Using the prover ANDP to simplify orthogonality

Dafa Li

Department of Mathematics, Tsinghua University, Beijing 100084, China

Received 6 July 2000; received in revised form 15 August 2002; accepted 10 June 2003

Communicated by S.N. Artemov

Abstract

In the 1920s, Heyting attempted at axiomatizing constructive geometry. Recently, von Plato used different concepts to axiomatize the geometry: he used 14 axioms to describe the axiomatization for apartness geometry. Then he added axioms A1 and A2 to his apartness geometry to get his affine geometry, then he added axioms O1, O2, O3 and O4 to the affine geometry to get orthogonality. In total, this gives 22 axioms. von Plato used four relations to describe the concept of orthogonality in O1, O2 and O4. That is, all the three relations of two lines, which are convergence, unorthogonality and difference, and the relation of a point and a line. ANDP is an automated natural deduction prover developed over the years at our institute. After doing a lot of experiments using ANDP, much shorter and more intuitive axioms were found for axioms O1, O2 and O4, respectively. For example, O2 can be replaced by one of its four conjuncts. This paper shows that it is enough to use two relations on lines, which are convergence and unorthogonality, to describe the concept of orthogonality.

Keywords: Automated theorem proving; Constructive geometry; Intuitionistic logic

1. Introduction

The problem of finding alternate axiomatizations of the same theories has received quite some attraction in automated theorem proving (ATP). For instance, Robbins algebra can be axiomatized by three axioms, and the longstanding open problem of whether or not every Robbins algebra is a Boolean algebra has been solved by Bill McCune’s
Certainly, this result will have an impact on the area of ATP. It is interesting to replace more axioms by less axioms, specially by shorter axioms [4–10,13], and doing so can be a fruitful application area for ATP. For example, McCune obtained single axioms for left group and right group calculi in [5], and Peterson got the shortest single axioms for the equivalential calculus in [10]. The present paper shows how von Plato’s orthogonality axioms can be simplified.

In the 1920s, Heyting attempted at axiomatizing constructive geometry [1]. Recently, von Plato used different concepts to axiomatize the geometry [1,12]. He used point apartness instead of point equality, and line apartness instead of line equality, convergence of two lines instead of parallelism, apartness of a point from a line instead of incidence of a point with a line. von Plato used 14 axioms to describe the axiomatization for apartness geometry. He added axioms: A1 (two axioms) and A2 to his apartness geometry to get his affine geometry, then he added the axioms O1, O2, O3 (two axioms) and O4 to the affine geometry to get orthogonality (see Section 2.1 for these axioms). Thus, there are 22 axioms for orthogonality. In [12], von Plato states that “The rules of logical inference are basically those of constructive logic in a natural deduction formulation.” In [11], van Dalen states that “the system of natural deduction is almost right” to get intuitionistic proofs. Our prover ANDP is an automated natural deduction prover.

We tried to simplify each one of the 22 axioms (some of the axioms for apartness geometry were simplified in [4]). The axioms O1, O2, O3 and O4 are carefully chosen by von Plato. In fact, O2 can be constructively rewritten as a conjunction of the form S1 & S2 & S3 & S4. We tried a lot of candidates for O1, O2, O3 and O4, including O2’s four conjuncts S1, S2, S3, and S4 and 33 theorems, lemmas and corollaries in [12], several of which use the relation URT. Beyond those candidates just mentioned, we made by hand many combinations of the five predicates DILN, DIPT, APT, CON and URT (cf. Section 4), and we chose some of the combinations, which are apparently valid, as candidates for O1, O2, O3 and O4. Some of them are listed in this paper (cf. Section 4). Since we did not know which of 22 axioms for orthogonality were useful to derive a candidate, ANDP was at first fed with all of 22 axioms for each candidate. If it failed for a candidate, then we removed some of axioms and tried again. To minimize premises we also removed some of 22 axioms. Because we did not know which of the 22 premises were redundant, we did many experiments for each candidate. In sum, for the orthogonality, the hard point is how to find new axioms. Clearly, theorem proving should include the two directions: (1) derive a conjecture from known premises, (2) find new axioms.

After many computer experiments by ANDP we obtained the following results (cf. Section 4).

**Result 1.** O4 can be replaced by NO4.

**Result 2.** O2 can be replaced by NO2* which is one of its four conjuncts, or NO2 or NO2**.

**Result 3.** O1 and O2 can be replaced by NO1 and NO2 simultaneously.
The new axioms are much simpler and shorter. From these results, it is not hard to see that it is enough to use only two relations of lines: convergence and unorthogonality, to describe the concept of orthogonality.

While von Plato used four relations to describe the concept of orthogonality in O1, O2 and O4. That is, all the three relations of two lines, which are convergence, unorthogonality and difference, and the relation of a point and a line. Thus, we got nine sets of axioms which are equivalent to von Plato’s orthogonality. Using the prover ANDP we obtained mechanical proofs in a natural deduction style of all theorems in this paper, and some of them are in the appendices below.

The rest of this paper is structured as follows. Section 2 is an introduction to von Plato’s orthogonality. Section 3 is an introduction to the prover ANDP. Section 4.1 contains the proofs of replacing O4 with the new axiom NO4. Section 4.2 contains the proofs of replacing O2 with its conjunct NO2*. Section 4.3 contains the proofs of replacing O2 with the new axiom NO2. Section 4.4 contains the relation among NO2, NO2* and NO2** and the proofs of replacing O2 with the new axiom NO2**. Section 4.5 contains the proofs of replacing both O1 and O2 with NO1 and NO2.

2. The axiomatization of von Plato’s orthogonality

von Plato used the DIPT, DILN, APT, CON and URT predicates for orthogonality [12], where

- \( DIPT \ x \ y \) means that \( x \) and \( y \) are distinct points,
- \( DILN \ x \ y \) means that \( x \) and \( y \) are distinct lines,
- \( CON \ x \ y \) means that \( x \) and \( y \) are convergent lines,
- \( APT \ x \ y \) means that point \( x \) is apart from line \( y \),
- \( URT \ x \ y \) means that lines \( x \) and \( y \) are unorthogonal.

He used the functions \( ln, pt, pa \) and \( rt \) [12], where

- \([ln x y]\) is the connecting line of points \( x \) and \( y \),
- \([pt x y]\) is the intersection point of lines \( x \) and \( y \),
- \([rt x y]\) is the orthogonal to line \( x \) through point \( y \),
- \([pa x y]\) is the parallel to line \( x \) through point \( y \).

We use \( \sim \), &, | and \( \rightarrow \) to stand for negation, conjunction, disjunction and implication, respectively. \( \forall \) denotes the universal quantifier. Bracketing [.] is used for syntactic disambiguation.
2.1. Twenty-two axioms of the orthogonality

To make this paper self-contained, we recall Plato’s axioms here. They can be organized as follows.

**Axiom group 1.** Apartness axioms for distinct points, distinct lines, and convergence lines:

1. (∀x) ¬DIPT x x,
2. (∀x) ¬DILN x x,
3. (∀x) ¬CON x x,
4. (∀x) (∀y) (∀z) [DIPT x y → [DIPT x z ∣ DIPT y z]],
5. (∀x) (∀y) (∀z) [DILN x y → [DILN x z ∣ DILN y z]],
6. (∀x) (∀y) (∀z) [CON x y → [CON x z ∣ CON y z]].

**Axiom group 2.** Axioms for connecting lines and intersection points:

1. (∀x) (∀y) [DIPT x y → ¬APT x [ln x y]],
2. (∀x) (∀y) [DIPT x y → ¬APT y [ln x y]],
3. (∀x) (∀y) [CON x y → ¬APT pt x y x],
4. (∀x) (∀y) [CON x y → ¬APT pt x y y].

**Axiom group 3.** Constructive uniqueness axiom for lines and points:

Axiom 3 (∀x) (∀y) (∀u) (∀v) [DIPT x y & DILN u v → [[APT x u ∣ APT x v] ∣ [APT y u ∣ APT y v]].

**Axiom group 4.** Compatibility of equality with apartness and convergence:

1. (∀x) (∀y) (∀z) [APT x y → [DIPT x z ∣ APT z y]],
2. (∀x) (∀y) (∀z) [APT x y → [DILN y z ∣ APT x z]],
3. (∀x) (∀y) (∀z) [CON x y → [DILN y z ∣ CON x z]].

**Affine geometry.** He added the following axioms A1 and A2 to his apartness geometry to get his affine geometry.

A1. Axioms for constructed parallels:

A1.1. (∀x) (∀y) ¬CON [pa x y] x,
A1.2. (∀x) (∀y) ¬APT x [pa y x].

A2. Constructive uniqueness axiom for parallels:

(∀x) (∀y) (∀z) [DILN x y → [[APT z x ∣ APT z y] ∣ CON x y]].

**Orthogonality.** Then, he added the following axioms O1, O2, O3 and O4 to the affine geometry to get orthogonality. Thus there are 22 axioms in total now.

O1. Compatibility of convergence and unorthogonality:

(∀x) (∀y) [CON x y ∣ URT x y].
O2. Apartness axiom for the conjunction of convergence and unorthogonality:

\[(\forall x)(\forall y)(\forall z)[\text{CON }x\text{ }y&\text{URT }x\text{ }y \rightarrow \text{CON }x\text{ }z\text{ }&\text{URT }x\text{ }z|\text{CON }y\text{ }z&\text{URT }y\text{ }z].\]

O3. Two axioms for the orthogonal construction:

O3.1 \((\forall x)(\forall y)\sim\text{URT}[\text{rt }x\text{ }y]x,\)

O3.2 \((\forall x)(\forall y)\sim\text{APT }x[\text{rt }y\text{ }x].\)

O4. Constructive uniqueness axiom for orthogonals:

\[(\forall x)(\forall y)(\forall z)(\forall u)[\text{DILN }x\text{ }y \rightarrow [\text{APT }u\text{ }x|\text{APT }u\text{ }y][\text{URT }x\text{ }z|\text{URT }y\text{ }z]].\]

2.2. Two rules of construction of von Plato’s apartness geometry

1. A rule for constructing a line from two distinct points

\[
\frac{a:\text{point} \quad b:\text{point} \quad \text{DIPT }a\text{ }b}{[ln\text{ }a\text{ }b]:\text{line}},
\]

[ln a b] means the connecting line of points a and b. So [ln a a] is not admissible in a constructive proof.

2. A rule for constructing a point from two convergent lines

\[
\frac{l:\text{line} \quad m:\text{line} \quad \text{CON }l\text{ }m}{[pt\text{ }l\text{ }m]:\text{point}},
\]

[pt l m] means the intersection point of two convergent lines. So [pt l l] is not admissible in a constructive proof.

3. Description of ANDP

3.1. Rules of inference

Please see Appendix D.

3.2. Two unification algorithms for quantifiers in natural deduction system

There are many rules of inference in natural deduction systems adapted from Gentzen’s system. The rules universal generalization (UG) and existential generalization (EG) are used to introduce quantifiers. The rules universal specialization (US) and existential specialization (ES) are used to eliminate quantifiers. The natural deduction calculus with the rules UG, EG, US, ES and an appropriate set of rules for connectives is complete.

We have two unification algorithms to handle quantifiers, one is for introducing quantifiers, the other is for eliminating quantifiers [2]. If two given formulas can become
equal by applying the rules of introducing quantifiers, then our algorithm can find the series of operations UG and EG and decide which occurrences of terms should be universally or existentially generalized upon [2]. Similarly, if two given formulas can become equal by applying the rules of eliminating quantifiers, then our algorithm can find the series of operations US and ES and decide what terms are used to replace the bound variables without using Skolem functions [2].

3.3. Search strategies

(1) First deduce theorems from bottom to top, then from top to bottom. This means that, if a present goal is of the form \(A \rightarrow B, A \& B\) or \(A \mid B\), then first apply the rules CP, CONJUNction or IMPlication to the goal, respectively (cf. Appendix D). Otherwise, it reasons from top to bottom.

(2) Prefer to apply the rules for propositional logic to ones for quantifiers.

3.4. Heuristics

1. Subsumption in natural deduction: If a formula can be derived from a formula in the preceding line by renaming the bound variables, then it will not be produced.

2. Tautology deletion: A wff is called tautologous if it can be obtained from a propositional tautology by substituting formulas for the propositional variables in the tautology and by renaming bound variables. For example, \(\exists x P x \rightarrow \exists y P y\) is tautologous. If a formula is tautologous, then the formula will not be produced. The algorithm for tautology checking is hard. To save time, only implications and disjunctions are checked for being tautologous, and only atoms with quantifiers are checked for subsumption.

3. Minimal scopes of quantifiers: Push quantifiers inside as much as possible to minimize scopes of quantifiers. Many equivalences for quantifiers are used for this purpose. For example, \((\forall x) [A(x) \& B(x)] = (\forall x) A(x) \& (\forall x) B(x)\) and so on. However, this technique is not needed for the proofs in the present paper.

3.5. The rule CASES

If the rule CASES is applied to a disjunction, then two disjuncts come up as new hypotheses. This possibly leads to new constants from the new hypotheses, and furthermore it may lead to many new Herbrand terms and irrelevant and redundant formulas. We use the following strategies to control the CASES inference rule:

1. The rule CASES is first applied to premises.

2. The rule CASES is then applied to the disjunctions from which it will not produce new constants.

3. The rule CASES is then applied to other formulas, thereby preferring short formulas.
3.6. How to use ANDP

3.6.1. Input to ANDP

Inputs accepted by the prover ANDP are well-formed formulas with quantifiers. There are five connectives and two quantifiers. The connectives are negation, conjunction, disjunction, implication and equivalence. The quantifiers are universal and existential quantifiers. We use $\sim, \&$, or, $\rightarrow$ and $\text{iff}$ for negation, conjunction, disjunction, implication and equivalence, and $(\forall x)$ and $(\exists x)$ for $\forall x$ and $\exists x$, respectively.

ANDP checks the input formulas for whether they are well formed, and it informs the user about possible errors in the input formulas.

3.6.2. Readable proof steps

Proof steps in natural deduction style are well readable. For example, in order to derive $Q$ from premises $P \rightarrow Q$ and $P$, we have the following proof steps:

1. $P$  Premise
2. $P \rightarrow Q$  Premise
3. $Q$  MP 1, 2

“Premise” justifies that the formula in the line is given as a premise. “MP 1, 2” justifies that the formula $Q$ is derived from lines 1 and 2 using the rule MP.

3.6.3. Automatically proving theorems

ANDP automatically applies the mentioned rules of inference. Users cannot interfere with ANDP once a proof attempt is started. If ANDP successfully finds a proof of a theorem, then it prints “I finished proving the theorem”. If it fails to apply any rule of inference to any formula, that is, it cannot derive any new result, then it prints “Please give me helps”.

For some formulas ANDP does not terminate.

4. Simplifying orthogonality

As said before, von Plato added the axioms O1, O2, O3 and O4 to the affine geometry to get orthogonality. In this section, we will report the following results on simplifying these axioms. For the sake of contrasting the old and the new versions of the axioms, the old versions are listed here as well.

Result 1. O4 can be replaced by NO4.

O4. Constructive uniqueness axiom for orthogonals (the relation among a point and three lines)

\[(\forall x) (\forall y) (\forall z) (\forall u) [\text{DILN} x y \rightarrow [\text{AP} u x | \text{AP} u y] [\text{URT} x z | \text{URT} y z]].\]

The axiom O4 means that if $x$ and $y$ are any different lines, then for any line $z$ and any point $u$, $u$ is apart from $x$ or $y$, or $x$ and $z$ or $y$ and $z$ are unorthogonal.
NO4. Convergence implying unorthogonality

$$(\forall x)(\forall y)(\forall z) \left[ \text{CON } x \ y \rightarrow \text{URT } x \ z | \text{URT } y \ z \right].$$

NO4 means that if $x$ and $y$ are any convergent lines, then for any line $z$, $x$ and $z$ or $y$ and $z$ are unorthogonal.

**Result 2.** O2 can be replaced by its conjunct NO2*, or by NO2, or by NO2**.

O2. Apartness axiom for the conjunction of convergence and unorthogonality

$$(\forall x)(\forall y)(\forall z) \left[ \text{CON } x \ y \land \text{URT } x \ y \rightarrow \text{CON } x \ z \land \text{URT } x \ z \right].$$

O2 means that if lines $x$ and $y$ are convergent and unorthogonal, then for any line $z$, $x$ and $z$ or $y$ and $z$ are convergent and unorthogonal.

O2 can be constructively rewritten in a conjunctive from S1 & S2 & S3 & S4. NO2* is just one of its conjuncts.

NO2*. The convergence and unorthogonality among three lines

$$(\forall x)(\forall y)(\forall z) \left[ \text{CON } x \ y \land \text{URT } x \ y \rightarrow \text{CON } x \ z \lor \text{URT } y \ z \right].$$

NO2* means that if lines $x$ and $y$ are convergent and unorthogonal, then for any line $z$, $x$ and $z$ are convergent or $y$ and $z$ are unorthogonal.

NO2. Unorthogonality implying convergence or unorthogonality

$$(\forall x)(\forall y)(\forall z) \left[ \text{URT } x \ y \rightarrow \text{CON } x \ z \lor \text{URT } y \ z \right].$$

NO2 means that if lines $x$ and $y$ are unorthogonal, then for any line $z$, $x$ and $z$ are convergent or $y$ and $z$ are unorthogonal.

NO2**. The three relations among three lines: unorthogonality, convergence and difference

$$(\forall x)(\forall y)(\forall z) \left[ \text{DILN } x \ y \land \text{URT } x \ y \rightarrow \text{CON } x \ z \lor \text{URT } y \ z \right].$$

NO2** means that if lines $x$ and $y$ are different and unorthogonal, then for any line $z$, $x$ and $z$ are convergent or $y$ and $z$ are unorthogonal.

**Result 3.** Replacing O1 and O2 by NO1 and NO2 simultaneously.

von Plato stated that it is possible to replace O1 by NO1 and NO2 together. Clearly, it is not desired to replace a short axiom by two axioms, a longer and a shorter one. However, it is desired and valuable to replace both O1 and O2 by the two shorter axioms: NO1 and NO2.

O1. Compatibility of convergence and unorthogonality

$$(\forall x)(\forall y) \left[ \text{CON } x \ y \mid \text{URT } x \ y \right].$$

O1 means that for any lines $x$ and $y$, they are convergent or unorthogonal.
Table 1

<table>
<thead>
<tr>
<th>Title</th>
<th>Proof steps</th>
<th>Time (CPU)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theorem 1</td>
<td>29</td>
<td>0.35</td>
</tr>
<tr>
<td>Theorem 2</td>
<td>51</td>
<td>7</td>
</tr>
<tr>
<td>Lemma 2</td>
<td>30</td>
<td>2.8</td>
</tr>
<tr>
<td>Lemma 3</td>
<td>37</td>
<td>2</td>
</tr>
<tr>
<td>Lemma 5</td>
<td>51</td>
<td>27</td>
</tr>
<tr>
<td>Lemma 6</td>
<td>52</td>
<td>28</td>
</tr>
<tr>
<td>Lemma 7</td>
<td>56</td>
<td>41</td>
</tr>
</tbody>
</table>

NO1. Irreflexivity

\((\forall x) \ URT \ x \ x.\)

NO1 means that for any line \(x\), \(x\) is unorthogonal to itself.

Thus, we got the following nine sets of axioms. Adding any one of the nine sets to affine geometry, we get a constructive axiomatization for orthogonality which is equivalent to von Plato’s one:

1. O1, O2, O3, NO4;
2. O1, NO2, O3, O4;
3. O1, NO2\(^*\), O3, O4;
4. O1, NO2\(^**\), O3, O4;
5. NO1, NO2, O3, O4;
6. NO1, NO2, O3, NO4;
7. O1, NO2, O3, NO4;
8. O1, NO2\(^*\), O3, NO4;
9. O1, NO2\(^**\), O3, NO4.

We also tested a lot of other combinations using ANDP, six of which are listed as follows. ANDP failed to prove their equivalence:

1. NO1, NO2\(^*\), O3, O4;
2. NO1, NO2\(^**\), O3, O4;
3. NO1, NO2\(^*\), O3, NO4;
4. NO1, NO2\(^**\), O3, NO4;
5. NO1, O2, O3, O4;
6. NO1, O2, O3, NO4.

Table 1 lists the time and proof steps for the automated proofs of theorems and lemmas in this paper.

We did lot of experiments to find shorter and more intuitive formulas to replace axioms O1 and O2 and O4. The 33 theorems, lemmas and corollaries in [12] and O2’s four conjuncts were chosen as candidates for O2 and O4. Except those above,
many statements for geometry were also chosen as candidates for them. Some of these statements are as follows:

\begin{align*}
F1: & \quad APT a \mid APT a m \rightarrow URT l m \\
F2: & \quad URT l m \rightarrow CON l m \mid URT m n \\
F3: & \quad CON l m \& URT l m \rightarrow DILN m [rt l [pt l m]] \\
F4: & \quad CON l m \& URT l m \rightarrow CON [rt l a] [rt m b] \& URT [rt l a] [rt m b] \\
F5: & \quad DILN l m \rightarrow APT a \mid APT a m \mid URT l n \mid URT m n \\
F6: & \quad URT l m \rightarrow DILN l n \mid URT m n
\end{align*}

After our experiments, however, we think that these statements cannot be used to equivalently replace the axioms O1 or O2 or O4.

4.1. The proof of replacing axiom O4 by NO4

The axiom O4 means that if \(x\) and \(y\) are any different lines, then for any line \(z\) and any point \(u\), \(u\) is apart from \(x\) or \(y\), or \(x\) and \(z\) or \(y\) and \(z\) are unorthogonal.

NO4 means that if \(x\) and \(y\) are any convergent lines, then for any line \(z\), \(x\) and \(z\) or \(y\) and \(z\) are unorthogonal.

Note that there are five occurrences of three different predicates in O4 which are DILN, APT and URT. There are three occurrences of two different predicates in NO4 which are CON and URT (cf. Table 2).

Note that NO4 is just Theorem 8.6 [12]. Axiom O4 and Theorem 3.2 [12] were used in the proof of the Theorem 8.6. And Theorem 3.2 was derived from axioms 4.3, 1.3 and 1.6. Therefore, Theorem 8.6 is a logical consequence of the axioms 4.3, 1.3, 1.5 and O4. Conversely, one might ask if O4 is a logical consequence of Theorem 8.6 and other axioms. It is not apparent and no evidence was given in [12]. After checking by ANDP a lot of candidates, which were Theorems 8.2, 8.3, 8.6 and Corollaries 8.4 and 8.5 [12], and other geometric statements made by hand, Theorem 8.6 was finally chosen to replace O4. What we need to do is to prove the equivalence between von Plato’s orthogonality and the new orthogonality obtained from von Plato’s one by replacing O4 by NO4.

**Theorem 1.** von Plato’s axiomatization and the new axiomatization obtained by replacing O4 by NO4 are constructively equivalent.

**Proof.** \(\Rightarrow\) We only need to show that NO4 can be derived from von Plato’s system. It is just Theorem 8.6 on p. 182 of [12].
Table 3
O2 vs. NO2*

<table>
<thead>
<tr>
<th>Axioms</th>
<th>The length of formulas</th>
<th>The relations used in</th>
</tr>
</thead>
<tbody>
<tr>
<td>O2</td>
<td>6</td>
<td>URT, CON</td>
</tr>
<tr>
<td>NO2*</td>
<td>4</td>
<td>URT, CON</td>
</tr>
</tbody>
</table>

We only need to show that O4 can be derived from the new axiomatization. The mechanical proof in natural deduction style obtained using the ANDP prover is put in Appendix A. It has 29 steps. The time consumed is 0.35 s. The proof is classical. The following is a constructive proof obtained by hand.

After eliminating quantifiers, let O4 be \( DILN \ l \ m \rightarrow \left[ \left[ APT \ a \ l \right] \left[ APT \ a \ m \right] \left[ URT \ l \ n \right] \left[ URT \ m \ n \right] \right] \). Assume \( DILN \ l \ m \). From axiom A2, after eliminating quantifiers, obtain \( DILN \ l \ m \rightarrow \left[ \left[ APT \ a \ l \right] \left[ APT \ a \ m \right] \left[ CON \ l \ m \right] \right] \), and then \( \left[ APT \ a \ l \right] \left[ APT \ a \ m \right] \left[ CON \ l \ m \right] \). If \( APT \ a \ l \left[ APT \ a \ m \right), then the conclusion follows. Otherwise, assume \( CON \ l \ m \). By NO4, after eliminating quantifiers, obtain \( CON \ l \ m \rightarrow \left[ URT \ l \ n \right] \left[ URT \ m \ n \right] \), then \( \left[ URT \ l \ n \right] \left[ URT \ m \ n \right] \), and so the conclusion follows.

4.2. The proof of replacing O2 by one of its four conjuncts: NO2*

O2 can be constructively rewritten in a conjunction form S1 & S2 & S3 & S4.

S1: \((\forall x) \ (\forall y) \ (\forall z) \left[ CON \ x \ y \& URT \ x \ y \rightarrow \left[ CON \ y \ z \right] \left[ CON \ x \ z \right] \right] \)
S2: \((\forall x) \ (\forall y) \ (\forall z) \left[ CON \ x \ y \& URT \ x \ y \rightarrow \left[ URT \ x \ z \right] \left[ URT \ y \ z \right] \right] \)
S3: \((\forall x) \ (\forall y) \ (\forall z) \left[ CON \ x \ y \& URT \ x \ y \rightarrow \left[ CON \ x \ z \right] \left[ URT \ y \ z \right] \right] \)
S4: \((\forall x) \ (\forall y) \ (\forall z) \left[ CON \ x \ y \& URT \ x \ y \rightarrow \left[ CON \ y \ z \right] \left[ URT \ x \ z \right] \right] \)

NO2* just is S3.

Axiom O2 means that if lines \( x \) and \( y \) are convergent and unorthogonal, then for any line \( z \), \( x \) and \( z \) or \( y \) and \( z \) are convergent and unorthogonal.

NO2* means that if lines \( x \) and \( y \) are convergent and unorthogonal, then for any line \( z \), \( x \) and \( z \) are convergent or \( y \) and \( z \) are unorthogonal.

Note that O2 describes the convergence and unorthogonality of any three lines using six occurrences of the predicates \( CON \) and \( URT \); however, NO2* is one of O2’s four conjuncts and describes the same two relations using the four occurrences of \( CON \) and \( URT \) (cf. Table 3).

Axiom O2 in von Plato’s axiomatization can be replaced, equivalently, by the much shorter formula NO2*. Then, we obtain a new axiomatization which includes axioms O1, NO2*, O3 and O4 and 17 axioms for affine geometry.

Before we establish the equivalence between von Plato’s axiomatization and the new axiomatization, we give three lemmas as follows.

Lemma 1. Axiom 1.6 \( \Rightarrow \) S1.

Proof. The proof is easy: after eliminating quantifiers, let S1 be \( CON \ l \ m \& URT \ l \ m \rightarrow \left[ CON \ l \ n \right] \left[ CON \ m \ n \right] \). Assume \( CON \ l \ m \& URT \ l \ m \). By axiom 1.6, we obtain \( CON \ l \ m \rightarrow CON \ l \ n \left[ CON \ m \ n \right] \). Then the conclusion follows.
**Lemma 2.** O4, 2.3, 2.4 and von Plato’s Theorem 3.2 ⇒ S2.

**Proof.** von Plato’s Theorem 3.2 [12] is \((\forall x)(\forall y)[CON\ x\ y\ \rightarrow\ DILN\ x\ y]\). Only axioms 4.3, 1.3 and 1.6 for apartness geometry are needed to prove it.

The mechanical proof of the lemma is constructive, and it is put in Appendix C. The proof has 30 steps and the CPU time consumed is 2.8 s.

The following symmetry of CON will be used below.

\((\forall x)(\forall y)[CON\ x\ y\ \rightarrow\ CON\ y\ x]\).

It needs axioms 1.6 and 1.3 to prove the symmetry. ☐

**Remark.** Clearly NO4 ⇒ S2.

**Lemma 3.** NO2* and axioms 1.3 and 1.6 ⇒ S4.

**Proof.** The mechanical proof in natural deduction style is omitted. It has 37 steps. The time consumed is 2 s. The proof is classical. We give a constructive proof by hand as follows. After eliminating quantifiers, let S4 be \(CON\ l\ m & URT\ l\ m \rightarrow [CON\ m\ n] URT\ l\ n\). Assume \(CON\ l\ m, URT\ l\ m\). By the symmetry of CON, obtain \(CON\ m\ l\). By NO2*, it follows that \(CON\ l\ m & URT\ l\ m \rightarrow [CON\ l\ l] URT\ m\ l\), and so we obtain \(CON\ l\ l & URT\ m\ l\). From axiom 1.3, we obtain \(\sim CON\ l l\). So, \(URT\ m\ l\) can be obtained. By \(CON\ m\ l\) and \(URT\ m\ l\) and NO2*, the conclusion follows. ☐

**Theorem 2.** von Plato’s axiomatization for the orthogonality is equivalent to the new axiomatization which includes axioms O1, NO2*, O3 and O4 and 17 axioms for affine geometry.

**Proof.** von Plato’s axiomatization ⇒ the new axiomatization.

The proof is trivial since NO2* is just S3.

von Plato’s axiomatization ⇔ the new axiomatization.

Please note that NO2* is just S3. Then, by Lemmas 1–3, the conclusion follows. ☐

**Challenge.** It is hard to derive O2 from the new orthogonality obtained from von Plato’s system by replacing O2 by NO2*.

**Conjecture.** Can O2 be replaced by its conjuncts S1 or S2 or S4? ANDP failed to prove the equivalence of von Plato’s orthogonality and the orthogonality obtained by replacing O2 by its conjuncts S1 or S2 or S4. We think that O2 cannot be replaced by S1 or S2 or S4.

4.3. The proof of replacing O2 by NO2

Axiom O2 means that if lines \(x\) and \(y\) are convergent and unorthogonal, then for any line \(z\), \(x\) and \(z\) or \(y\) and \(z\) are convergent and unorthogonal.
Table 4
O2 vs. NO2

<table>
<thead>
<tr>
<th>Axioms</th>
<th>The length of formulas</th>
<th>The relations used in</th>
</tr>
</thead>
<tbody>
<tr>
<td>O2</td>
<td>6</td>
<td>URT, CON</td>
</tr>
<tr>
<td>NO2</td>
<td>3</td>
<td>URT, CON</td>
</tr>
</tbody>
</table>

NO2 means that if lines \(x\) and \(y\) are unorthogonal, then for any line \(z\), \(x\) and \(z\) are convergent or \(y\) and \(z\) are unorthogonal.

Note that there are six occurrences of two different predicates in O2 which are \(CON\) and \(URT\), and three occurrences of the two predicates in NO2 (cf. Table 4).

**Theorem 3.** von Plato’s axiomatization and the new axiomatization obtained by replacing O2 and O4 by NO2 and NO4 are constructive equivalent.

**Proof.** \(\Rightarrow\) We only need to show that NO2 can be derived from von Plato’s system. This is just Theorem 8.3 on p. 182 of [12]. By Theorem 1 we finished the proof of this direction.

\(\Leftarrow\) We only need to show that O2 can be derived from the new axiomatization. Then, by Theorem 1 we can finish the proof of this direction.

The mechanical proof of O2 in natural deduction style from the new axiomatization is put in Appendix B. It has 51 steps. The time consumed is 7s. The proof is classical. Fortunately, we found a constructive proof of O2 from the mechanical proof as follows.

From the mechanical proof, we know only four axioms of the new axiomatization are used to get the proof. They are axioms 1.3, 1.6, NO2 and NO4. We give by hand a constructive proof as follows.

After eliminating quantifiers, O2 is of the form \(CON \ l \ m \& URT \ l \ m \rightarrow [CON \ l \ n \& URT \ l \ n][CON \ m \ n \& URT \ m \ n]\). Assume \(CON \ l \ m, URT \ l \ m\). By NO4, after eliminating quantifiers, we obtain that \(CON \ l \ m \rightarrow [URT \ l \ n]URT \ m \ n\)...(1). By NO2, after eliminating quantifiers we obtain \(URT \ l \ m \rightarrow [CON \ l \ n]URT \ m \ n\), then it follows easily that \(CON \ l \ n]URT \ m \ n\)...(2). By (1) and (2) using the distributive law it follows that \([CON \ l \ n & URT \ l \ n]URT \ m \ n\)...(3). By axiom 1.6, after eliminating quantifiers we obtain \(CON \ l \ m \rightarrow [CON \ l \ n]CON \ m \ n\), then it follows that \(CON \ l \ n]CON \ mn\)...(4). By NO2, it follows that \(URT \ l \ m \rightarrow [CON \ l \ l]URT \ m \ l\). By the assumption \(URT \ l \ m\), then it follows that \(CON \ l \ l]URT \ m \ l\). From axiom 1.3, after eliminating quantifiers we obtain \(~CON \ l \ l\), then we obtain \(URT \ m \ l\). By NO2, we obtain \(URT \ m \ l \rightarrow [CON \ m \ n]URT \ l \ n\). By \(URT \ m \ l\), we obtain \(CON \ m \ n\]URT \ l \ n\)...(5).

Applying the distributive law to (4) and (5), we obtain that \([CON \ l \ n & URT \ l \ n]]CON \ m \ n\)...(6). Applying distributive law to (3) and (6), the conclusion follows.

4.4. The proof of replacing O2 by NO2** and the relations among NO2, NO2* and NO2**

NO2** means that if lines \(x\) and \(y\) are different and unorthogonal, then for any line \(z\), \(x\) and \(z\) are convergent or \(y\) and \(z\) are unorthogonal.
There are four occurrences of three predicates in $NO_2^{**}$ $DILN$, $CON$ and $URT$ (cf. Table 5).

$NO_2$, $NO_2^*$ and $NO_2^{**}$ are all implications, and their conclusions are the same. Their antecedents are different. The antecedents are $URT\; x\; y$, $CON\; xy\&URT\; x\; y$ and $DILN\; x\; y\&URT\; x\; y$, respectively. Clearly, $CON\; xy\&URT\; x\; y$ is more restricted than $DILN\; x\; y\&URT\; x\; y$, which is more restricted than $URTxy$. We discuss the relations among $NO_2$, $NO_2^*$ and $NO_2^{**}$ as follows.

**Lemma 4.** Constructively,

\[ NO_2 \Rightarrow NO_2^*, \]
\[ NO_2 \Rightarrow NO_2^{**}. \]

$NO_2^{**}$ and von Plato’s Theorem 3.2 $\Rightarrow NO_2^*$.

The proofs are trivial.

**Lemma 5.** $NO_2^*$ and axioms 1.3, 1.6 and $O_1 \Rightarrow NO_2$.

**Proof.** The mechanical proof obtained using the prover ANDP is omitted. The proof has 51 steps and is classical. The time consumed is 27 s. We give a constructive proof by hand as follows.

The following proof is similar to the one of Theorem 8.3 in [12]. After eliminating quantifiers, $NO_2$ is of the form $URT\; m\; n \rightarrow CON\; l\; m\; URT\; m\; n$. Assume $URT\; m\; n$. By $O_1$, obtain $CON\; m\; n\; URT\; m\; n$. If $URT\; m\; n$ holds, then the conclusion follows. Assume $CON\; m\; n$. By axiom 1.6, obtain $CON\; m\; n \rightarrow CON\; m\; l\; CON\; n\; l$. Then $CON\; m\; l\; CON\; n\; l$. If $CON\; n\; l$, then $CON\; l\; n$ by the symmetry of $CON$, and then the conclusion follows. If $CON\; m\; l$, then $CON\; l\; m$ by the symmetry of $CON$, and then obtain $CON\; l\; m\&URT\; l\; m$. By $NO_2^*$, it follows that $CON\; l\; m\&URT\; l\; m \rightarrow CON\; l\; n\; URT\; m\; n$. Then the conclusion follows. □

**Lemma 6.** $NO_2^*$ and axioms 1.3, 1.6 and $O_1 \Rightarrow NO_2^{**}$.

**Proof.** The mechanical proof obtained using the prover ANDP is omitted. The proof has 52 steps and is classical. The time consumed is 28 s. We give a constructive proof by hand as follows.

<table>
<thead>
<tr>
<th>Axioms</th>
<th>The length of formulas</th>
<th>The relations used in</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_2$</td>
<td>6</td>
<td>$URT, CON$</td>
</tr>
<tr>
<td>$NO_2^{**}$</td>
<td>4</td>
<td>$URT, CON, DILN$</td>
</tr>
</tbody>
</table>
It is obvious from Lemmas 4 and 5, and similar to proof of Lemma 5 above. The difference is that NO$^2_{**}$ is of the form DILN $l \ m \ &URT \ l \ m \rightarrow CON \ l \ n \ &URT \ m \ n$ and we need to assume that DILN $l \ m$ and $URT \ l \ m$. The rest of the proof carried out analogously to the proof of Lemma 5. □

**Lemma 7.** NO$^2_{**}$, Theorem 3.2 [12] and axioms 1.3, 1.6 and O1 $\Rightarrow$ NO2.

**Proof.** The mechanical proof obtained using the prover ANDP is omitted. The proof has 56 steps and is classical. The time consumed is 41 s. We give a constructive proof as follows.

It is obvious from Lemmas 4 and 5. In fact it is easy to get a proof by modifying the proof of Lemma 5 above. We modify the last sentence of the proof of Lemma 5 as follows. If CON $ml$, then CON $lm$ by the symmetry of CON. Then, by von Plato’s Theorem 3.2, obtain DILN $l \ m$. Then, by DILN $l \ m$ and assumption $URT \ l \ m$ and NO$^2_{**}$, the conclusion follows. □

Though NO$^2_{*}$ and NO$^2_{**}$ are logical consequences of NO2, the axiomatizations obtained by replacing O2 by NO2, NO$^2_{*}$ and NO$^2_{**}$, respectively, are equivalent. The formula $URT \ l \ l \rightarrow CON \ l \ n \ URT \ l \ n$ is an instance of NO2. It is equivalent to O1 since $URT \ l \ l$ is true. O1 shows the convergence and orthogonality of two lines. NO2 shows the convergence and orthogonality of three lines.

**Theorem 4.** Axiom O2 in von Plato’s orthogonality can be replaced equivalently by NO2 or NO$^2_{*}$ or NO$^2_{**}$.

**Proof.** It is obvious from Lemmas 4–7 and Theorem 2. □

**Theorem 5.** von Plato’s orthogonality is equivalent to the axiomatizations obtained by adding O1, O3, NO4 and NO2 or NO$^2_{*}$ or NO$^2_{**}$ to affine geometry.

**Proof.** Obvious from Lemmas 4–7, and Theorems 1–4. □

4.5. *The proof of replacing O1 and O2 by NO1 and NO2 simultaneously*

Axiom O1 means that any two lines $x$ and $y$ are convergent or unorthogonal.

NO1 means that for any line $x$, $x$ is not unorthogonal to itself. Please see the following Table 6.

**Lemma 8.** NO1, NO2 $\Rightarrow$ O1.

**Proof.** Trivial. □

**Theorem 6.** von Plato’s axiomatization for orthogonality is equivalent to the axiomatization obtained by adding NO1, NO2, O3 and NO4 to affine geometry.
Table 6
O1 and O2 vs. NO1 and NO2

<table>
<thead>
<tr>
<th>Axioms</th>
<th>The length of formulas</th>
<th>The relations used in</th>
</tr>
</thead>
<tbody>
<tr>
<td>O1, O2</td>
<td>8</td>
<td>URT, CON</td>
</tr>
<tr>
<td>NO1, NO2</td>
<td>4</td>
<td>URT, CON</td>
</tr>
</tbody>
</table>

**Proof.** ($\Rightarrow$) By Theorems 8.2, 8.3 and 8.6 in [12] it is obvious.

($\Leftarrow$) By Lemma 8 and Theorem 3 above, it is obvious. $\square$

We did a lot of computer experiments to obtain the results in this subsection.

*The problems and computer experiments related to O1:*

1. Can O1 be replaced by a short axiom?
2. NO1 is just Theorem 8.2 [12]. It just means that NO1 is a logical consequence of von Plato’s system. Does it mean that NO1 itself can replace one of von Plato’s axioms O1, O2, O3 and O4?
3. After knowing that each of NO2*, NO2** and NO2 can replace O2, then can each of the combinations: NO1 and NO2*, NO1 and NO2** and NO1 and NO2, replace both O1 and O2?
4. After knowing that NO4 can replace O4, then can NO1 and NO4 together replace both O1 and O4?

After computer experiments using ANDP, we obtained the following results.

1. We think O1 cannot be replaced by a single shorter axiom, for example NO1. O1 describes the convergence and unorthogonality of two lines. By contrast O1 is the shortest and “nicest” among the axioms relating to the orthogonality of two lines.
2. We think that NO1 itself cannot replace any one of von Plato’s O1, O2, O3 and O4. ANDP failed to find any mechanical proofs related.
3. von Plato did state on p. 182 of [12] that “it is possible to replace O1 by Theorems 8.2 and 8.3”. It means to replace a short axiom by two new axioms: one is longer and one is shorter. Clearly, it is not desired to add Theorems 8.2 and 8.3, O2, O3 and O4 rather than O1, O2, O3 and O4 to the affine geometry to get orthogonality.

Can O2 be replaced by Theorems 8.2 and 8.3? No evidence was given in [12], and, further, O2 was used in the proof of Theorem 8.3 on p. 182 of [12]. After testing many combinations by ANDP we found that NO1 and NO2 together can replace not only O1 but also O2. But it failed to replace both O1 and O2 by NO1 and NO2* or NO1 and NO2**.
4. We think that NO1 and NO4 together cannot replace both O1 and O4.

Thus we got the following conjecture.

**Conjecture.** We conjecture that the axiomatizations obtained by adding each of six sets of axioms listed in the former of the section to affine geometry are not equivalent
Table 7
Summary

<table>
<thead>
<tr>
<th>Axioms</th>
<th>The length of formulas</th>
<th>The relations used in</th>
</tr>
</thead>
<tbody>
<tr>
<td>O1, O2, O4</td>
<td>13</td>
<td>URT, CON, DILN, APT</td>
</tr>
<tr>
<td>NO1, NO2, NO4</td>
<td>7</td>
<td>URT, CON</td>
</tr>
</tbody>
</table>

to von Plato’s system for orthogonality. We could not find any mechanical proof of axiom O1 from the axiomatizations using the prover ANDP.

5. Conclusions

To sum up, we obtain nine sets of axioms for orthogonality, which are listed in Section 4. Adding any one of these nine sets to affine geometry, we get a constructive axiomatization for orthogonality which is equivalent to von Plato’s one. \{NO1, NO2, O3, NO4\} is the shortest and nicest one. For example, adding NO1, NO2, O3 and NO4 instead of von Plato’s O1, O2, O3 and O4 to his affine geometry we get a new axiomatization of orthogonality. There are seven predicate occurrences in NO1, NO2 and NO4, while there are 13 predicate occurrences in von Plato’s axioms O1, O2 and O4.

Four relations, consisting of the three relations of two lines, which are convergence (CON), unorthogonality (URT) and difference (DILN), and the relation (APT) of a point and line were used in O1, O2 and O4. However only two relations, convergence and unorthogonality, are used in NO1, NO2 and NO4. This means that two relations, namely difference (DILN) of two lines and the relation (APT) of a point and a line, are not necessary to describe the concept of orthogonality. Please see Table 7.

6. Discussion

6.1. Towards constructive proofs

The prover ANDP produces constructive proofs for some theorems. If a theorem is an implication, then usually the mechanical proof in natural deduction style obtained with the prover ANDP is constructive. If a theorem is a disjunction, then the mechanical proofs obtained with the prover ANDP sometimes is classical. However, in each of the results in this paper it is not hard to transform a mechanical proof obtained with the prover ANDP into a constructive proof for the following reasons.

1. From a mechanical proof obtained using the prover ANDP it is easy to see which of von Plato’s 22 axioms are used to get the proof. Usually, only three or four axioms of the 22 axioms are used to get the results in this paper.
2. Usually, only one or two steps in the proofs obtained with ANDP are classical.
6.2. The order of premises

From our experience, we know that the order of premises is important to find a proof. Some orders make it easy to find proofs, and some make it hard. So far, we do not know how to order premises. However, it is obvious to list the shortest premise at the top and the longest at the bottom. That is, shorter premises are tried first.

It is also obvious to list useful premises for a proof at the top. Unfortunately we do not know which of premises are needed for a proof, before a proof is found. For example, there are 22 axioms for von Plato’s orthogonality, but in many cases not all of them are needed. However, it is easy to see that if some predicates in a premise appear in the goal, perhaps the premise is useful for a proof.

Acknowledgements

The paper is supported by NSFC, partially by Intelligence Technology and Systems Lab of Tsinghua University. The author wants to thank Prof. D. Kapur for his invitation and discussion with him and Dr. P. Baumgartner for improving English and editing and many helpful comments.

The experiments whose results are reported in this paper were executed at RRL of department of computer science of SUNY Albany.

Appendix A

Axioms NO4 and A2 => 04
The time consumed is 0.35 s.

1. A2 & NO4 \hspace{1cm} \text{ASSUMED-PREMISE}

2. $(Ax)(Ay)(Az) [DILN \ x \ y \rightarrow [APT \ z \ x \ | \ APT \ z \ y] \ | \ CON \ x \ y]$ \hspace{1cm} \text{SIMP 1}

3. $(Ax)(Ay)(Az) [CON \ x \ y \rightarrow URT \ x \ z \ | \ URT \ y \ z]$ \hspace{1cm} \text{SIMP 1}

4. DILN $v1 \ v2$ \hspace{1cm} \text{ASSUMED-PREMISE}

5. $\neg APT \ v4 \ v1$ \hspace{1cm} \text{ASSUMED-PREMISE}

6. $\neg APT \ v4 \ v2$ \hspace{1cm} \text{ASSUMED-PREMISE}

7. $\neg URT \ v1 \ v3$ \hspace{1cm} \text{ASSUMED-PREMISE}

8. $(Ay)(Az) [CON \ v1 \ y \rightarrow URT \ v1 \ z \ | \ URT \ y \ z]$ \hspace{1cm} \text{US (v1 x) 3}

9. $(Ay)(Az) [DILN \ v1 \ y \rightarrow [APT \ z \ v1 \ | \ APT \ z \ y] \ | \ CON \ v1 \ y]$ \hspace{1cm} \text{US (v1 x) 2}

10. $(Az)[DILN \ v1 \ v2 \rightarrow [APT \ z \ v1 \ | \ APT \ z \ v2] \ | \ CON \ v1 \ v2]$ \hspace{1cm} \text{US (v2 y) 9}

11. DILN $v1 \ v2 \rightarrow [APT \ v4 \ v1 \ | \ APT \ v4 \ v2] \ | \ CON \ v1 \ v2$ \hspace{1cm} \text{US (v4 z) 10}

12. $[APT \ v4 \ v1 \ | \ APT \ v4 \ v2] \ | \ CON \ v1 \ v2$ \hspace{1cm} \text{MP 11 4}

13. APT $v4 \ v1$ | $[APT \ v4 \ v2 \ | \ CON \ v1 \ v2]$ \hspace{1cm} \text{IMPLICATION 12}

14. APT $v4 \ v2$ | $CON \ v1 \ v2$ \hspace{1cm} \text{LDS 13 5}

15. $CON \ v1 \ v2$ \hspace{1cm} \text{LDS 14 6}

16. $(Az)[CON \ v1 \ v2 \rightarrow URT \ v1 \ z \ | \ URT \ v2 \ z]$ \hspace{1cm} \text{US (v2 y) 8}

17. $CON \ v1 \ v2 \rightarrow URT \ v1 \ v3 \ | \ URT \ v2 \ v3$ \hspace{1cm} \text{US (v3 z) 16}
18. $\text{URT} \ v_1 \ v_3 \ | \ \text{URT} \ v_2 \ v_3$  
19. $\text{URT} \ v_2 \ v_3$  
20. $\text{URT} \ v_2 \ v_3$  
21. "$\text{URT} \ v_1 \ v_3 \ \rightarrow \ \text{URT} \ v_2 \ v_3$"  
22. $\text{URT} \ v_1 \ v_3 \ | \ \text{URT} \ v_2 \ v_3$  
23. "$\text{APT} \ v_4 \ v_2 \ \rightarrow \ \text{URT} \ v_1 \ v_3 \ | \ \text{URT} \ v_2 \ v_3$"  
24. $\text{APT} \ v_4 \ v_2 \ | \ \text{URT} \ v_1 \ v_3 \ | \ \text{URT} \ v_2 \ v_3$  
25. "$\text{APT} \ v_4 \ v_1 \ \rightarrow \ \text{APT} \ v_4 \ v_2 \ | \ \text{URT} \ v_1 \ v_3 \ | \ \text{URT} \ v_2 \ v_3$"  
26. $\text{APT} \ v_4 \ v_1 \ | \ \text{APT} \ v_4 \ v_2 \ | \ \text{URT} \ v_1 \ v_3 \ | \ \text{URT} \ v_2 \ v_3$  
27. $\text{DILN} \ v_1 \ v_2$  
   $\rightarrow$ $\text{[APT} \ \ v_4 \ v_1 \ | \ \text{APT} \ v_4 \ v_2 \ | \ \text{URT} \ v_1 \ v_3 \ | \ \text{URT} \ v_2 \ v_3]$  
28. $(\text{Ax})(\text{Ay})(\text{Az})(\text{Au})[\text{DILN} \ x \ y$  
   $\rightarrow$ $\text{[APT} \ u \ x \ | \ \text{APT} \ u \ y \ | \ \text{URT} \ x \ z \ | \ \text{URT} \ y \ z\text{]}]$  
29. $\text{A2} \ & \ \text{NO4} \ \rightarrow \ \text{O4}$

Appendix B

Axioms 1.3 & 1.6 & NO2 & NO4 => O2

The proof has 51 steps. The time consumed is 7 s.

1. 1.3 & 1.6 & NO2 & NO4 \text{ASSUMED-PREMISE}
2. $(\text{Ax})(\text{Ay})\neg\text{CON} \ x \ x$ \text{SIMP} 1
3. $(\text{Ax})(\text{Ay})(\text{Az})[\text{CON} \ x \ y \ \rightarrow \ \text{URT} \ x \ z \ | \ \text{URT} \ y \ z]$ \text{SIMP} 1
4. $(\text{Ax})(\text{Ay})(\text{Az})[\text{URT} \ x \ y \ \rightarrow \ \text{CON} \ x \ z \ | \ \text{URT} \ y \ z]$ \text{SIMP} 1
5. $(\text{Ax})(\text{Ay})(\text{Az})[\text{CON} \ x \ y \ \rightarrow \ \text{CON} \ x \ z \ | \ \text{CON} \ y \ z]$ \text{SIMP} 1
6. $\text{CON} \ v_1 \ v_2 \ \text{URT} \ v_1 \ v_2$ \text{ASSUMED-PREMISE}
7. $\neg[\text{CON} \ v_1 \ v_3 \ & \ \text{URT} \ v_1 \ v_3]$ \text{ASSUMED-PREMISE}
8. $\text{CON} \ v_1 \ v_2$ \text{SIMP} 6
9. $\text{URT} \ v_1 \ v_2$ \text{SIMP} 6
10. $\neg\text{CON} \ v_1 \ v_3 \ | \ \neg\text{URT} \ v_1 \ v_3$ \text{DE.MORGAN} 7
11. $(\text{Ay})(\text{Az})[\text{CON} \ v_1 \ y \ \rightarrow \ \text{CON} \ v_1 \ z \ | \ \text{CON} \ y \ z]$ \text{US} $(v_1 \ x)$ 5
12. $(\text{Ay})(\text{Az})[\text{URT} \ v_1 \ y \ \rightarrow \ \text{CON} \ v_1 \ z \ | \ \text{URT} \ y \ z]$ \text{US} $(v_1 \ x)$ 4
13. $(\text{Ay})(\text{Az})[\text{CON} \ v_1 \ y \ \rightarrow \ \text{URT} \ v_1 \ z \ | \ \text{URT} \ y \ z]$ \text{US} $(v_1 \ x)$ 3
14. $(\text{Ay})\neg\text{CON} \ v_1 \ v_1$ \text{US} $(v_1 \ x)$ 2
15. $\neg\text{CON} \ v_1 \ v_1$ \text{US} $(v_1 \ y)$ 14
16. $(\text{Az})[\text{URT} \ v_1 \ v_2 \ \rightarrow \ \text{CON} \ v_1 \ z \ | \ \text{URT} \ v_2 \ z]$ \text{US} $(v_2 \ y)$ 12
17. $(\text{Az})[\text{CON} \ v_1 \ v_2 \ \rightarrow \ \text{URT} \ v_1 \ z \ | \ \text{URT} \ v_2 \ z]$ \text{US} $(v_2 \ y)$ 13
18. $(\text{Az})[\text{CON} \ v_1 \ v_2 \ \rightarrow \ \text{CON} \ v_1 \ z \ | \ \text{CON} \ v_2 \ z]$ \text{US} $(v_2 \ y)$ 11
19. $\text{URT} \ v_1 \ v_2 \ \rightarrow \ \text{CON} \ v_1 \ v_1 \ | \ \text{URT} \ v_2 \ v_1$ \text{US} $(v_1 \ z)$ 16
20. $\text{CON} \ v_1 \ v_1 \ | \ \text{URT} \ v_2 \ v_1$ \text{MP} 19 9
21. $\text{URT} \ v_1 \ v_2$ \text{LDS} 20 15
22. $(\text{Ay})(\text{Az})[\text{URT} \ v_2 \ y \ \rightarrow \ \text{CON} \ v_2 \ z \ | \ \text{URT} \ y \ z]$ \text{US} $(v_2 \ x)$ 4
23. $(\text{Az})[\text{URT} \ v_2 \ v_1 \ \rightarrow \ \text{CON} \ v_2 \ z \ | \ \text{URT} \ v_1 \ z]$ \text{US} $(v_1 \ y)$ 22
24. $\neg\text{CON} \ v_1 \ v_3$ \text{CASE2} 10
25. $\neg\text{URT} \ v_1 \ v_3$ \text{CASE1} 10
26. $\text{CON} \ v_1 \ v_2 \ \rightarrow \ \text{URT} \ v_1 \ v_3 \ | \ \text{URT} \ v_2 \ v_3$ \text{US} $(v_3 \ z)$ 17
27. $\text{URT} \ v_1 \ v_3 \ | \ \text{URT} \ v_2 \ v_3$ \text{MP} 26 8
28. $\text{URT} \ v_2 \ v_3$ \text{LDS} 27 25
29. $\text{URT} \ v_1 \ v_2 \ \rightarrow \ \text{CON} \ v_1 \ v_3 \ | \ \text{URT} \ v_2 \ v_3$ \text{US} $(v_3 \ z)$ 16
30. $\text{CON} \ v_1 \ v_3 \ | \ \text{URT} \ v_2 \ v_3$ \text{MP} 29 9
31. $\text{URT} \ v_2 \ v_3$ \text{LDS} 30 24
32. $\neg\text{CON} \ v_1 \ v_3$ \text{CASE2} 10
33. \(~\text{URT} v_1 v_3\)  
34. \(\text{URT} v_2 v_1 \rightarrow \text{CON} v_2 v_3 \; | \; \text{URT} v_1 v_3\)  
35. \(\text{CON} v_2 v_3 \; | \; \text{URT} v_1 v_3\)  
36. \(\text{CON} v_2 v_3\)  
37. \(\text{CON} v_1 v_2 \rightarrow \text{CON} v_1 v_3 \; | \; \text{CON} v_2 v_3\)  
38. \(\text{CON} v_1 v_3 \; | \; \text{CON} v_2 v_3\)  
39. \(\text{CON} v_2 v_3\)  
40. \(\text{CON} v_2 v_3\)  
41. \(\text{CON} v_2 v_3\)  
42. \(\text{URT} v_2 v_3\)  
43. \(\text{URT} v_2 v_3\)  
44. \(\text{URT} v_2 v_3\)  
45. \(\text{CON} v_2 v_3\)  
46. \(\text{CON} v_2 v_3 \; \& \; \text{URT} v_2 v_3\)  
47. \(~\text{CON} v_1 v_3 \; \& \; \text{URT} v_1 v_3\)\(\rightarrow\) \(\text{CON} v_2 v_3 \; \& \; \text{URT} v_2 v_3\)  
48. \(\text{CON} v_1 v_3 \; \& \; \text{URT} v_1 v_3 \; | \; \text{CON} v_2 v_3 \; \& \; \text{URT} v_2 v_3\)  
49. \(\text{CON} v_1 v_2 \; \& \; \text{URT} v_1 v_2\)  
40. \(~\text{CON} v_1 v_2\)\(\rightarrow\) \(\text{CON} v_1 v_3 \; | \; \text{CON} v_2 v_3 \; \& \; \text{URT} v_2 v_3\)  
50. \((\text{Ax})(\text{Ay})(\text{Az})(\text{Au})\;[\text{DILN} v_1 y \rightarrow [\text{APT} u v_1 | \text{APT} u v_2] \; | \; [\text{URT} v_1 z \; | \; \text{URT} y z]]\)  
51. \(1.3 \; \& \; 1.6 \; \& \; \text{NO}2 \; \& \; \text{NO}4 \rightarrow \text{O}2\)  

Appendix C

Axioms 2.3, 2.4 and 04 and von Plato's Theorem 3.2 \(\rightarrow\) S2.
The proof has 30 steps. The time consumed is 2.8 s.

1. Axioms 2.3, 2.4 and 04 and von Plato's Theorem 3.

ASSUMED-PREMISE

2. \((\text{Ax})(\text{Ay})[\text{CON} x y \rightarrow \text{DILN} x y]\)  
3. \((\text{Ax})(\text{Ay})[\text{CON} x y \rightarrow \neg \text{APT}[\text{pt} x y] x]\)  
4. \((\text{Ax})(\text{Ay})[\text{CON} x y \rightarrow \neg \text{APT}[\text{pt} x y] y]\)  
5. \((\text{Ax})(\text{Ay})(\text{Az})(\text{Au})[\text{DILN} x y \rightarrow [\text{APT} u x \; | \; \text{APT} u y] \; | \; [\text{URT} x z \; | \; \text{URT} y z]]\)  

6. \(\text{CON} v_1 v_2 \; \& \; \text{URT} v_1 v_2\)  

ASSUMED-PREMISE

7. \(\text{CON} v_1 v_2\)  
8. \((\text{Ay})[\text{CON} v_1 y \rightarrow \neg \text{APT}[\text{pt} v_1 y] y]\)  
9. \((\text{Ay})[\text{CON} v_1 y \rightarrow \neg \text{APT}[\text{pt} v_1 y] v_1]\)  
10. \((\text{Ay})[\text{CON} v_1 y \rightarrow \text{DILN} v_1 y]\)  
11. \(\text{CON} v_1 v_2 \rightarrow \neg \text{APT}[\text{pt} v_1 v_2] v_2\)  
12. \(\neg \text{APT}[\text{pt} v_1 v_2] v_2 \; \& \; \text{MP} 11\)  
13. \(\text{CON} v_1 v_2 \rightarrow \neg \text{APT}[\text{pt} v_1 v_2] v_1\)  
14. \(\neg \text{APT}[\text{pt} v_1 v_2] v_1 \; \& \; \text{MP} 13\)  
15. \(\text{CON} v_1 v_2 \rightarrow \text{DILN} v_1 v_2\)  
16. \(\text{DILN} v_1 v_2\)  
17. \((\text{Ay})(\text{Az})(\text{Au})[\text{DILN} v_1 y \rightarrow [\text{APT} u v_1 \; | \; \text{APT} u v_2] \; | \; [\text{URT} v_1 z \; | \; \text{URT} y z]]\)  
18. \((\text{Az})(\text{Au})[\text{DILN} v_1 v_2 \rightarrow [\text{APT} u v_1 \; | \; \text{APT} u v_2] \; | \; [\text{URT} v_1 z \; | \; \text{URT} v_2 z]]\)  
19. \((\text{Au})[\text{DILN} v_1 v_2 \rightarrow [\text{APT} u v_1 \; | \; \text{APT} u v_2] \; | \; [\text{URT} v_1 v_{15} \; | \; \text{URT} v_2 v_{15}]\)
Appendix D. Rules of inference used in Natural deduction

D.1. Rules of inference for propositional logic

The axiom and basic rules:
(a) \( \Gamma, A \vdash A \), premise or assumption,
(b) \( \frac{A \vdash \Gamma}{\Gamma, A \vdash B} \).

Rules for connectives:
(a) \( \frac{\Gamma \vdash A \quad \Gamma \vdash \Gamma \vdash A \rightarrow B}{\Gamma \vdash B} \) MP (modus ponens),
\( \frac{\Gamma \vdash \neg B \quad \Gamma \vdash A \rightarrow B}{\Gamma \vdash \neg A} \) MT (modus tollens),
(b) \( \frac{\Gamma \vdash A \land B}{\Gamma \vdash A} \) LDS (left disjunctive syllogism),
\( \frac{\Gamma \vdash A \land B}{\Gamma \vdash B} \) RDS (right disjunctive syllogism),
(c) \( \frac{\Gamma \vdash A \rightarrow C \quad \Gamma \vdash \neg C \rightarrow A}{\Gamma \vdash B} \) CP (conditional proof),
(d) \( \frac{\Gamma \vdash A \land B \quad \Gamma \vdash A \rightarrow \Gamma \vdash B}{\Gamma \vdash A \rightarrow \Gamma \vdash B \rightarrow \Gamma \vdash B} \) ADD (addition),
(e) \( \frac{\Gamma \vdash A \quad \Gamma \vdash \neg A \rightarrow \neg B}{\Gamma \vdash B} \) CONJUN (conjunction),
(f) \( \frac{\Gamma \vdash A \land B \quad \Gamma \vdash \neg (A \land B)}{\Gamma \vdash \neg A \rightarrow \neg B} \) SIMP (simplification),
(g) \( \frac{\Gamma \vdash A \quad \Gamma \vdash \neg A \rightarrow \neg B}{\Gamma \vdash B} \) IP (indirect proof),
\( \frac{\Gamma \vdash A \quad \Gamma \vdash A \rightarrow \neg A}{\Gamma \vdash \neg A} \) Elimination,
(h) \( \frac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A \rightarrow \neg B}{\Gamma \vdash B} \) EQUIV (equivalence),
(i) \( \frac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A \rightarrow \neg B}{\Gamma \vdash B} \) DILEMMA (or cases),
(j) \( \frac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A \rightarrow \neg B}{\Gamma \vdash \neg A} \) IMP (implication),
(k) \( \frac{\Gamma \vdash A \quad \Gamma \vdash A \rightarrow \neg A}{\Gamma \vdash \neg A} \) double negation
D.2. Rules for quantifiers

(a) \( \frac{\Gamma, A(x)}{\Gamma, \forall x A(x)} \) UG (universal generalization),
where \( x \) is not free in any member of \( \Gamma \).

(b) \( \frac{\Gamma, \forall x A(x)}{\Gamma, A(t)} \) US (universal specialization),
where \( t \) is free for \( x \) in \( A(x) \).

(c) \( \frac{\Gamma, A(t)}{\Gamma, \exists x A(x)} \) EG (existential generalization),
where \( t \) is free for \( x \) in \( A(x) \).

(d) \( \frac{\Gamma, \exists x A(x), \Gamma, A(a) \vdash C}{\Gamma, C} \) ES (or EE, Eliminating Existential quantifier),
where \( a \) is an individual constant not occurring in any member of \( \Gamma \), in \( \exists x A(x) \) or in \( C \).

Substitution rule:
\( \frac{\Gamma, A(x)}{\Gamma, A(t)} \) SUB (Substitution) where variable \( x \) is not free in any member of \( \Gamma \) and \( t \) is free for \( x \) in \( A(x) \).

De Morgan rules:
\( \frac{\Gamma, \neg [A \land B]}{\Gamma, \neg A \lor \neg B} \), \( \frac{\Gamma, \neg [A \lor B]}{\Gamma, \neg A \land \neg B} \), \( \frac{\Gamma, \neg [A \rightarrow B]}{\Gamma, \neg A \lor \neg B} \), \( \frac{\Gamma, \neg A \land \neg B}{\Gamma, \neg A \lor \neg B} \), \( \frac{\Gamma, \neg \forall x A(x)}{\Gamma, \exists x \neg A(x)} \), \( \frac{\Gamma, \neg \exists x A(x)}{\Gamma, \forall x \neg A(x)} \), \( \frac{\Gamma, \forall x A(x)}{\Gamma, \neg \exists x \neg A(x)} \), \( \frac{\Gamma, \exists x A(x)}{\Gamma, \neg \forall x \neg A(x)} \).

References

[3] Li Dafa, Three axioms of von Plato’s axiomatization of constructive projective geometry are not independent, AAR Newsletter, No. 37, August, 1997.