Random Matrix Theory at Nonzero μ and T

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We review applications of random matrix theory to QCD at nonzero temperature and chemical potential. The chiral phase transition of QCD and QCD-like theories is discussed in terms of eigenvalues of the Dirac operator. We show that for QCD at $\mu \neq 0$, which has a sign problem, the discontinuity in the chiral condensate is due to an alternative to the Banks-Casher relation. The severity of the sign problem is analyzed in the microscopic domain of QCD.

§1. Introduction

Starting from its introduction in nuclear physics by Wigner,[1)] random matrix theories have been applied to a wide range of problems ranging from the physics of proteins[2)] to quantum gravity (see[3),4)] for a historical review). Three reasons for the ubiquity of random matrix theory come to mind. First, eigenvalues of large random matrices have universal properties determined by symmetries. Second, random matrices are models for disorder present in many physical systems. Third, random matrix theories have a topological expansion which is important for applications to quantum field theory. One of the attractive features of random matrix theory is that analytical information can be obtained for complex systems which otherwise only can be studied experimentally or numerically.

In this review we discuss applications of random matrix theory to QCD at nonzero temperature and chemical potential. Since the order parameter for the chiral phase transition[5),6)] and the deconfining phase transition[7),8)] are determined by the infrared behavior of the eigenvalues of the Dirac operator, these eigenvalues are essential for the phase transitions in QCD. Remarkably, the distribution of the smallest Dirac eigenvalues is given by universal functions[9)–13)] that depend only on one or two parameters, the chiral condensate and the pion decay constant. This offers an alternative way to measure these constants on the lattice.[14)–22)]

§2. Random Matrix Theory in QCD

Chiral Random Matrix Theory (chRMT) is a theory with the global symmetries of QCD, but matrix elements of the Dirac operator replaced by random numbers[9),10)]

$$D = \begin{pmatrix} m & iW \\ iW^\dagger & m \end{pmatrix}, \quad P(W) \sim e^{-N\mathrm{Tr}W^\dagger W}. \tag{2·1}$$

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This random matrix model has the global symmetries and topological properties of QCD. It is confining in the sense that only color singlets have a nonzero expectation value. It is now well understood that fluctuations of low-lying eigenvalues of the Dirac operator are described by chRMT (see[23)–28)] for lectures and reviews). Philosphically, this is important because of the realization that chaotic motion dominates the dynamics of quarks at low energy. Practically, this is important because we can use powerful random matrix techniques to calculate physical observables.

The condition for the applicability of chRMT is that the Compton wavelength of Goldstone bosons associated with the mass scale z of these eigenvalues is much larger than the size of the box. With the squared mass of the associated Goldstone boson given by $2z\Sigma/F_\pi^2$, this condition reads[29)]

$$\frac{2z\Sigma}{F_\pi^2} \ll \frac{1}{\sqrt{V}} \ll \Lambda^2. \tag{2·2}$$

The second condition is necessary to factorize the partition function into a contribution from the lightest degrees of freedom and all heavier degrees of freedom. These two conditions determine the microscopic domain of QCD. We stress that z is a scale in the Dirac spectrum so that, for sufficiently large volumes, we always have eigenvalues in the domain (2·2) where eigenvalues fluctuate according to chRMT. This can be shown rigorously from the following two observations.[30),31)] First, the infrared Dirac spectrum follows from a (partially quenched) chiral Lagrangian determined by chiral symmetry, and the inequality (2·2) is the condition for factorization of the partition function into a factor containing the constant modes and another factor containing the nonzero momentum modes. Second, the factor with the constant modes is equal to the large N limit of chiral random matrix theory.

In[32),33)] the condition (2·2) was imposed on the quark masses and was the bases for a systematic expansion of the chiral Lagrangian known as the ϵ expansion.

One feature that underlies universal properties of eigenvalues is that they behave as repulsive confined charges. This follows from the joint probability distribution $\sim \prod_k \lambda_k \prod_{k<l}(\lambda_k^2 - \lambda_l^2)^2 \exp(-N\sum_k \lambda_k^2)$. It can be shown that eigenvalues correlations at the micrsocopic scale are universal.[34)] The reason is spontaneous symmetry breaking and a mass gap so that they can be described in terms of a chiral Lagrangian.

2.1. *Chiral Random Matrix Theory at $\mu \neq 0$ and $T \neq 0$*

A nonzero temperature does not change the fluctuating behavior of the Dirac eigenvalues provided that chiral symmetry remains broken. However, a transition to a different universality class takes place at the critical temperature. A random matrix model that reproduces this universal behavior of QCD is obtained by replacing the off-diagonal elements in (2·1) by[35)]

$$iW \to iW + t, \qquad iW^\dagger \to iW^\dagger - t \qquad \text{with} \quad t = \mathrm{diag}(-\pi T, \pi T). \tag{2·3}$$

This model has been studied elaborately in the literature (see e.g.[35)–40)]).

A nonzero chemical potential can be introduced analogously to the quark mass. The requirement is that the small μ behaviour of the QCD partition function should

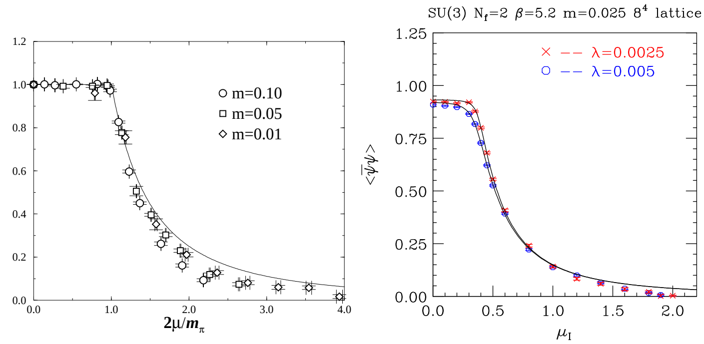

Fig. 1. Lattice results for $N_c = 2$ (taken from[55)]) and phase quenched QCD with $N_c = 3$ (taken from[56)])

be reproduced by the random matrix partition function. This achieved by modifying (2·1) by[41)]

$$iW \to iW + \mu, \qquad iW^\dagger \to iW^\dagger + \mu, \tag{2·4}$$

resulting in a nonhermitean Dirac operator with eigenvalues scattered in the complex plane. The prescription (2·4) is not unique. A random matrix model that has had a strong impact on recent developments is defined by[42)]

$$iW \to iW + \mu H, \qquad iW^\dagger \to iW^\dagger + \mu H \qquad \text{with} \quad H^\dagger = H, \tag{2·5}$$

where H is drawn from a Gaussian ensemble of random matrices. This model is in the same universality class as (2·4) but is technically simpler since it can be worked out by means of the complex orthogonal polynomial method.[42)–46)]

There are other types of random matrix models that have been applied to QCD. For example models with random gauge fields such as the Eguchi-Kawai model[47)] or its 2-dimensional version.[48)] QCD in 1 dimension[49),50)] is a random matrix model as well, with universally fluctuating Dirac eigenvalues. Also models with random Wilson loops[51),52)] have attracted significant interest.

§3. Phases of QCD and RMT

QCD-like theories with charged Goldstone bosons have a critical chemical potential equal to $m_\pi/2$. The phase transition to the Bose condensed phase can therefore be described completely in terms of a chiral Lagragian. At the mean field level,[53)] the kinetic terms of this chiral Lagrangian do not contribute, so that these results can also be obtained from chiral random matrix theory. Indeed, the static part of the chiral Lagrangian[53),54)]

$$\mathcal{L} = \frac{1}{4}F_\pi^2\mu^2 \mathrm{Tr}[U,B][U^\dagger,B] - \frac{1}{2}\Sigma \mathrm{Tr}(MU + MU^\dagger). \tag{3·1}$$

can also be obtained from the large N limit of the models (2·4) or (2·5).

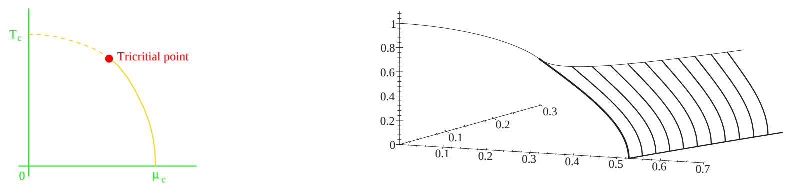

Fig. 2. QCD phase diagram in the $\mu T m$-space (taken from[58])

In Fig. 1 we display lattice results for QCD with $N_c = 2$[55] and phase quenched QCD.[56] They show an impressive agreement with the results from (3·1) given by the solid curves in both figures.

3.1. *Schematic RMT Phase Diagram*

The phase transition in QCD with $N_c = 3$ at $\mu_c = m_N/3$ cannot be analyzed by means of chiral Lagrangians. Because of the sign problem lattice studies are not possible either. In such situation there is long tradition to analyze the same problem in a much simpler theory in the hope of obtaining at least a qualitative understanding of the problem. For example, one dimensional QCD,[49),50] or more recently, super Yang-Mills theory and AdS-CFT duality,[57] been explored as toy models for QCD. We will use random matrix theory at $T \neq 0$ and $\mu \neq 0$, introduced in (2·3) and (2·4) to obtain a qualitive understanding of the QCD phase diagram. Lattice QCD simulations show that the chiral phase transition at $\mu = 0$ is of second order or a steep cross-over. At $T = 0$ we expect a first order phase transition at $\mu_c = m_N/3$. It is natural that the first order line ends in a critical end point or joins the second order critical line at the tricritical point (see Fig. 3.1, left). This is indeed what is observed in random matrix theory[58),59] (see Fig. 3.1, right). A similar phase diagram has also been obtained from the NJL model.[60)–62]

Another scenario that was discovered in RMT is the splitting of the first order line into two at nonzero isospin chemical potential.[63] This behavior was also found in a NJL model[64),65] but might not be stable against flavor mixing interactions.[66]

§4. Dirac Spectrum in Theories Without a Sign Problem

Since the spectrum of the Dirac operator determines the chiral condensate, phase transitions in QCD can be understood in terms of its spectral flow. In this section we discuss theories with a positive fermion determinant such as QCD with two colors and phase quenched QCD, where a probabilistic interpretation of the eigenvalue density is possible. The relation between chiral symmetry breaking and Dirac spectra is much more complicated when the fermion determinant is complex and its discussion will be postponed to the next section.

The spectrum of an anti-Hermitean Dirac operator is purely imaginary with an eigenvalue density that is proportional to the volume. If chiral symmetry is broken spontaneously, the chiral condensate becomes discontinuous across the imaginary axis in the thermodynamic limit. Chiral symmetry is restored if such discontinuity

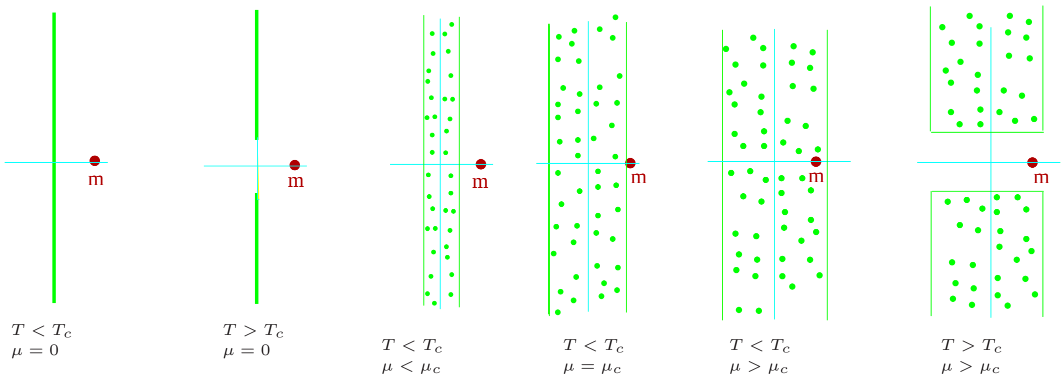

Fig. 3. Critical behavior of the Dirac spectrum. $\mu_c = m_\pi/2$ for $T = 0$ and increases with T.

is absent for example by the formation of a gap in the Dirac spectrum, see eg.[71] .

For $\mu \neq 0$, the Dirac spectrum broadens into a strip of width $4\mu^2 F_\pi^2/\Sigma$.[49),67)] The chemical potential becomes critical when the quark mass hits the edge of this strip. At this point the chiral condensate starts rotating into a pion condensate. Chiral symmetry restoration takes place when a gap forms at zero. A schematic picture of the critical behavior of Dirac eigenvalues is shown in Fig. 3 and the spectral flow of the Dirac eigenvalues with respect to increasing μ and T is summarized in Fig. 4. One conclusion from this behavior is that $T_c(\mu)$ is a concave function of μ, and that $\mu_c(T)$ is a convex function of T. The spectral flow discussed in this section is supported by lattice simulations at $T \neq 0$ and $\mu \neq 0$ (See Fig. 5)

4.1. *Dirac spectrum in the μ-plane*

We could equally well have diagonalized the Dirac operator in a representation where $\mu\gamma_0$ is proportional to the identity,

$$\det(D + m + \mu\gamma_0) = \det(\gamma_0(D + m) + \mu). \qquad (4\cdot1)$$

These eigenvalues are relevant to the baryon number density. A gap in the spectrum develops at $m \neq 0$ (see Fig. 6), and the chemical potential becomes critical, $\mu = m_\pi/2$ when it hits the inner edge of the domain of eigenvalues.

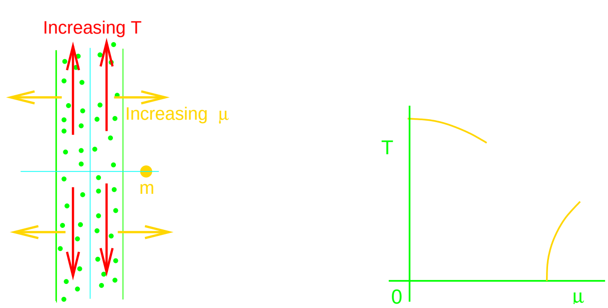

Fig. 4. Spectral flow of the Dirac spectrum (left) and phase diagram (right) with respect to μ and T in phase quenched QCD and QCD with two colors.

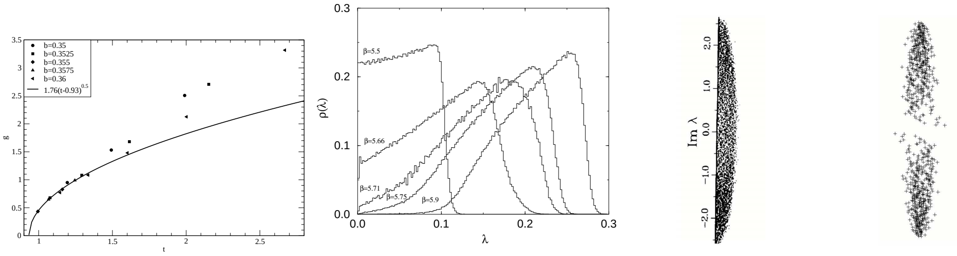

Fig. 5. Temperature and chemical potential dependence of Dirac eigenvalues. From left to right taken from.[70),72)–74)]

4.2. *Quenched Lattice QCD Dirac Spectra at* $\mu \neq 0$

Small Dirac eigenvalues at $\mu \neq 0$ have been computed in quenched QCD. The analytical formulas for the average density of the small Dirac eigenvalues are available.[68),69)] They were first derived[68)] by exploiting the Toda lattice hierarchy in the flavor index. Comparisons of random matrix predictions[68)] for the radial spectral density and lattice QCD results[75),76)] are shown in the left panel of Fig. 7. In other cases, such as the overlap Dirac operator[77)] and QCD with $N_c = 2$,[78)] a similar degree of agreement was found. Both the spectral density and two-point correlations can be derived from the Lagrangian (3·1), i.e. they are determined by two parameters, F_π and Σ. This can be exploited to extract these low-energy constants. For example, F_π and Σ were determined[19),21)] (see also[20)]) from the correlators shown in the two right panels of Fig. 7.

§5. Chiral Symmetry Breaking at $\mu \neq 0$

The full QCD partition function at $\mu \neq 0$ which is the average of

$$\det(D + m + \mu\gamma_0) = |\det(D + m + \mu\gamma_0)|e^{i\theta}, \qquad \theta \neq 0, \tag{5·1}$$

has properties which are drastically different from the phase quenched partition function where the phase factor is absent. In particular, $\mu_c = m_N/3$ instead of $m_\pi/2$, so that the free energy remains μ-independent until $\mu = m_N/3$. For $\mu < m_N/3$ the

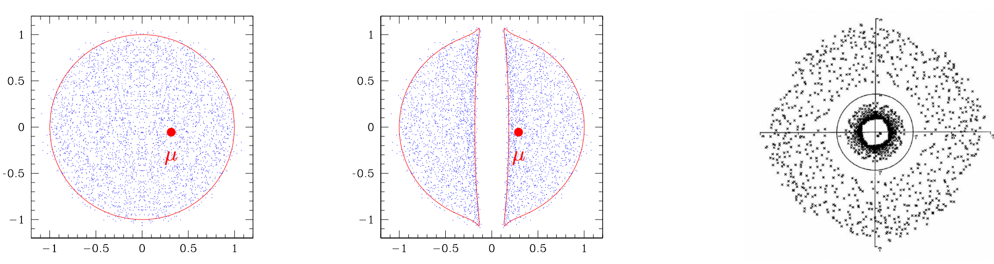

Fig. 6. Eigenvalues of $\gamma_0(D + m)$ for a random matrix Dirac operator at $m = 0$ (left), $m \neq 0$ (middle) (both taken from[79)]), and lattice QCD at $m \neq 0$ (right, taken from[49)]).

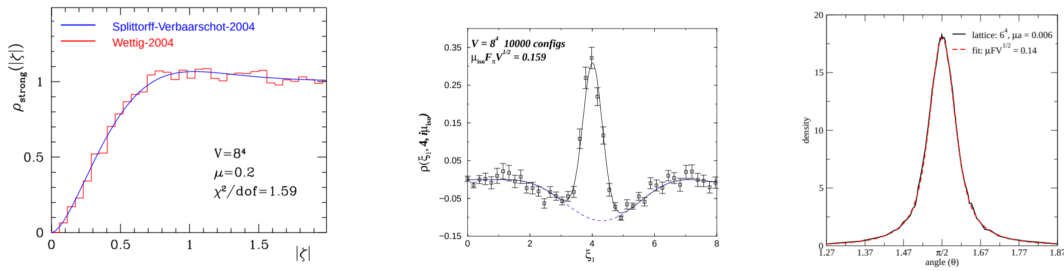

Fig. 7. The radial spectral density for (left, taken from[75),76)]) and two-point correlations (middle taken from[19)] and right taken from[21)]).

chiral condensate remains discontinuous at $m = 0$, whereas the chiral condensate of the phase quenched theory approaches zero for $m \to 0$ (see Fig. 5). The only difference between the phase quenched partition function and the full QCD partition function is the phase of the fermion determinant. We conclude that the phase factor is responsible for the discontinuity of the chiral condensate. How can this happen if for each configuration the support of the spectrum is approximately the same? This problem known as the "Silver Blaze Problem"[80)] was solved in.[6)]

5.1. *Unquenched Spectral Density*

The spectral density for QCD with dynamical fermions is given by

$$\rho_{N_f}(\lambda) = \langle \sum_k \delta^2(\lambda - \lambda_k) \det{}^{N_f}(D + m + \mu\gamma_0) \rangle. \tag{5·2}$$

Because of the phase of the fermion determinant, this density is in general complex and can be decomposed as $\rho_{N_f}(\lambda) = \rho_{N_f=0}(\lambda) + \rho_U(\lambda)$. The chiral condensate can then be decomposed as $\Sigma_{N_f}(m) = \Sigma_{N_f=0}(m) + \Sigma_U(m)$, so that the discontinuity in $\Sigma(m)$ is due to ρ_U. Asymptotically it behaves as[81)]

$$\rho_U \sim e^{\frac{2}{3}\mu^2 F^2 V} e^{\frac{2}{3} i \mathrm{Im}(\lambda) \Sigma V}$$

and vanishes outside an ellips starting at $\mathrm{Re}(\lambda) = m$ (see Fig. 9).[6)] In the right part of this figure we show the real part of the spectral density for QCD with one flavor at nonzero chemical potential.

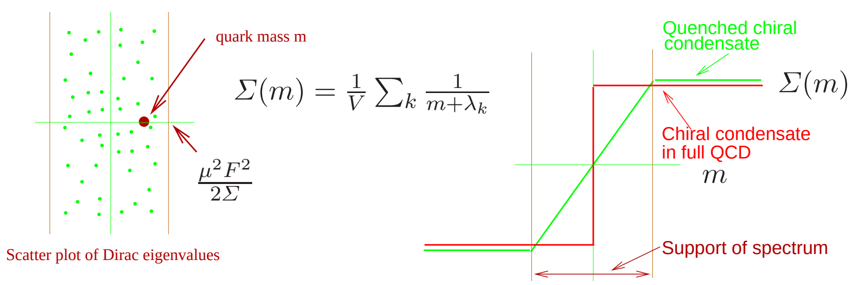

Fig. 8. Chiral condensate of quenched and full QCD.

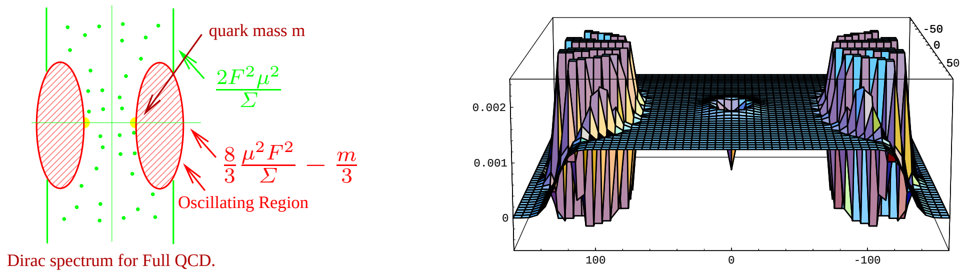

Fig. 9. Support (left) and real part (right, taken from[27]) of Dirac spectral density for QCD with $N_f = 1$ and $\mu \neq 0$.

This result explains the mechanism of chiral symmetry breaking at nonzero chemical potential. The phase of the fermion determinant rotates the pion condensate back into a chiral condensate, but it does so in an unexpected way.[6] The same mechanism is at play for 1d QCD at $\mu \neq 0$.[82]

§6. Phase of the Fermion Determinant

The magnitude of the sign problem can be measured by means of the expectation value of the phase factor of the fermion determiant which can be defined in two ways

$$\langle e^{2i\theta}\rangle_{N_f} = \frac{1}{Z_{N_f}}\left\langle \frac{\det(D+\mu\gamma_0+m)}{\det^*(D+\mu\gamma_0+m)}\det{}^{N_f}(D+\mu\gamma_0+m)\right\rangle, \quad \langle e^{2i\theta}\rangle_{1+1^*} = \frac{Z_{N_f=2}}{Z_{1+1^*}}.$$

The average $\langle\cdots\rangle$ is with respect to the Yang-Mills action. The sign problem is managable when the average phase factor remains finite in the thermodynamic limit. In the microscopic domain it is possible to obtain exact analytical expressions for the average phase factor by exploiting the equivalence between QCD and RMT in this domain. For $\mu < m_\pi/2$ the free energy of both QCD and phase quenched QCD are independent of μ. This does not imply that the average phase factor is μ-independent. The μ-dependence originates from the charged Goldstone bosons with mass $m_\pi \pm 2\mu$, and for N_f flavors the mean field result[83),84] for $\langle\exp(2i\theta)\rangle$ reads $(1-4\mu^2/m_\pi^2)^{N_f+1}$. The exact result for the average phase factor for $N_f = 2$ is shown in Fig. 10 (right), where lattice results[85] are also shown (left). The exact result has an essential singularity at $\mu = 0$, but its thermodyanmic limit agrees with the mean result.

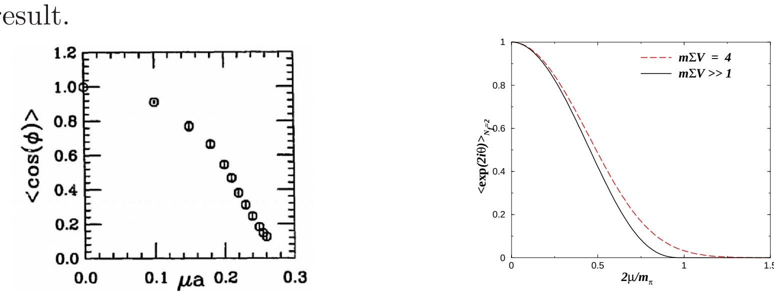

Fig. 10. Average phase factor. Lattice QCD results are shown left (taken from[85]) and the exact microscopic result[83] is shown right.

§7. Conclusions

The equivalence of chiral random matrix theory and QCD has been exploited succesfully to derive a host of analytical results. Among others, eigenvalue fluctuations predicted by chRMT have been observed in lattice simulations, the phases of QCD can be understood in terms of spectral flow, observables can be extracted from the fluctuations of the smallest eigenvalues, the sign problem is not serious when the quark mass is outside the domain of the eigenvalues, and mean field results can be obtained from random matrix theory. Summarizing, chiral random matrix theory is a powerful tool for analyzing the infrared domain of QCD.

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