Estimation of Mobile Vehicle Range & Position Using The Tobit Kalman Filter

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Abstract—Censored measurements arise frequently in engineering applications in the form of saturation, limit of detection, and occlusion effects. Censored data regression uses a formulation known as the Tobit model. We have recently presented a novel extension of the Kalman filter that allows for optimal state tracking in the presence of censored measurements. We call this formulation the Tobit Kalman filter. In this paper, we present an application of the Tobit Kalman filter to mobile vehicle position estimation using received signal strength from a number of antennas. Received signal strength drops below the noise floor as distance increases, and is therefore a censored measurement. We will briefly introduce the Tobit Kalman filter and compare its performance on the position estimation problem with the standard Kalman filter. Stable closed-loop control will be demonstrated by use of the Tobit Kalman filter as an observer in a linear-quadratic-Gaussian regulator.

I. INTRODUCTION

Censored measurements present themselves frequently in many engineering applications. Sensor measurements can often become saturated due to dynamics or interference. Limit of detection deficiencies can reduce the useable range of many measurements and occlusion may hinder the ability of some detectors to generate accurate measurements. Often the effect of censored data leads to either engineering the system away from regions where censoring may occur, treating censored measurements as normal measurements, or rejecting censored measurements completely. One common example of this is the censoring of received signal strength (RSS), which results in a drastically reduced ability to accurately estimate vehicle position via range estimates. We will demonstrate how the Tobit Kalman filter can be used to produce a more accurate position estimate despite censoring on RSS, and how this estimate can be used as feedback in a linear-quadratic-Gaussian (LQG) regulator for position reference tracking.

Although receiving limited attention in the field of estimation and tracking, censored data regression has been extensively studied in the fields of reliability and econometrics. The Tobit Model for censored data, as given by (1), was first introduced by Economist James Tobin in 1958 for accurately modeling household expenditure [1].

\[ y_t = \begin{cases} \beta x_t + u_t, & \beta x_t + u_t > T \\ \frac{T}{T}, & \beta x_t + u_t \leq T \end{cases} \]  

Further work has resulted in many off-line methods for estimating Tobit model parameters [2]-[8]. We have previously introduced a formulation of the standard Kalman Filter which is capable of providing a recursive, unbiased state estimate from censored data [2], [3].

Outside of the Tobit Model, there has been limited previous work in attempting to use censored data for estimation. In [4] censored data and uncensored data is explicitly separated and treated differently, and the resulting formulation is an iterative maximum likelihood solution which requires knowledge of all past measurements. The computational burden and memory requirements are unbounded and a significant advantage of the Kalman filter is lost. Treating censored measurements as intermittent measurements will allow use of a Kalman filter as described in [5], [6]. The method provides a minimum state error variance given that missing measurements are not correlated to the state value and provides reasonable estimation when used with an informative state transition model. However, with censored data, the probability of censored measurements is correlated with the state value, and therefore use of [5] with censored data will result in a biased estimate. With censored measurements we have the difficulty of having a non-Gaussian measurement noise distribution near the censoring region, which violates one of the principal assumptions of the Kalman filter. In [7] an iterative Kalman filter was developed to solve for this nonlinearity. The Tobit Kalman filter will avoid this issue by interpolating a linear Kalman filter near the censoring region.

In the following sections we will briefly derive the Tobit Kalman filter, with a full derivation with proof available in [2]. The equivalence of the Tobit Kalman filter to the standard Kalman filter in the uncensored case will be shown. Effectiveness of the Tobit Kalman filter will be shown through simulation of mobile vehicle position estimation using received signal strength (RSS). Given range to multiple transmitters with known locations, a position estimate for the receiver can be generated. However, due to the non-linear and noisy nature of RSS, previous methods of estimating receiver position often artificially censor RSS measurements below a certain limit due to lack of confidence in the measurement [8]. Also, for the same reasons, many RSS indicators will only report to a defined lower threshold. We will show that despite censored RSS measurements, the Tobit Kalman filter will be able to produce effective distance to transmitter estimates, and therefore allow for more accurate position estimates with a larger range than typically seen before. By implementing the Tobit Kalman filter as an observer we will implement output feedback and control the vehicle to regions...
unattainable using the standard Kalman filter.

II. PROBLEM FORMULATION

To define the problem in the Tobit sense, consider the scalar system as given by

\[
x_k = A x_{k-1} + B u_{k-1} + G w_{k-1}
\]

\[
y_k' = C x_k + v_k
\]

\[
y_k = \begin{cases} y_k', & y_k' > T \\ T, & y_k' \leq T \end{cases}
\] (2)

Using (2), we define \( y_k \) as the censored observation and \( y_k' \) as the latent variable. The variance of the expected measured value is derived in [9] and can be written as:

\[
\text{Var}[y_k|x_k, \sigma] = \sigma^2[1 - \vartheta(\gamma)]
\] (3)

where \( \lambda(\gamma) \) is the inverse Mills ratio [10]

\[
\lambda(\gamma) = \frac{\phi(\gamma)}{[1 - \Phi(\gamma)]}
\]

\[
\gamma = \left( \frac{T - C x_k}{\sigma} \right)
\] (4)

and

\[
\vartheta(\gamma) = \lambda(\gamma)[\lambda(\gamma) - \gamma]
\] (5)

with \( \phi() \) and \( \Phi() \) being the normal p.d.f and c.d.f. respectively.

III. THE TOBIT KALMAN FILTER

As shown in detail in [2], the derivation of the Tobit Kalman filter is similar to the standard Kalman filter except measurement censoring leads to redefinitions for the measurement residual, the optimal Kalman gain, and the state estimate covariance. The notation for the hidden Markov model to be used forthright is given below with the state \( x_k \in \mathbb{R}^{n \times 1} \) being hidden, \( y_k' \in \mathbb{R}^{m \times 1} \) the latent variable, and \( y_k \in \mathbb{R}^{m \times 1} \) being the observed measurement.

\[
x_k = A x_{k-1} + B u_{k-1} + w_{k-1}
\]

\[
y_k' = C x_k + v_k
\]

\[
y_k = \begin{cases} y_k', & y_k' > T \\ T, & y_k' \leq T \end{cases}
\] (6)

Here \( A \in \mathbb{R}^{m \times n} \) is the system matrix, \( C \in \mathbb{R}^{m \times n} \) is the measurement model, and \( u_k \) and \( v_k \) are zero mean white Gaussian noise with covariance matrices \( Q \in \mathbb{R}^{n \times n} \) and \( R \in \mathbb{R}^{m \times m} \) respectively.

We introduce a Bernoulli random variable to model the occurrence of a censored measurement vs an uncensored measurement. The measurement model is now written as

\[
p_k(l) = \begin{cases} 1, & C x_k(l) + u_k(l) > T(l) \\ 0, & C x_k(l) + u_k(l) \leq T(l) \end{cases}
\] (7)

Now measurement \( l \) represents the state \( C x_k(l) + u_k(l) \) with probability \( E(p_k(l)) \). Diagonal matrix \( p_k \in \mathbb{R}^{m \times m} \) represents this Bernoulli process in matrix notation and yields

\[
y_k = p_k(C x_k + v_k) + (I_{m \times m} - p_k)T
\] (8)

Under assumptions detailed in [2], [3], the optimal Tobit Kalman filter is:

\[
\bar{x} = A \hat{x} + B u
\]

\[
\hat{\Psi} = A \hat{\Psi} A^T + Q
\]

\[
\bar{Y} = C \bar{x} + D u
\]

\[
\gamma_i = \left( \frac{\bar{Y}_i - Y_i}{\sigma_i} \right)
\]

\[
L = R_{xe}(R_{ee})^{-1}
\]

\[
\hat{x} = \bar{x} + L(Z - E(y))
\]

\[
\hat{\Psi} = (I_{N \times N} - L E(p_k C) \hat{\Psi})
\]

\[
X(k + 1) = A X(k) + B u(k) + w(k)
\]

\[
Y(k) = C X(k) + D u(k) + v(k)
\]

\[
E(w w') = Q, \ E(v v') = R
\]

\[
\bar{x} = A \hat{x} + B u
\]

\[
\hat{\Psi} = A \hat{\Psi} A^T + Q
\]

\[
L = \hat{\Psi} C^T (C \hat{\Psi} C^T + R)^{-1}
\]

\[
\hat{x} = \bar{x} + L(Z - \hat{x})
\]

\[
\hat{\Psi} = (I_{N \times N} - L C) \hat{\Psi}
\]

where \( \bar{x}, \hat{x} \) are the a priori and a posteriori state estimates, \( \hat{\Psi}, \hat{\Psi} \) are the a priori and a posteriori state covariance estimates, and

\[
E_{pk} = \text{diag} \left( \begin{array}{c} \Phi(\gamma_1) \\ \Phi(\gamma_2) \\ \vdots \end{array} \right)
\]

\[
R_{xe} = \hat{\Psi} C E_{pk}^T E_{pk}^T + V
\]

\[
R_{ee} = E_{pk} C \hat{\Psi} C^T E_{pk}^T + V
\]

\[
V = \text{diag} \left( \begin{array}{c} \sigma_1^2 (1 - IMR(1)^2 - IMR(1)^2) \\ \sigma_2^2 (1 - IMR(2)^2 - IMR(2)^2) \\ \vdots \end{array} \right)
\]

\[
\text{IMR} = \begin{pmatrix} \phi(\gamma_1) \\ \phi(\gamma_2) \\ \vdots \end{pmatrix}
\]

and \( \text{Var}[y_k(i)|x_{k-1}(i), \sigma(i)] \) is calculated according to (3).

As shown in [2], the Tobit Kalman filter will converge to the standard Kalman filter when the state value is far away from the censoring region, and it is a generalization of the standard Kalman filter.

IV. EXPERIMENT

A demonstration of the usefulness of the Tobit Kalman filter can be seen in position estimation and control using censored range-to-transmitter measurements. Estimating receiver position using range measurements is notoriously unreliable due to signal strength measurement noise creating little confidence the estimated true position. As can be seen in [8], [11], it is common for RSS measurements below a certain threshold to be ignored during localization because...
of their unreliability, or for commercial devices to simply not report an RSS below a certain level. We will show how using the Tobit Kalman filter we can allow for this thresholding to still be present while maintaining an improved range-to-transmitter estimate, and therefore an improved position estimation for a mobile vehicle.

A. Mobile Vehicle Motion Model

For simulation purposes the simplified motion model shown below was used to generate vehicle position over time. The model is a two dimensional double integrator with a circular reference input \( r \) with radius \( \zeta \) which starts at \((0, \zeta)\) with frequency \( \omega \). LQG regulation was used to implement a closed-loop controller, with performance of the standard Kalman filter and the Tobit Kalman filter as observers being compared. Position control input is defined as

\[
U_p = \bar{N} \hat{r} - K_{lqf} \hat{X}_p
\]  

(15)

, with \( \bar{N} \) being a scaling factor, \( \hat{X}_p \) being \((\hat{x}_p, \hat{y}_p)\) state estimates given by (36), and \( K_{lqf} \) being the optimal solution to the LQG control problem for the motion model outlined by (16,17). Weighting of output power vs. control effort is determined by the user. By use of the separation principal, we design the feedback gain \( K_{lqf} \) separately from that of the observer gain \( L \). We will show that when the Tobit Kalman filter is used as an observer closed-loop control remains stable even when signal strength measurements are heavily censored, allowing for stable vehicle motion into regions unreachable by use of the standard Kalman filter alone.

The vehicle motion model can now be fully described by the following :

\[
\begin{align*}
X_p(k+1) &= A_p X_p(k) + B_p U_p(k) + G_p \sigma_q \\
Y_p(k) &= C_p X_p(k) + H_p \sigma_r
\end{align*}
\]  

(16)

with

\[
\begin{bmatrix}
\dot{x}_p \\
\dot{y}_p \\
\dot{x}_v \\
\dot{y}_v
\end{bmatrix} = 
\begin{bmatrix}
1 & \Delta & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & \Delta \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_p \\
y_p \\
x_v \\
y_v
\end{bmatrix} + 
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
U_{p1} \\
U_{p2}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & \frac{1}{2} \Delta^2 & 0 & 0 \\
0 & \frac{1}{2} \Delta^2 & 0 & 0 \\
0 & 0 & \frac{1}{2} \Delta^2 & 0 \\
0 & 0 & 0 & \Delta
\end{bmatrix}
\sigma_q
\]

\[
\begin{bmatrix}
x_p \\
y_p
\end{bmatrix} = 
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
x_p \\
y_p \\
x_v \\
y_v
\end{bmatrix} + 
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\sigma_r
\]  

(17)

(18)

At each time step \( k \) the range to transmitter \( i \) is defined as

\[
d_i(k) = \sqrt{(x_p(k) - x_i)^2 + (y_p(k) - y_i)^2}
\]  

(19)

B. Radio Frequency Propagation Modeling

At true distance \( d_i \) the received signal strength (RSS) relative to transmitter \( i \) is modeled as shown in [12] as

\[
\text{RSS}_i(k) = P_{tx} - PL(d_0) - 10\alpha \log_{10}\left(\frac{d_i(k)}{d_0}\right) + \eta_{RSS}
\]  

(20)

Where \( P_{tx} \) is the transmitter power in dBm, \( \alpha \) is the path-loss exponent, and

\[
PL(d_0) = P_{tx}(1 - \text{FSPL}(d_0))
\]

is the power loss at reference distance \( d_0 \), which is modeled as the free space path loss at \( d_0 \) given by

\[
\text{FSPL}(d_0) = \left(\frac{4\pi d_0 f}{c}\right)^2
\]

Interference, multi-path errors, and fluctuations in transmitter and receiver power are accounted for by \( \eta_{RSS} \), which has been shown to be accurately modeled as white Gaussian noise independent of \( d_i \) [12].

C. Model for Tracking

In order to arrange this system into a suitable form for tracking with a Tobit Kalman filter we need to assume a linear state transition model for \( d_i \). For this, we use a simple Brownian motion model as given below because it represents the least informative model possible, and highlights the Tobit Kalman filter’s ability to provide accurate state estimation despite the lack of strong information about the true system.

\[
d_i^*(k) = \beta_i d_i^*(k-1) + \sigma_d
\]  

(21)

Letting \( \beta = 1 \), taking the \( \log_{10} \) of (21) and combining it with (20) yields a linear approximation of the system in state space form that is now suitable for Kalman filtering :

\[
\log_{10}(d_i^*(k)) = \log_{10}(d_i^*(k)|k-1) + \eta_d
\]

\[
\text{RSS}_i(k) = -10\alpha \log_{10}(d_i^*(k)) + (P_{tx} - PL(d_0)) + \eta_{RSS}
\]  

(22)

The constant term \((P_{tx} - PL(d_0)) \approx 0\), and may be omitted. In order to accurately propagate the state estimate for \( \log_{10}(d_i^*) \) between time steps a correction factor \( U_{di} \) may be incorporated. We can find a formulation for \( U_{di} \) by noticing that

\[
\begin{align*}
d_i^*(k+1) &= \sqrt{(x_p(k+1) - x_i)^2 + (y_p(k+1) - y_i)^2} \\
&\approx ((x_p(k) + \Delta x_v(k)) - x_i)^2 \\
&+ ((y_p(k) + \Delta y_v(k)) - y_i)^2 \frac{1}{2} \\
&= \beta_i d_i^*(k)
\end{align*}
\]

(23)

with

\[
\beta_i = \frac{d_i^*(k+1)}{d_i^*(k)}
\]

\[
= \sqrt{\frac{((x_p(k) + \Delta x_v(k)) - x_i)^2 + ((y_p(k) + \Delta y_v(k)) - y_i)^2}{(x_p(k) - x_i)^2 + (y_p(k) - y_i)^2}}
\]

(24)
Therefore,

\[
\log_{10}(d^*_i(k + 1)) = \log_{10}(\beta_i d^*_i(k)) + \eta_d
\]

and we can now define

\[
f_u(X_p) = \frac{1}{2} \log_{10}(\beta_i)
\]

Therefore, by taking \( U_{d_i} = f_u(X_p) = \frac{1}{2} \log_{10}(\beta_i) \) the log of distance can be more accurately transitioned between time steps. However, the simplified linear model of the system may be retained by setting \( U_{d_i} = 0 \), at the expense of reduced performance during censoring due to increased uncertainty in \( \log_{10}(d_i) \) state propagation.

This leads to the following model for tracking the distances to three transmitters:

\[
\begin{align*}
X_d(k + 1) &= A_d \tilde{X}_d(k) + B_d \tilde{U}_d(k) + G_d \eta_{\text{rss}} \\
Y_d(k) &= C_d X_d(k) + H_d \eta_{\text{rss}}
\end{align*}
\]

\[
\begin{bmatrix}
\hat{\log}_{10}(d_1) \\
\hat{\log}_{10}(d_2) \\
\hat{\log}_{10}(d_3)
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\log_{10}(d_1) \\
\log_{10}(d_2) \\
\log_{10}(d_3)
\end{bmatrix} + \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
f_{u_1}(X_p) \\
f_{u_2}(X_p) \\
f_{u_3}(X_p)
\end{bmatrix} + \begin{bmatrix}
100 \\
100 \\
100
\end{bmatrix} \eta_d
\]

\[
\begin{bmatrix}
\text{RSS}_1 \\
\text{RSS}_2 \\
\text{RSS}_3
\end{bmatrix} = \begin{bmatrix}
-10\alpha & 0 & 0 \\
0 & -10\alpha & 0 \\
0 & 0 & -10\alpha
\end{bmatrix} \begin{bmatrix}
\hat{\log}_{10}(d_1) \\
\hat{\log}_{10}(d_2) \\
\hat{\log}_{10}(d_3)
\end{bmatrix} + \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \eta_{\text{rss}}
\]

As shown in [12], \( \sigma_d \) can be modeled as

\[
\sigma_d(k) = d_i(k) \frac{\sigma_{\text{rss}}}{10\alpha}
\]

which, when substituted into (21), yields

\[
d_i(k + 1) = d_i(k) + \sigma_d(k)
\]

Taking the \( \log_{10} \) of both sides yields

\[
\log_{10}(d_i(k + 1)) = \log_{10}(d_i(k)) + \log_{10}(1 + \frac{\sigma_{\text{rss}}}{10\alpha})
\]

and therefore an approximation for \( \eta_d \) is made as

\[
\eta_d = \log_{10}(1 + \frac{\sigma_{\text{rss}}}{10\alpha})
\]

The full combined system model for tracking vehicle position, velocity, and distance to transmitters can now formed as

\[
\begin{align*}
\begin{bmatrix}
\tilde{X}_p(k + 1) \\
\tilde{Y}_p(k + 1)
\end{bmatrix} &= \begin{bmatrix}
A_p & 0 \\
0 & A_d
\end{bmatrix} \begin{bmatrix}
\tilde{X}_p(k) \\
\tilde{X}_d(k)
\end{bmatrix} + \begin{bmatrix}
B_p & 0 \\
0 & B_d
\end{bmatrix} \begin{bmatrix}
\tilde{U}_p(k) \\
\tilde{U}_d(k)
\end{bmatrix} \\
+ \begin{bmatrix}
G_p & 0 \\
0 & G_d
\end{bmatrix} \begin{bmatrix}
\sigma_q \\
\eta_{\text{rss}}
\end{bmatrix}
\end{align*}
\]

Finally, estimating range to transmitter \( i \) is accomplished by transforming the state \( \log_{10}(d_i(k)) \) by

\[
\hat{d}_i(k) = 10^{\hat{\log}_{10}(d_i(k))} = d^*_i(k)
\]

D. Estimating Position Using Range Estimates

The circle upon which the receiver lies when at position \( (x_p, y_p) \), with range \( d_i \) from transmitter \( i \) located at position \( (x_i, y_i) \) is given by

\[
d_i^2 = (x_p - x_i)^2 + (y_p - y_i)^2
\]

By using multiple transmitters with known locations, one can estimate the unknown position of the receiver by using range estimates \( \hat{d}_i \) to solve for the intersection of these overlapping circles. Expanding (35) for multiple transmitters, three for simplicity, yields a linear system as given below.

\[
R = A \tilde{X}_p
\]

\[
\begin{bmatrix}
\hat{d}_1^2 - \hat{d}_2^2 \\
\hat{d}_1^2 - \hat{d}_3^2 \\
\hat{d}_2^2 - \hat{d}_3^2
\end{bmatrix} = \begin{bmatrix}
(2x_2 - 2x_1) \\
(2x_3 - 2x_1) \\
(2x_3 - 2x_2)
\end{bmatrix} \begin{bmatrix}
(2y_2 - 2y_1) \\
(2y_3 - 2y_1) \\
(2y_3 - 2y_2)
\end{bmatrix} \begin{bmatrix}
(x_1^2 + y_1^2 - x_2^2 - y_2^2) \\
(x_1^2 + y_1^2 - x_3^2 - y_3^2) \\
(x_2^2 + y_2^2 - x_3^2 - y_3^2)
\end{bmatrix}
\]

\[
\tilde{X}_p = \begin{bmatrix}
\hat{x}_p \\
\hat{y}_p \\
1
\end{bmatrix}
\]

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Where \( \mathbf{R} \) is a vector of squared range estimate differences and \( \mathbf{A} \) is a constant matrix dependent on the transmitter locations. Solving the system for \( \mathbf{X}_p \) in a least-squares sense yields an estimate \( (\hat{x}_p, \hat{y}_p) \) of true receiver position \( (x_p, y_p) \). Matrices \( \mathbf{R} \) and \( \mathbf{A} \) can be expanded to \( N \) transmitters by following the above form.

In simulation \( \mathbf{X}_p \) is then combined with simulated measurements \( \text{RSS}_{m_i}, \) calculated according to (20), to form the censored measurement vector \( \mathbf{Z} \) used for the innovation process of (9).

\[
\mathbf{Z} = \begin{bmatrix}
\hat{x}_p \\
\hat{y}_p \\
\text{RSS}_{m_1} \\
\text{RSS}_{m_2} \\
\text{RSS}_{m_3}
\end{bmatrix}
\]

\[
\text{RSS}_{m_i} = \begin{cases}
\text{RSS}_{m_i}, & \text{RSS}_{m_i} > T \\
T, & \text{RSS}_{m_i} \leq T
\end{cases}
\]

The distance estimation states will be affected by any censoring imposed on the measured \( \text{RSS}_{m_i} \), then affecting the raw distance estimate \( d_i(k) \), which in turn degrades the localization estimate. By censoring the measured \( \text{RSS}_{m_i} \), the Tobit Kalman filter does use the information that measurements are censored to its full potential and provides an optimal, real-time, recursive state estimation given the available information.

\[K\]

E. Simulation Setup

For the following simulation it was desired to show the capabilities of the Tobit Kalman filter in a general system, not to provide an optimal representation of RSS and its use directly for position estimation and control under realistic dynamics. Therefore, for simplicity, desired vehicle motion was restricted to a circle with constant radius \( \xi = 27m \) with \( \omega \) set to traverse approximately one cycle in the given time period. The RF model has \( P_{tx} = 20 \text{ dBm}, d_0 = 1, \alpha = 6, \) and \( \eta_{\text{RSS}} = 2 \). The number of transmitters was fixed to three with \( (x_1, y_1) = (0, 10), (x_2, y_2) = (10, -10), (x_3, y_3) = (-10, -10) \), and \( \mathbf{A} \) in (36) was constant at all times according to these positions. Initial conditions were set such that velocity \( (x_v(0), y_v(0)) = 0 \), position \( (x_p(0), y_p(0)) = (0, 0) \), \( \log_{10}(d_i) \) were set to their true values, \( \sigma_q = .1 \) and \( \sigma_r = .05 \), and covariance \( \Psi = .1 \). For solution of the LQG problem for feedback gain \( K_{\text{lqr}}, \) output control vs. control effort was weighted 1000:1 in the cost minimization, meaning that higher control inputs were favored in exchange for faster response.

As the given example is only meant to be a demonstration for the effectiveness of the Tobit Kalman filter in a censored situation, non-linearities in the formulation of \( U_p \) and \( d_i \) are ignored. Both effects, while not trivial, are presented to both the standard Kalman filter and the Tobit Kalman filter equally, and therefore a direct comparison between the two remains valid. The following simulation is meant to be a novel approach to the localization problem and does not claim to be the optimal solution.

F. Results

We see in fig. 1 a 2D representation of the scenario, in which the mobile vehicle has traversed the space, attempting to follow the circular reference input using LQG regulation with a Tobit filter based observer. The locations of the three transmitters are noted by the labeled cross marks, with the range at which each transmitter’s measured RSS by the vehicle will be thresholded (in the absence of \( \eta_{\text{RSS}} \)). A standard Kalman filter observer was run in parallel for comparison, but it’s state estimate was not used in control feedback. It is readily apparent that the standard Kalman filter can only accurately estimate position when well within the -90 dBm range of all three transmitters, as dictated by (36). As the mobile vehicle approaches the thresholded region range to transmitter estimates become heavily biased due to RSS censoring and the least squares position approximation degrades. This censoring on the RSS measurements in shown in fig. 2, and it’s effect on the estimation of \( \log_{10}(d_i^*) \) for \( i = 1 \) is shown in fig. 3.

Position estimation error is shown in fig. 4 as a comparison between the presented filters. For the short period of time in which all three transmitters are uncensored we see that the filter errors converge as the Tobit Kalman filter converges to an uncensored standard Kalman filter. However, for a large majority of the time at least one transmitter is out of range and will read as a censored RSS measurement by the vehicle. The standard Kalman filter behaves poorly because its ignorance of censored measurements causes it to trust all measurements with equal weight, even when censored for long periods of time, resulting in a biased estimate. The Tobit Kalman filter is able to generate a state estimate well outside the range of the three overlapping transmitters, and thus is able to maintain a lower position estimate error at nearly all times. Although it is not explicitly shown here, attempting reliable closed-loop LQG control with a standard Kalman filter observer severely restricts the maximum radius \( \xi \) of the reference trajectory to within approximately 19m with the given parameters, a nearly 33% reduction in radius compared to that attainable by the Tobit Kalman filter. The bias introduced into the standard Kalman estimate by censoring causes the closed-loop to go unstable if the tracking radius is increased beyond this limit.

The Tobit Kalman filter cannot completely overcome the inherent lack of information that a censored measurement presents, and therefore cannot entirely account for the dynamics of the vehicle’s motion with respect to a censored transmitter. At this point the system model becomes an important factor, with more informative models yielding improved state estimates than models with an inherent lack of information. Without an informative dynamic model when measurements are censored the full system behavior is unobservable and cannot be accounted for, but the Tobit Kalman filter does use the information that measurements are censored to its full potential and provides an optimal, real-time, recursive state estimation given the available information.
V. CONCLUSIONS AND FUTURE WORKS

A. Conclusions

We have presented a representative application of a novel formulation of the Kalman filter. When the true state of the system is near the censoring region the Tobit Kalman filter provides an unbiased state estimate that far outperforms the standard Kalman filter. The effectiveness of the Tobit Kalman filter and a potential application for its use is demonstrated by estimation and control of vehicle position using censored received signal strength as a range to transmitter estimate. We see that the Tobit Kalman filter is able to estimate vehicle position with lower mean-squared error despite censored measurements and high noise. We have shown that in cases where the censoring model is well understood there is a significant amount of information present in what cannot be observed, which is information that can be exploited by the Tobit Kalman filter and has previously largely been ignored.

B. Future Works

In future work we will expand the Tobit Kalman filter’s formulation for non-linear systems and explore its applications to a more diverse set of systems. Coupling of the Tobit Kalman filter with more advanced position estimation techniques will enable even further performance increases over the straightforward method presented. Furthermore,
the 2D case will explored, especially with the presence of occlusion, and its application towards computer vision based algorithms. The integration of the Tobit Kalman filter with optimal control techniques will be studied in order maximize the information that censored data presents. It is believed that new techniques can be defined to enable optimal control strategies for improved battery life, increased sensor dynamic range, minimal actuation, and time varying censor limits with cost.

VI. ACKNOWLEDGMENTS
This work is supported by the Army Research Office under grant number W911NF-10-1-0386.

VII. REFERENCES