A Quantitative Analysis of Implicational Paradoxes in Classical Mathematical Logic

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ABSTRACT

Classical mathematical logic includes a lot of "implicational paradoxes" as its logic theorems. On the other hand, relevant logics and strong relevant logics have rejected those implicational paradoxes as their logical theorems. This paper uses the property of strong relevance as the criterion to identify implicational paradoxes in logical theorems of classical mathematical logic, and count the number of logical theorem schemata of classical mathematical logic that do not satisfy the strong relevance. Our results quantitatively shows that classical mathematical logic is by far not a suitable logical basis for automated forward deduction.

Categories and Subject Descriptors

F.4.1 [Mathematical Logic and Formal Languages]: Mathematical Logic; I.2.3 [Artificial Intelligence]: Deduction and Theorem Proving—Deduction

General Terms

Automated forward deduction, Relevant logics

Keywords

Knowledge representation and reasoning, Automated forward deduction, Relevant logics, Strong relevance

1. INTRODUCTION

A forward deduction engine is an indispensable component for any knowledge-based system to discover new knowledge or predict future incidents. Since any forward deduction for discovery or prediction has no explicitly specified proposition or theorem given previously as goal, intrinsically, to apply all inference rules to all given premises and previously deduced conclusions is the only way to deduce new knowledge or predictions. This naturally requires that a forward deduction engine deduces only conclusions that are certainly relevant to given premises.

Within the framework of classical mathematical logic, the conclusion of a valid deduction is not necessarily relevant to its premises, because classical mathematical logic, CML for short, is established based on the classical account of validity. It is well known that logical theorems of CML include a lot of "implicational paradoxes" [1, 2].

On the other hand, relevant logics and strong relevant logics have rejected those implicational paradoxes as their logical theorems and been selected as logical basis for automated forward deduction [3, 4]. However, until now no one investigated quantitatively how CML is "bad" and/or how relevant logics are "good" for automated forward deduction. In order to give a quantitative reference standard for designers and developers of applied intelligent systems, this paper presents the result of our quantitative analysis of implicational paradoxes in CML and explains its implications. The paper uses the property of strong relevance as the criterion to identify implicational paradoxes in logical theorems of classical mathematical logic, and count the number of logical theorem schemata of CML that do not satisfy the strong relevance. Our results quantitatively shows that CML is by far not a suitable logical basis for automated forward deduction. This is the first claim with numerical grounds useful to designers and developers of applied intelligent systems.

2. IMPLICATIONAL PARADOXES

CML and its various conservative extensions have the well-known "implicational paradox problem" [1, 2]. In CML and its various conservative extensions, the notion of conditional is represented by the truth-functional extensional notion of material implication (denoted by → in this paper) that is defined as "A → B =df (A ∧ ¬B) or A → B =df ¬A ∨ B", where ∧, ∨, and ¬ denote the notion of conjunction, disjunction, and negation, respectively. It is no more than an extensional truth-function of its antecedent and consequent but does not require that there is a necessarily relevant and conditional relation between its antecedent and consequent. Consequently, in the framework of CML and its various conservative extensions, even if a reasoning is valid in the sense of CML, neither the necessary relevance between its premises and conclusion nor the truth of its conclusion in the sense of conditional can be guaranteed necessarily.

Relevant logics, RL for short, were constructed to obtain a notion of implication which is free from the so-called "paradoxes" of material and strict implication [1, 2]. However, although RL have rejected those paradoxes of implication, there still exist some "conjunction-implicational paradoxes"
and “disjunction-implicational paradoxes” in logical theorems of RL [3, 4]. Cheng has proposed some strong relevant logics, SRL for short, which do not include the paradoxes [3, 4].

In the framework of SRL, if a reasoning is valid, then both the relevance between its premises and conclusions and the validity of its conclusions in the sense of conditional can be guaranteed in a certain sense of strong relevance. Strong relevance, SR for short, is one of principles in RL and SRL: every sentential variable (or pattern variable) in a formula (or formula schema) occurs at least once as an antecedent part and at least once as a consequent part. For the definition of antecedent part and consequent part, refer to [1].

On the other hand, implicational paradoxes in CML do not satisfy SR, because SR is a principle which formally guarantees the relationship between antecedent and consequent. Therefore, we can distinguish implicational paradoxes from axioms and logical theorems of CML by whether a formula or a formula schema satisfies SR or not.

3. A QUANTITATIVE ANALYSIS

In this paper, we focus on the axiomatic system of CML with only the connectives of implication and negation on propositional calculus. We count the number of implicational paradoxes in \( k^{th} \) degree logical theorem schemata fragment of CML. In order to forecast the tendency of the increase of the difference between the number of implicational paradoxes and that of paradox-free theorems in \( k^{th} \) degree logical theorem schemata fragment of CML, we also count the number of formula schemata satisfying SR in \( k^{th} \) degree formula schemata fragment of CML.

In this paper, a formula schema is a schema that replaced all propositional variable in a formula with pattern variables. A pattern variable is a variable for which it can substitute a certain sentential variable. We use \( U \) to denote the set of all formula schemata of CML. It consists of only following formula schemata: \( p, \neg p, (A \Rightarrow B), \neg(A \Rightarrow B) \in U \), where \( A, B \in U \), and \( p \) and \( q \) are pattern variables.

A \( k^{th} \) degree formula schema is defined as follows: let \( \text{deg}_-(A) = k \) denotes that the degree of nested material implication \( \rightarrow \) of a formula schema \( A \) is \( k \), and both \( A \) and \( B \) are formula schema.

1. If there is no occurrence of \( \rightarrow \) in \( A \), then \( \text{deg}_-(A) = 0 \),
2. \( \text{deg}_-(\neg A) = \text{deg}_-(A) \),
3. \( \text{deg}_-(A \rightarrow B) = 1 + \max(\text{deg}_-(A), \text{deg}_-(B)) \).

\( k^{th} \) degree formula schemata fragment of \( U \) is a set which includes all \( j^{th} \) degree formula schemata of \( U \) \( (1 \leq j \leq k) \), and denoted by \( F_k \). \( F_{S_k} \) denotes a set which includes all formula schemata satisfying SR in \( F_k \). \( k^{th} \) degree logical theorem schemata fragment of CML is a set which includes all \( j^{th} \) degree logical theorem schemata of CML \( (1 \leq j \leq k) \), and denoted by \( ThS_k \). \( ThS_k \) denotes a set which is \( F_{S_k} \cap Th_k \). \( IP_k \) denotes a set of all implicational paradoxes in \( ThS_k \), i.e., \( IP_k = ThS_k - ThS_k \).

First, we produced all elements of \( F_k \) by our program, then we distinguished logical theorem schemata from produced formula schemata by the tableau method. After that, we judged whether this logical theorem schema satisfies SR or not. Table 1 shows the number of elements of \( IP_k \) and \( ThS_k \) \( (1 \leq k \leq 3) \).

Second, we implemented an algorithm to calculate the number of elements of \( F_k \) and \( FS_k \), and some programs based on the algorithm. Our experiment shows the number of elements of \( F_k - FS_k \) increases exponentially when \( k \) becomes larger. We will give the algorithm and detail calculation results in our full paper.

Our analysis results showed that the number of implicational paradoxes in the set of all \( 1^{st} \sim 3^{rd} \) degree logical theorem schemata of CML is 16.13 times as many as that of paradox-free theorem schemata of the set. We conclude that this rate becomes larger exponentially as the degree of nested implications becomes larger lineally.

4. CONCLUDING REMARKS

Implicational paradoxes spoil the validity of forward deduction, and unnecessarily lengthen the execution time of automated forward deduction. However, classical mathematical logic, CML for short, has implicational paradoxes by far more than paradox-free logical theorems in its logical theorems.

On the other hand, all logical theorems of relevant logics, RL for short, and strong relevant logics, SRL for short, with only the connectives of implication and negation satisfy strong relevance. Hence they are elements of the set of paradox-free logical theorems of CML, if we regard material implication in CML and entailment in RL and SRL as a same connective to represent the notion of conditional. Therefore, CML has implicational paradoxes by far more than all logical theorems of RL and SRL with only the connectives of implication and negation.

Consequently, as the logic systems underlying forward deduction, SRL and RL are quantitatively more suitable by far than CML and its various conservative extensions. Thus, designers and developers of knowledge-based systems should implement a forward deduction system on the basis of SRL or RL.

5. REFERENCES