quadratic control [20] in the early 1960s. The existing results about MJSs cover a large variety of problems such as stochastic stability [21–25] stochastic controllability [26–28] and the references therein. In recent years, the filtering schemes for MJSs have gained a great deal of attention, and a large number of results [29–38] are also available. It is worth noticing that towards each case above, more details are related to the regular $H_{\infty}$ filtering problems and the system time-delay is the retarded one which only contains time-delay in its states. In practice, with respect to modeling errors, unknown inputs and distributed time-delayed dynamic systems may be more reasonable to account for the realistic process. This motivates our research of this topic.

In this paper, we consider the design problem of unbiased $H_{\infty}$ filtering for a class of distributed time-delayed MJSs. We first construct the full-order linear $H_{\infty}$ filter with distributed time-delays of original MJSs. Subsequently, under the unbiased filtering condition, the dynamics of filtering error are developed. Then, in order to guarantee the robustness against disturbances, the $H_{\infty}$ filtering problem is formulated to minimize the influences of the unknown disturbances. Furthermore, by using the constructed Lyapunov–Krasovskii functional and linear matrix inequalities (LMIs) [39] approach, the sufficient condition on the solution of unbiased $H_{\infty}$ filter are presented and proved. Finally, the unbiased $H_{\infty}$ filtering problem is formulated as an optimization algorithm. Simulation results illustrate the effectiveness of the developed techniques.

In the sequel, if not explicitly states, matrices are assumed to have compatible dimensions. The notations used throughout this paper are quite standard. The symbols $\mathbb{R}^n$ and $\mathbb{R}^{n \times m}$ stand for an $n$-dimensional Euclidean space and the set of all $n \times m$ real matrices, respectively. $A^T$ and $A^{-1}$ denote the matrix transpose and matrix inverse, $\text{diag} \{A, B\}$ represents the block-diagonal matrix of $A$ and $B$. $\sigma_{\min}(\cdot)$, $\sigma_{\max}(\cdot)$ mean the minimum and the maximum eigenvalue of the corresponding matrices. $E[\cdot]$ stands for the mathematics statistical expectation of the stochastic process or vector and $\|\cdot\|$ is the Euclidean vector norm. $L_2^c[0, +\infty)$ is the space of $n$-dimensional square integrable function vector over $[0, +\infty)$. $P > 0$ stands for a positive-definite matrix, $I$ is the unit matrix with appropriate dimensions. $0$ is the zero matrix with appropriate dimensions. In symmetric block matrices, we use “*” as an ellipsis for the terms that are introduced by symmetry.

2. System description

Given a probability space $(\Omega, F, P)$ where $\Omega$ is the sample space, $F$ is the algebra of events and $P$ is the probability measure defined on $\Omega$. Let us consider a class of linear distributed time-delayed MJSs defined in the probability space $(\Omega, F, P)$ and described by the following differential equations:

$$
\begin{align*}
\dot{x}(t) & = A_1(\delta(t))x(t) + A_2(\delta(t))x(t - \tau) + A_3(\delta(t))f(t) + B_1(\delta(t))w(t) \\
y(t) & = C_1(\delta(t))x(t) + D_1(\delta(t))w(t) \\
z(t) & = C_2(\delta(t))x(t) + D_2(\delta(t))w(t) \\
\dot{z}(t) & = z(t) - \xi(t), t \in [t_0 - \tau, t_0] 
\end{align*}
$$

(1)

where $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^l$ is the measured output, $w(t) \in L_2^c[0, +\infty)$ is the unknown disturbance, $z(t) \in \mathbb{R}^m$ is the controlled output, $\xi(t)$ is a vector-valued initial continuous function defined on the interval $[t_0 - \tau, t_0]$ and $\xi(t)$ is the initial mode, the time delays $\tau > 0$, $\eta > 0$ and $r > 0$ are assumed to be known and

$$
H = \max \left\{ h, \eta, \tau \right\}, A(\delta), A_1(\delta), A_2(\delta), A_3(\delta), B(\delta), C_s(\delta), C_2(\delta), D_s(\delta), D_2(\delta)
$$

are known mode-dependent constant matrices with appropriate dimensions.

The jump parameter $\delta(t)$ in (1) represents a continuous-time discrete-state Markov stochastic process taking values on a finite set $\mathbb{M} = \{1, 2, \ldots, N\}$ with transition rate matrix $\Pi = (\pi_{ij})$, $i, j \in \mathbb{M}$ and has the following transition probability from mode $i$ at time $t$ to mode $j$ at time $t + \Delta t$ as

$$
P_{ij} = P_t[\delta(t + \Delta t) = j | \delta(t) = i] = \left\{ \begin{array}{ll}
\pi_{ij}\Delta t + o(\Delta t), & i \neq j \\
1 + \pi_{ij}\Delta t + o(\Delta t), & i = j
\end{array} \right.
$$

(2)

where $\Delta t > 0$ and $\lim(\Delta t)/\Delta t \rightarrow 0$. $\pi_{ij} \geq 0$ is the transition probability rates from mode $i$ at time $t$ to mode $j$ at time $t + \Delta t$, and $\sum_{j=1}^N\pi_{ij} = -\pi_{ii}$.

Remark 1. To simplify the study, we assume that the time-delays in (1) are constant and only dependent of the system structure, and they are not dependent on the defined stochastic process. Furthermore, we will take the initial time $t_0 = 0$ and let the initial values $\{\delta(t)\}_{t \in [-d, 0]}$ and $\{\xi(t) = \delta(t)\}_{t \in [-d, 0]}$ be fixed. At each mode, we assume that the time-delay MJSs have the same dimension. For convenience, when $\delta_i = 1$, $A(\delta_i), A_1(\delta_i), A_2(\delta_i), A_3(\delta_i), B(\delta_i), C_s(\delta_i), C_2(\delta_i), D_s(\delta_i), D_2(\delta_i)$ are respectively denoted as $A_i, A_{1i}, A_{2i}, A_{3i}, B_i, C_{si}, C_{2i}, D_{si}, D_{2i}$.

Remark 2. For the relevant estimation problems of the distributed time-delayed MJSs (1), the matrices parameters $A_{2i}, A_{3i}$, satisfy $A_{2i} \neq 0$, $A_{3i} \neq 0$. In order to guarantee the stability of the differential operator $\chi(x(t)) = x(t) - A_{3i}x(t - \tau)$, we always set $\|A_{3i}\| < 1$, where $\|\cdot\|$ is the relevant matrix norm of every modes. In fact, Markov jump systems with mixed delays (constant, varying, distributed or neutral time-delays) have received some attention and several research results have been proposed [1–6].

In this paper, our aim is to develop techniques of unbiased $H_{\infty}$ filtering problem for time-delay MJSs (1) and to obtain an estimate $\hat{z}(t)$ of the signal $z(t)$ such that the defined index performance is minimized in an unbiased estimation error sense. Thus, we construct the
following full-order linear filter:

\[
\begin{align*}
\dot{x}(t) &= A_f \dot{x}(t-r) + A_1 \dot{x}(t-h) + A_{21} \int_{t-\tau}^{t} \dot{x}(s)ds + B_f \nu(t) \\
\dot{z}(t) &= C_f \dot{x}(t) + D_f \nu(t) \\
\hat{z}(t) &= \varphi(t), \quad \delta_t = \zeta(t), \quad t \in [-H, 0]
\end{align*}
\]  

(3)

where \( \dot{x}(t) \in \mathbb{R}^n \) is the filter state, \( z(t) \in \mathbb{R}^n \), \( \varphi(t) \) is a continuous vector-valued initial function and \( A_f, B_f, C_f, D_f \) are the unknown filter parameters to be designed for each value \( i \in M \).

Define \( e(t) = x(t) - \dot{x}(t) \) and \( r(t) = z(t) - \hat{z}(t) \). The dynamic filtering error system can be reconstructed as:

\[
\begin{align*}
\dot{e}(t) &= (A_i - B_f C_yi) \dot{e}(t) + A_1 \dot{e}(t-h) + A_{21} \int_{t-\tau}^{t} \dot{e}(s)ds + B_f \nu(t) \\
r(t) &= (C_i - D_f C_yi) \dot{e}(t) - C_{fi} \dot{x}(t) + (D_i - D_{fi}) \nu(t)
\end{align*}
\]  

(4)

Considering the following unbiased condition \([15, 16, 18, 19]\),

\[
A_{fi} = A_i - B_f C_{yi}, \quad C_{fi} = C_{i} - D_{fi} C_{yi}.
\]  

(5)

It follows from the filtering error system (4) that,

\[
\begin{align*}
\dot{e}(t) &= A_i \dot{e}(t) + A_1 \dot{e}(t-h) + A_{21} \int_{t-\tau}^{t} \dot{e}(s)ds + B_f \nu(t) \\
r(t) &= \tau \dot{e}(t) + D_f \nu(t) \\
\dot{e}(t) &= \sigma(t) - \varphi(t), \quad \delta_t = \zeta(t), \quad t \in [-H, 0]
\end{align*}
\]  

(6)

where \( A_i = A_{fi}, \quad B_i = B_i - B_f D_{yi}, \quad C_i = C_{fi} = C_{i} - D_{fi} C_{yi}, \quad D_i = D_{i} - D_{fi} D_{yi} \).

Remark 3. By the so-called unbiased conditions, the state variables \( x(t) \) of the original time-delay MJS is (1) do not influence the filtering error system (3). And the stability of filtering system (9) only relates to unbiased filtering system (3) since \( A_i = A_i \). Therefore, this filtering approach can be considered as a sensible filtering technique for unstable system. By comparison with the conventional \( H_\infty \) filtering, it has been shown that the dimensions are reduced into half and the design approaches are then becoming feasible and simplifier.

In order to propose the unbiased \( H_\infty \) filtering problem of time-delay MJS in (1), the following definitions are required.

Definition 1. The dynamic filtering error system (6) (setting \( \nu(t) \equiv 0 \)) is said to be stochastically stable, if for any initial state \( e(t) = \epsilon(t) \) and mode \( \delta_t = \zeta(t) \), then

\[
\lim_{t \to \infty} \mathbb{E} \left\{ \int_{0}^{T} (e(t, \epsilon(t), \zeta(t)))^2 dt \right\} < \infty.
\]  

(7)

Definition 2. (Mao [22]) In the Euclidean space \( \mathbb{R}^n \times M \times \mathbb{R}_+ \), we introduce a stochastic positive functional as

\[
V(e(t), \delta_t = i, \quad t > 0), \quad \text{the weak infinitesimal operator of which satisfies}
\]

\[
\mathbb{E} \left\{ V(e(t+\Delta t), \delta_t + \Delta t, t+\Delta t) \right\} - V(e(t), i).
\]  

(8)

Definition 3. Consider the filtering error system (6). Given a scalar \( \gamma > 0 \), the mapping from \( \nu(t) \) to \( r(t) \) is said to have \( L_2 \)-gain less than or equal to \( \gamma \), that is, the time-delay MJSs (6) possesses \( \gamma \)-disturbance attenuation property with unbiased estimation, if for any \( \nu(t) \in L^2_\nu (0, +\infty) \), \( r(t) \neq 0 \),

\[
\mathbb{E} \left\{ \int_{0}^{T} r(t)^2 dt \right\} \leq \gamma^2 \int_{0}^{T} \nu(t)^2 dt
\]  

(9)

holds for all \( T > 0 \) under zero initial condition, i.e., \( \epsilon(t) = 0 \). Let

\[
\|r(t)\|_2 = \left\{ \int_{0}^{T} r(t)^2 dt \right\}^{1/2}, \quad \|\nu(t)\|_2 = \left\{ \int_{0}^{T} \nu(t)^2 dt \right\}^{1/2},
\]  

(10)

Hence, relation (13) follows that \( \|r(t)\|_2 < \gamma \). In other words, \( \gamma \)-disturbance attenuation implies \( \gamma \)-suboptimal stochastic \( H_\infty \) filtering with unbiased estimation.

Remark 4. It should be pointed out that the unknown disturbance \( \nu(t) \) in this paper is not considered as a stochastic process, thus the relevant expression in (10) is defined as a regular one. In the design of delay-dependent unbiased \( H_\infty \) filter, the unknown input \( \nu(t) \) is assumed to be arbitrary deterministic signal of bounded energy, and the problem of this paper is to design a filter that guarantees a prescribed bounded for the infinite-time interval induced \( L_2 \) norm of the operator from the unknown input \( \nu(t) \) to the output error signal \( r(t) \); then the stochastic \( H_\infty \) norm of \( r_{\nu, w} \) is,

\[
\|r_{\nu, w}\|_E = \sup_{\nu(t) \in L^2_\nu (0, +\infty)} \|r(t)\|_2.
\]  

3. Unbiased \( H_\infty \) filter design

In this section, we will analyze the unbiased \( H_\infty \) filtering problem of the jump system with distributed time-delays.

Lemma 1. [40] Given a positive-define symmetric matrix \( \Delta \), a positive scalar \( l > 0 \), and a time-varying vector function \( \mu(t) \), the following relation will be hold,

\[
I \int_{t-l}^{t} \mu(t) \Delta \mu(t) dt \geq \left( \int_{t-l}^{t} \mu(t) dt \right)^2 \Delta \left( \int_{t-l}^{t} \mu(t) dt \right).
\]  

(11)

Theorem 1. Given a scalar \( \eta > 0 \), the filtering error dynamic MJSs (6) is stochastically stable, if there exist a set of mode-dependent positive-define symmetric matrices \( P_i > 0 \), and
positive-definite symmetric matrix $Q > 0$, $R > 0$, $S > 0$, such that the following inequality holds for all $i \in M$,

$$
\begin{bmatrix}
\Psi_i & P_i \Gamma_{1i} & P_i \Gamma_{3i} & \eta P_i \Gamma_{2i}
\end{bmatrix}
\begin{bmatrix}
-\tilde{Q} & 0 & 0 & A_{1i}^T
-\tilde{R} & 0 & A_{3i}^T
-\eta S & \eta A_{2i}^T
-\tilde{R}
\end{bmatrix} < 0
$$

(12)

where $\Psi_i = P_i \Gamma_{1i} + A_{3i}^T P_i + \sum_{j=1}^{N} \pi_j P_j + Q + \eta S$.

**Proof.** Let the mode at time $t$ be $i$; that is, $\delta_t = i \in M$. Take the stochastic Lyapunov–Krasovskii functional $V(e(t), \delta_t, t > 0) : \mathbb{R}^N \times M \times \mathbb{R}_+ \to \mathbb{R}_+$ to be

$$
V(e(t), \delta_t) = e^T(t)P_i e(t) + \int_{t-h}^{t} e^T(s)Qe(s)ds
\nonumber
\left. + \int_{t-h}^{t} e^T(s)R(s)Re(s)ds + \int_{t-h}^{t} (s-\theta + \eta e^T(s)Se(s)ds
\right|

\text{where $P_i > 0$ is the given mode-dependent symmetric positive-definite matrix for each modes $i \in M$, and $Q > 0$, $R > 0$, $S > 0$ are the symmetric positive-definite matrices.}

Recalling **Definition 2** and along the trajectories of time-delayed MJSs (7), the weak infinitesimal operator of the stochastic process $\{e(t), \delta_t = i \}_{t \geq 0}$ is given by

$$
\begin{align*}
3V(e(t), i) &= 2e^T(t)P_i e(t) + e^T(t)Qe(t) - e^T(t-h)Qe(t-h) + e^T(t)R e(t) - e^T(t-h)R e(t-h) + \eta e^T(t)Se(t) \\
&+ \int_{t-h}^{t} e^T(s)R e(s)ds + \int_{t-h}^{t} (s-\theta + \eta e^T(s)Se(s)ds
\end{align*}
$$

with

$$
\begin{align*}
2e^T(t)P_i e(t) &= 2e^T(t)P_i \left[ \tilde{A}_i e(t) + A_{1i} e(t-h) \right] \\
&+ A_{2i} \int_{t-h}^{t} e(s)ds + A_{3i} e(t-h)
\end{align*}
$$

$$
\begin{align*}
e^T(t)R e(t) &= \left[ \tilde{A}_i e(t) + A_{1i} e(t-h) \right]^T \\
&+ A_{2i} \int_{t-h}^{t} e(s)ds + A_{3i} e(t-h)
\end{align*}
$$

$$
\begin{align*}
R \left[ \tilde{A}_i e(t) + A_{1i} e(t-h) + A_{2i} \int_{t-h}^{t} e(s)ds + A_{3i} e(t-h) \right]^T.
\end{align*}
$$

Since $\sum_{j=1}^{N} \pi_j = 0$, we have

$$
\begin{align*}
\sum_{j=1}^{N} \pi_j \int_{t-h}^{t} e^T(s)Qe(s)ds &= \sum_{j=1}^{N} \pi_j \left( \int_{t-h}^{t} e^T(s)Qe(s)ds \right) = 0,
\end{align*}
$$

$$
\begin{align*}
\sum_{j=1}^{N} \pi_j \int_{t-h}^{t} e^T(s)R e(s)ds &= \sum_{j=1}^{N} \pi_j \left( \int_{t-h}^{t} e^T(s)R e(s)ds \right) = 0.
\end{align*}
$$

Recalling **Lemma 1**, it not difficult to get the following relation,

$$
\begin{align*}
\int_{t-h}^{t} e^T(s)Se(s)ds &\leq \left( \frac{1}{\eta} \int_{t-h}^{t} e(s)ds \right)^T \left( \frac{1}{\eta} \int_{t-h}^{t} e(s)ds \right).
\end{align*}
$$

Thus, it follows that,

$$
3V(e(t), i) = X^T(t) \Gamma_i X(t) < 0
$$

where

$$
X(t) = \begin{bmatrix}
e(t) & e(t-h) & \dot{e}(t-h) & \frac{1}{\eta} \int_{t-h}^{t} e(s)ds
\end{bmatrix},
$$

$$
\Gamma_i = \begin{bmatrix}
\Gamma_{1i} & \Gamma_{2i} & \Gamma_{3i}
\end{bmatrix},
$$

with

$$
\Gamma_{1i} = \begin{bmatrix}
\Psi_i & \tilde{A}_i^T \Gamma_{1i} & \tilde{A}_i^T \Gamma_{3i} & \eta \tilde{A}_i^T \Gamma_{2i}
\end{bmatrix};
$$

$$
\Gamma_{2i} = \begin{bmatrix}
P_i \Gamma_{1i} & \eta \tilde{A}_i^T \Gamma_{2i}
\end{bmatrix};
$$

$$
\Gamma_{3i} = \begin{bmatrix}
- \eta S & \eta A_{2i}^T \Gamma_{2i}
\end{bmatrix}.
$$

Applying Schur complements, it is obvious that $\Gamma_i < 0$ is equivalent to inequality (12).

If $\Gamma_i < 0$ holds, there will exist matrix $\Phi_i > 0$, such that

$$
3V(e(t), i) = -X^T(t) \Phi_i X(t).
$$

Since $3V(e(t), i) < 0$, we can get

$$
E\left[ V(e(t), \delta_0) \right] < V(e(0), \delta_0) + \int_{0}^{T} e^T(s)Qe(s)ds
$$

$$
+ \int_{0}^{T} \left. e^T(s)Re(s)ds + \int_{0}^{T} (s+\theta)e^T(s)Se(s)ds. \right|

Then, the following relation holds

$$
3V(e(t), i) = -X^T(t) \Phi_i X(t).
$$

Define $M_1 = E\left[ X(\delta_0) \right]^2$, $M_2 = \sup_{-\theta \leq \delta \leq 0} E\left[ e(\delta) \right]^2$, $M_3 = \sup_{-\theta \leq \delta \leq 0} E\left[ \dot{e}(\delta) \right]^2$, $\sigma_Q = \max_{i \in M} \sigma_Q(i)$, $\sigma_R = \sigma_{\max}(R)$, $\sigma_S = \sigma_{\max}(S)$, $\sigma_P = \max_{i \in M} \sigma_P(i)$, $\sigma_{\Phi} = \min_{i \in M} \sigma_{\Phi}(i)$, then we have

$$
X^T(t) \Phi_i X(t) \geq M_1 \sigma_{\Phi}.
$$

$$
V(e(0), \delta_0) \leq \sigma_P M_1 + h \sigma_Q M_2 + r \sigma_R M_3 + \sigma_S M_2 \left( \int_{0}^{T} (s+\theta)ds \right)
$$

$$
= \sigma_P M_1 + (0.5 \sigma^2 \sigma_S + h \sigma_Q) M_2 + r \sigma_R M_3.
$$

Therefore, for the given a minus positive number $\sigma > 0$, we can get

$$
3V(e(t), i) = -X^T(t) \Phi_i X(t)
$$

$$
E\left[ V(e(t), \delta_0) \right] \leq \frac{M_1 \sigma_{\Phi}}{M_1 \sigma_P + (0.5 \sigma^2 \sigma_S + h \sigma_Q) M_2 + r \sigma_R M_3} = -\sigma.
$$

Since $M_1 > 0$, $M_2 > 0$, $M_3 > 0$, $\sigma_Q > 0$, $\sigma_R > 0$, $\sigma_S > 0$, $\sigma_P > 0$, $\sigma_{\Phi} > 0$, we have

$$
3V(e(t), i) \leq -\sigma E\left[ V(e(t), \delta_0) \right].
$$

That is

$$
E\left[ V(e(t), \delta_0) \right] \leq \exp \left( \sigma t \right) E\left[ V(e(0), \delta_0) \right].
$$
Letting $\rho = \sigma_pM_1 + (0.5d^2\sigma_S + h\sigma_D)M_2 + \tau\sigma_RM_3$, we can obtain that, for a given small positive scalar $\lambda > 0$,

$$\lambda E[|e(t)|^2] \leq E[|V(e(t), i)|^2] \leq \rho \exp(-\sigma t).$$

(13)

Letting $t$ go to infinity implies that,

$$\lim_{t \to \infty} E[|e(t)|^2] = 0.$$ 

(14)

Taking limit as $T \to \infty$, one obtains from (13),

$$\lim_{T \to \infty} E\left\{\int_0^T |e^T(t)\|dt; \xi(t)\right\} \leq \lim_{t \to \infty} \left\{\frac{1}{\lambda} (1 - \exp(-\sigma t))\right\}$$

$$= \frac{1}{\lambda} < \infty$$

which implies the filtering error MJSSs (6) is stochastically stable by Definition 1. This completes the proof.

**Remark 5.** In fact, in order to get the sufficient condition of Theorem 1, we always choose the weighting matrices $Q > 0$, $R > 0$, $S > 0$ as mode-dependent ones. For more details of this case, we refer readers to [2,6,18,19,23–25,27–31] and the references therein. But when we use MJSSs to model practical systems, some time-delays might be mode-dependent. In order to decrease the conservatism, the weighting matrices of the relevant Lyapunov–Krasovski functional can be defined as mode-dependent ones. According to the definitions in [2,6,18,19,27,28,31], we know that the filtering error system (6) is mean square stable when the filtering error $e(t)$ satisfies relation (14). By the main proof of Theorem 1, we can see that mean square stable implies stochastically stable. It also means almost surely (asymptotically) stable by the main results in [2,6,18,19,27,28,31] and the references therein. Note that if the filtering error system (6) without jump parameters, the main results will reduce to the result in [14–17].

**Theorem 2.** Given a scalar $\eta > 0$, the filtering error dynamic MJSSs (6) is stochastically stable, and the mapping from $w(t)$ to $r(t)$ has a $L_2$-gain less than or equal to $\gamma$ for all $T > 0$ and $w(t) \in L_2(0, \infty)$ with unbiased estimation condition, if there exist a set of mode-dependent positive-definite symmetric matrices $P_i > 0$, positive-definite symmetric matrix $Q > 0$, $R > 0$, $S > 0$, and mode-dependent matrices $X_i$, $B_f$, $D_f$, such that the following LMIs hold for all $i \in M$,

$$\Xi_i = \begin{bmatrix} \Xi_{1i} & \Xi_{2i} \\ * & \Xi_{3i} \end{bmatrix} < 0,$$

(15)

where

$$\Xi_{1i} = \begin{bmatrix} \Xi_{11i} & \Xi_{12i} \\ * & \Xi_{13i} \end{bmatrix} = \begin{bmatrix} P_iA_{i1} & P_iA_{i2} \\ * & -Q & 0 & 0 \\ * & * & -R & 0 \\ * & * & * & -\eta S \end{bmatrix};$$

$$\Xi_{2i} = \begin{bmatrix} P_iB_i - X_iD_{fi} & C_{j_i} - C_{j_{fi}}D_{fi}^T \\ 0 & 0 & A_{j_i}^T \\ 0 & 0 & 0 \end{bmatrix};$$

$$\Xi_{3i} = \begin{bmatrix} -\gamma^2 I & D_{fi}^T - D_{fi}^TD_{fi}^T \\ * & -I \\ * & * & 0 \end{bmatrix};$$

$$\Xi_{11i} = P_iA_{i1} + A_{i1}^TP_i - C_{j_{fi}}^TC_{j_{fi}} - X_iC_{j_{fi}} + \sum_{j=1}^{n_i} \eta_jP_j + Q + \eta S.$$
Thus, \( J(T) < 0 \) can be guaranteed by \( \Sigma_i < 0 \), which is equivalent to the following inequality by applying Schur complements,

\[
\begin{bmatrix}
\Gamma_{ii} & P_i A_{i1} & P_i A_{i2} & \eta P_i A_{i2} & P_i B_i & A_i^T C_i & C_i^T \\
* & -Q & 0 & 0 & A_{i1}^T & 0 \\
* & * & -R & 0 & 0 & A_{i2}^T & 0 \\
* & * & * & -\eta S & 0 & \eta A_{i2}^T & 0 \\
* & * & * & * & -\gamma^2 I & B_i^T & D_i \\
* & * & * & * & * & -R & 0 \\
* & * & * & * & * & -I & 0 \\
\end{bmatrix} < 0.
\]

Substituting the correlative matrices into above inequality and using some changes of variables with \( X_i = P_i B_i \), we can get LMI (15). It is clear that LMI (15) will reduce to inequality (12) for \( \omega(t) = 0 \), i.e., the dynamic filtering error system (6) is stochastically stable.

On the other hand, for \( T \to \infty \), \( \Sigma_i < 0 \) results in \( J(\infty) < -V(\infty) < 0 \), that is \( \|S_{r,w}\|_E \leq \gamma \). This completes the proof.

**Remark 6.** To obtain an optimized unbiased \( H_\infty \) filtering performance against unknown inputs, time-delays and model errors, the attenuation lever \( \gamma \) can be reduced to the minimum possible value such that (15) is satisfied. The optimization problem can be described as follows:

\[
\begin{align*}
\min_{P_i, X_i, B_i, D_i, \eta, \gamma} & \quad \rho \\
\text{s.t.} & \quad \text{LMI (15) with } \rho = \gamma^2.
\end{align*}
\]

**Remark 7.** Theorem 2 presents the stability criteria and unbiased \( H_\infty \) filtering performance of neutral MJSSs (6). It is necessary to note that LMI (15) is independent of delay parameters \( h \) and \( r \). To solve iteratively the matrix inequalities of Theorem 2 with respect to \( \eta \), we can obtain the maximum allowable value of \( \eta \) for guaranteeing the stochastic stability of filtering error system (4).

**Remark 8.** In [18], the operator \( D(e(t)) \) was used to transform the original system and construct the Lyapunov–Krasovskii functional. Without the aid of the selected stochastic operator, this paper provided an easier unbiased filtering scheme for a class of more complex systems, i.e., distributed time-delayed MJSSs. By using the Matlab LMI Toolbox, it is straightforward to check the feasibility of LMI (15). In order to illustrate the effectiveness of the developed techniques, we will give a numerical example about neutral jump system with time-delays in the following Section 4.

### 4. Simulation examples

**Example 1.** Consider the following distributed time-delayed MJSSs with parameters described as follows:

**Mode 1:**

\[
\begin{align*}
A_1 &= \begin{bmatrix}
-2 & 0 \\
-1 & -3
\end{bmatrix},
A_{11} &= \begin{bmatrix}
0.1 & -0.2 \\
-0.1 & 0.3
\end{bmatrix}, \\
A_{21} &= \begin{bmatrix}
-0.2 & 0 \\
0 & -0.2
\end{bmatrix},
A_{31} &= \begin{bmatrix}
-0.1 & 0.1 \\
0 & -0.2
\end{bmatrix}.
\end{align*}
\]

\[
B_1 = \begin{bmatrix}
0.1 \\
-0.1
\end{bmatrix},
C_{y_1} = \begin{bmatrix}
-0.1 & 0.2
\end{bmatrix},
D_{y_1} = \begin{bmatrix}
-0.2
\end{bmatrix},
D_{z_1} = \begin{bmatrix}
0.2
\end{bmatrix}.
\]

**Mode 2:**

\[
\begin{align*}
A_2 &= \begin{bmatrix}
-2 & 4 \\
-2 & 2
\end{bmatrix},
A_{12} &= \begin{bmatrix}
-0.2 & 0.3 \\
-0.1 & -0.2
\end{bmatrix}, \\
A_{22} &= \begin{bmatrix}
0.1 & -0.2 \\
0.2 & -0.1
\end{bmatrix},
A_{32} &= \begin{bmatrix}
-0.1 & 0.1 \\
-0.2 & 0.1
\end{bmatrix}, \\
B_2 &= \begin{bmatrix}
-0.1 \\
0.2
\end{bmatrix},
C_{y_2} = \begin{bmatrix}
0.1 & -0.1
\end{bmatrix},
C_{z_2} = \begin{bmatrix}
0.2 & 0.1
\end{bmatrix},
D_{y_2} = \begin{bmatrix}
0.1
\end{bmatrix},
D_{z_2} = \begin{bmatrix}
-0.2
\end{bmatrix}.
\end{align*}
\]

Let \( \eta = 1.0, \gamma = 0.9 \) and define the transition rate matrix as \( \Pi = \begin{bmatrix}
-0.5 & 0.5 \\
1 & -1
\end{bmatrix} \).

By solving the LMI (15), we can get the following mode-dependent unbiased \( H_\infty \) filtering parameters:

\[
\begin{align*}
A_{f_1} &= \begin{bmatrix}
-1.05 & 0.10 \\
-0.95 & -3.10
\end{bmatrix},
B_{f_1} = \begin{bmatrix}
-0.5000
\end{bmatrix},
C_{f_1} = \begin{bmatrix}
0 & 0.3
\end{bmatrix},
D_{f_1} = \begin{bmatrix}
-0.1
\end{bmatrix}.
\end{align*}
\]

![Fig. 1. The jumping modes \( \delta(t) \).](image1)

![Fig. 2. The system state \( x_1(t) \).](image2)
bounded for the induced the presented unbiased filter guarantees a prescribed mate state can track the real state smoothly. Meanwhile, the unknown inputs to the output error with attenuation \( \gamma \) as follows:

\[
A_f = \begin{bmatrix} 0.9499 & -1.8997 \\ -0.9500 & 2.9001 \end{bmatrix}, \quad B_f = \begin{bmatrix} -0.5015 \\ 0.4995 \end{bmatrix},
\]

\[
C_f = [0, 0.3], \quad D_f = -1.0,
\]

\[
A_g = \begin{bmatrix} 4.1001 & 1.8999 \\ -2.1998 & 2.1998 \end{bmatrix}, \quad B_g = \begin{bmatrix} -1.0008 \\ 1.9984 \end{bmatrix},
\]

\[
C_g = [0.4, -0.1], \quad D_g = -2.0.
\]

Assume the mixed time-delays are \( d = 1.0, \ h = 0.2 \), the initial conditions are \( x_1(0) = x_1(0) = 1.0, \ x_2(0) = x_2(0) = 0.8, \delta_0 = 1 \) and the unknown inputs are selected as:

\[
w(t) = \begin{cases} 
0.6e^{-0.2t\sin(100t)} & t \geq 0 \\
0 & t < 0.
\end{cases}
\]

The jumping mode, the system states (real states and estimate states), and system output signal are shown in Figs. 1–4.

From the above Figs. 2–4, we can see that the estimate state can track the real state smoothly. Meanwhile, the presented unbiased filter guarantees a prescribed bounded for the induced \( H_\infty \) norm of the operator from the unknown inputs to the output error with attenuation \( \gamma = 0.9 \).

**Example 2.** In order to illustrate the effectiveness of the developed techniques in this paper, we proposed the unstable MJSs with parameters described as \( A_1 = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}, A_2 = \begin{bmatrix} 4 & 2 \\ -2 & 2 \end{bmatrix} \), the modes, transition rate matrix and other matrices parameters are defined similarly as **Example 1**.

Similar to **Example 1**, we can get the resulting unbiased \( H_\infty \) filter gain with a disturbance attenuation value of \( \gamma = 0.9 \) as follows:

\[
A_f = \begin{bmatrix} 0.9499 & -1.8997 \\ -0.9500 & 2.9001 \end{bmatrix}, \quad B_f = \begin{bmatrix} -0.5015 \\ 0.4995 \end{bmatrix},
\]

\[
C_f = [0, 0.3], \quad D_f = -1.0,
\]

\[
A_g = \begin{bmatrix} 4.1001 & 1.8999 \\ -2.1998 & 2.1998 \end{bmatrix}, \quad B_g = \begin{bmatrix} -1.0008 \\ 1.9984 \end{bmatrix},
\]

\[
C_g = [0.4, -0.1], \quad D_g = -2.0.
\]

**Remark 9.** To illustrate the efficiency of the proposed methods, we present a two-mode distributed time-delayed MJSs. By means of LMIs techniques, He and Liu [18] studied the unbiased \( H_\infty \) filter for a class of linear MJSs with neutral time-delays using the differential operator \( D(e(t)) \). But the main results in this paper deals with the unbiased estimation problem for a class of more complex MJSs with distributed delays, including constant time-delay, differential time-delay and integral time-delay. Moreover, without using the differential operator \( D(e(t)) \), the relevant results are more feasible and achievable. From the simulation results, we see that the estimate states can track the real states smoothly, which proves the modified method given in this paper is quite effective.

**Remark 10.** The applications of time-delays are comprehensive in industrial control processes, for example, chemical engineering process, pneumatic systems with long transmission lines, neural network, inferred grinding model, etc. It should be observed out that the contributions of this paper are mainly theoretical aspects. As one of the important problems in control theory, state estimation and filtering problems are widely investigated in theoretical and practical aspects. In this paper, we succeeded in designing the unbiased \( H_\infty \) filter for a class of MJSs with distributed time delays. In our future work about this problem, we will study the relevant problems considering the differential operator \( D(e(t)) \).

**5. Conclusions**

In the paper, we have studied the unbiased \( H_\infty \) filter design problems of distributed time-delayed MJSs. It ensures the asymptotically stability for the overall dynamic error system and a prescribed bound on the \( H_\infty \) gain from the unknown disturbances to the estimation error. By using the appropriate Lyapunov–Krasovskii functional methods and applying matrix transformation and variable substitution, sufficient conditions that the solution of unbiased \( H_\infty \) filter is existed are presented and proved. The main results are presented in the form of LMIs. The performance of the
filter design to two numerical results is evaluated by the simulation examples.

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