Soft-Constrained Stochastic Nash Games for Multimodeling Systems via Static Output Feedback Strategy

Hiroaki Mukaidani, Hua Xu and Vasile Dragan

Abstract—In this paper, we discuss infinite-horizon soft-constrained stochastic Nash games involving state-dependent noise in multimodeling systems (SMS) via static output feedback strategy. First, we derive the conditions for the existence of robust equilibria that have been derived based on the solutions of sets of cross-coupled stochastic algebraic Riccati equations (CSAREs). After establishing an asymptotic structure along with positive definiteness for cross-coupled stochastic multimodeling algebraic Riccati equations (CSMAREs) solutions, we propose a parameter independent strategy. The numerical example for a multimachine power system is given to demonstrate the efficiency and feasibility of the proposed algorithm.

I. INTRODUCTION

Robust equilibria in indefinite linear quadratic differential games under a deterministic disturbance input affecting the systems have been discussed [10], [11]. These results are based on the steady-state feedback saddle-point solution for soft-constrained Nash games. Meanwhile, the theoretical and numerical aspects by extending the previous results of [10], [11] in the deterministic case to the soft-constrained stochastic Nash games governed by Ito's differential equations with state-dependent noise has been investigated in [17]. Although the results in [10], [11], [17] are very elegant in theory and despite it being easy to obtain a strategy pair by solving cross-coupled stochastic algebraic Riccati equations (CSAREs), static output feedback strategy is an issue that remains to be considered. From a practical point of view, static output feedback control is extremely attractive since states are not always available for feedback.

The problem of designing a feedback strategy for a multimodeling system has been the subject of many papers during the past three decades (see e.g., [6], [7], [8], [9]). In order to solve these important problems, the time-scale decomposition method can be used. This method allows us to avoid high dimensionality and ill conditioning in computation. Furthermore, even if the singular perturbation parameter is partially unknown, such method can be used.

H. Mukaidani is with Graduate School of Education, Hiroshima University, 1-1-1 Kagamiyama, Higashi-Hiroshima, 739-8524 Japan. mukaidai@hiroshima-u.ac.jp
H. Xu is with Graduate School of Business Sciences, The University of Tsukuba, 3-29-1, Otsuka, Bunkyo-ku, Tokyo, 112-0012 Japan. xu@mbaib.gsbs.tsukuba.ac.jp
V. Dragan is with Institute of Mathematics of the Romanian Academy, 1-764, Ro-70700, Romania. Vasile.Dragan@imar.ro

This research was supported by FY 2007 Researcher Exchange Program between Japan Society for the Promotion of Science (JSPS) and Natural Sciences and Engineering Research Council of Canada (NSERC), the Electric Technology Research Foundation of Chugoku of Japan and the Grant-in-Aid for Scientific Research (C)-20500014 from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

Recent advance in theory of the descriptor systems approach has allowed a revisiting of the Nash games for the multiparameter singularly perturbed systems [16]. However, the issue of robustness against stochastic uncertainty and deterministic disturbance has not been considered. On the other hand, although infinite-horizon soft-constrained stochastic Nash games involving state-dependent noise in weakly coupled large-scale systems have been tackled, the static output feedback strategy has not been discussed in [17].

In this paper, the linear quadratic infinite-horizon soft-constrained Nash games for stochastic multimodeling systems (SMS) via static output feedback strategy are investigated. Although the output feedback Nash equilibrium for the infinite-horizon LQ differential games has been tackled in [12], the soft-constrained stochastic Nash games under the output measurements have not been treated. Moreover, the nature of stochastic uncertainty considered here is quite different from that of the existing results [9]. Indeed, we consider the SMS that is governed by Ito’s differential equation with deterministic disturbance, while the authors of [9] have investigated the LQG Nash game with white Gaussian noise. The contributions of this paper are as follows. First, the problem of the soft-constrained stochastic Nash games by means of static output feedback strategy is defined. Second, the soft-constrained problem for the one-player case is given as the saddle-point solution. Moreover, in order to guarantee the existence of strategy sets, sets of CSAREs are introduced for the first time. After introducing the SMS, the asymptotic structure of the solutions to the cross-coupled stochastic multimodeling algebraic Riccati equations (CSMAREs) is derived. Using this structure, a parameter independent stochastic Nash strategy set is established. Furthermore, the degradation of the cost by means of the proposed strategy is also investigated. Finally, in order to demonstrate the efficiency of the proposed algorithm, a numerical example for a multimachine power system is solved.

Notation: The notations used in this paper are fairly standard. $I_n$ denotes the $n \times n$ identity matrix. $\| \cdot \|$ denotes its Euclidean norm for a matrix. block diag denotes the block diagonal matrix. vec$M$ denotes the column vector of the matrix $M$. $\otimes$ denotes the Kronecker product. $\oplus$ denotes the Kronecker sum such that $M \oplus N := M \otimes I_n + I_m \otimes N$. $M \in \mathbb{R}^{m \times m}$, $N \in \mathbb{R}^{n \times n}$. $E[\cdot]$ denotes the expectation operator. The space of the $\mathbb{R}^k$-valued functions that are quadratically integrable on $(0, \infty)$ are denoted by $L^2_{\mathbb{R}}(0, \infty)$. Finally, throughout this paper we have used the notation $\|x(t)\|^2_R$ instead of $x^T(t)Rx(t)$ for a real positive semidefinite symmetric matrix $R$ and vector $x(t)$. 
II. PRELIMINARY RESULT

Let us consider the following SMS that consist of $N$-fast subsystems with specific structure of lower level interconnected through the dynamics of a higher level slow subsystem.

$$dx_0(t) = \left[ \sum_{j=0}^{N} A_{0j}x_j(t) + \sum_{j=1}^{N} [B_{0j}u_j(t) + E_{0j}v(t)] \right] dt$$

$$+ \sum_{p=1}^{M} A_{p00}x_0(t) + \mu \sum_{j=1}^{N} A_{pj0}x_j(t) \right] dw_p(t),$$

$$\varepsilon_i dx_i(t) = [A_{0i}x_0(t) + A_{ii}x_i(t) + B_{ii}u_i(t) + E_{ii}v(t)] dt$$

$$+ \varepsilon^\delta \sum_{p=1}^{M} [A_{pi0}x_0(t) + A_{pii}x_i(t)] dw_p(t),$$

$$x_i(0) = x_0^0, \quad y_i(0) = 0, \quad y_i(t) = C_{i0}x_0(t) + C_{ii}x_i(t), \quad i = 1, \ldots, N,$$

where $x_i(t) \in \mathbb{R}^{n_i}$, $i = 1, \ldots, N$ represent the $i$-th state vectors. $u_i(t) \in \mathbb{R}^{m_i}$, $i = 1, \ldots, N$ represent the $i$-th control inputs. $y_i(t) \in \mathbb{R}^{e_i}$, $i = 1, \ldots, N$ represent the $i$-th output measurement vectors. $v(t) \in \mathbb{R}^{v}$ represents the disturbance. $w_p(t) \in \mathbb{R}$, $p = 1, \ldots, M$ is a one-dimensional standard Wiener process defined in the filtered probability space [4], [13], [14]. Note that one of the fast state matrices $A_{ii}$, $i = 1, \ldots, N$ may be singular. $\varepsilon_i > 0, \varepsilon_i = 1, \ldots, N$ and $\mu$ are small parameters and $\delta > 1/2$ is independent of $\varepsilon := \min\{\varepsilon_1, \ldots, \varepsilon_N\}$ [13], [14]. It should be noted that the parameters $\mu$ and $\delta$ have been introduced in [13], [14] for the first time. Moreover, $\delta = 1/2$ was demonstrated in [9].

It is assumed that the ratios of the small positive parameters $\varepsilon_i, i = 1, \ldots, N$ and $\mu$ are bounded by some positive constants $k_{ij}, k_{ij}, l_i$ and $l$ and only these bounds are assumed to be known [6], [7]. In other words, they have the same order of magnitude.

$$0 < k_{ij} \leq \alpha_{ij} \equiv \frac{\varepsilon_j}{\varepsilon_i} \leq k_{ij} < \infty, \quad 0 \leq l \leq \frac{\mu}{\varepsilon_i} \leq l < \infty.$$ (2)

Note that one of the fast state matrices $A_{ii}$, $j = 1, \ldots, N$ may be singular. The performance criterion is given by

$$J_i(u_1, \ldots, u_N, v, x(0))$$

$$= E\left[\int_0^\infty [y_i^T(t)Q_{ii}y_i(t) + u_i^T(t)R_iu_i(t) - v^T(t)V_{ii}v(t)] dt, \right.$$ (3)

for all $i = 1, \ldots, N$, $\bar{n} := \sum_{j=0}^{N} n_{jj}$, $x(t) := \left[ x_0^T(t) \quad x_1^T(t) \quad \ldots \quad x_N^T(t) \right]^T \in \mathbb{R}^{\bar{n}}$, $Q_i = \left[ \begin{array}{c} Q_{00} \clubdot \clubdot \clubdot \quad Q_{0i} \\ Q_{i0} \quad Q_{ii} \clubdot \clubdot \clubdot \quad \end{array} \right], \quad Q_{0i} := \left[ 0 \cdots 0 \quad Q_{0i} \quad 0 \cdots 0 \right], \quad Q_{ii} := \text{block diag} \left( 0 \quad 0 \quad Q_{iii} \quad 0 \quad \ldots \quad 0 \right), \quad R_i := R_i > 0, \quad V_{ii} := \text{block diag} \left( \mu^{-(1-\delta_1)}V_{i1} \quad \mu^{-(1-\delta_2)}V_{i2} \quad \ldots \quad \mu^{-(1-\delta_N)}V_{iN} \right), \quad V_{ii} = V_{ii}^T > 0.$$

It should be noted that we do not assume the positive semidefiniteness of $Q_i$.

These conditions are quite natural since at least one control agent has to be able to control and observe unstable modes. Our purpose is to find a linear feedback strategy set $(u_1^*, \ldots, u_N^*)$ such that

$$J_i(u_1^*, \ldots, u_N^*) \leq \bar{J}_i(u_1, \ldots, u_{i-1}^*, u_i, u_{i+1}^*, \ldots, u_N^*),$$

where

$$\bar{J}_i(u_1, \ldots, u_N) := \sup_{v \in L_2^2(0, \infty)} J_i(F_{i}e^{y_1}, \ldots, F_{Ne^{y_N}}, v, x(0)), \quad \bar{l} := \sum_{i=1}^{N} l_i.$$ (4)

The decision makers are required to select the closed loop strategy $u_i^*(t)$, if they exist, such that (4) holds. Moreover, each player uses the strategy $u_i^*(t)$ such that the closed-loop system is asymptotically mean square stable (AMSS) 1 for sufficiently small $\varepsilon_i$.

It is noteworthy that in this study, the strategies $u_i^*$ are restricted as output feedback strategies such as

$$u_i(t) := F_{ie}e^{y_i}(t) = F_{ie}C_i x(t),$$

where $C_i := \left[ C_{i0} \quad 0 \cdots 0 \quad C_{ii} \quad 0 \cdots 0 \right].$

Let $F_{Ne}$ denote the set of all $(F_{1e}, \ldots, F_{Ne})$ such that the following closed-loop stochastic system is AMSS.

$$dx(t) = \left[ A_e + \sum_{j=1}^{N} B_{je}F_{je}C_i \right] x(t) dt + \sum_{p=1}^{M} A_{pe}x(t) dw_p(t),$$ (6)

where $A_e := \Phi^{-1}_e A, \quad A_{pe} := \Phi^{-1}_e A_p, \quad B_{ie} := \Phi^{-1}_e B_i$ with

$$\Phi_e := \text{block diag} \left( I_{n_e}, \Pi_e \right), \quad \Pi_e := \text{block diag} \left( \varepsilon_1I_{n_1}, \ldots, \varepsilon_NI_{n_N} \right), \quad A := \left[ \begin{array}{c} A_{00} \quad A_{0f} \\ A_{f0} \quad A_f \end{array} \right], \quad A_p := \left[ \begin{array}{c} A_{00} \quad \mu A_{p0f} \\ \varepsilon^\delta A_{p0f} \quad \varepsilon^\delta A_{pf} \end{array} \right], \quad A_{0f} := \left[ A_{01} \cdots A_{0N} \right], \quad A_{f0} := [A^T_{T_0} \cdots A^T_{TN}]^T, \quad A_f := \text{block diag} \left( A_{11} \cdots A_{NN} \right), \quad A_{p0} := \left[ A_{p01} \cdots A_{p0N} \right], \quad A_{p0f} := \left[ A^T_{T_0} \cdots A^T_{TN} \right]^T, \quad A_{pf} := \text{block diag} \left( A_{p11} \cdots A_{pNN} \right), \quad B_1 := \left[ B^T_{T_1} \quad B^T_{T_2} \quad 0 \cdots 0 \right]^T, \quad B_2 := \left[ B^T_{T_0} \cdots B^T_{Ti} \cdots 0 \right]^T, \quad B_N := \left[ B^T_{TN} \quad 0 \cdots B^T_{NN} \right]^T.$$

First, a one-player case is discussed. The result obtained for that particular case is used as the basis for the derivation of results for the general $N$-player case.

1 For definition of stability we refer to [1].
Consider a linear time-invariant stochastic stabilizable system
\[
dx(t) = [A_e x(t) + B_{1e} u_1(t) + E_e v(t)] dt + \sum_{p=1}^{M} A_{pe} x(t) dw_p(t), \quad x(0) = x^0, \tag{7}
\]
where \(u_1(t, x) := F_{1e} C_1 x(t), F_{1e} \in \mathcal{F}_1\) and
\[
E_e := \Phi_{e}^{-1} E_e, E_e := \begin{bmatrix} E_{0d} & E_T \end{bmatrix}^T, \\
E_{0f} := \begin{bmatrix} E_{01} & \cdots & E_{0N} \end{bmatrix}, \\
E_f := \text{block diag} \left\{ E_{11}, \ldots, E_{NN} \right\}.
\]
The cost function is given below.
\[
J(u_1, v, x(0)) = E \int_0^\infty \left( \|y(t)\|^2_{Q_1} + \|u(t)\|^2_{R_{11}} - \|v(t)\|^2_{V_{11}} \right) dt. \tag{8}
\]
Let us define the strategy spaces \(\Gamma_u := \{u_1(t, y_1) := F_{1e} y_1(t) \mid F_{1e} \in \mathcal{F}_1\}\) and \(\Gamma_v := \{v(t) \mid v(t) \in L^1(0, \infty)\}\).

**Definition 1:** A strategy pair \((u_1^*, v^*) \in \Gamma_u \times \Gamma_v\) is in saddle-point equilibrium if
\[
J(u_1^*, v^*, x(0)) \leq J(u_1, v^*, x(0)) \leq J(u_1^*, v, x(0)) \tag{9}
\]
for all \((u_1, v) \in \Gamma_u \times \Gamma_v\) and \((u_1, v^*) \in \Gamma_u \times \Gamma_v\).

The following theorem generalizes the existing results of [17] these results are very important in deterministic soft-constrained Nash games for a stochastic case.

**Theorem 1:** Assume that for all \((u_1, v) \in \Gamma_u \times \Gamma_v\) the closed-loop system is AMSS. Suppose the following stochastic algebraic Riccati equation (SARE)(10) has a solution \(P_e \geq 0\) and \(F_{ie}^*\) such that \(F_{ie}^* C_1 = -R_{11}^{-1} B_{ie}^T P_e\).

\[
P_e A_e + A_e^T P_e + \sum_{p=1}^{M} A_{pe}^T P_e A_{pe} - \mu E_{1e} U_{1e} P_e + \tilde{Q}_1 = 0, \tag{10}
\]
where \(U_{1e} := S_{ie} - M_{ie}, S_{ie} := B_{ie} R_{11}^{-1} B_{ie}^T, M_{ie} := E_c V_{1e}^{-1} E_{1e}^T, \tilde{Q}_1 := C_1^T Q_1 C_1\). Furthermore, suppose there exists a real symmetric matrix \(W_e\) that satisfies matrix inequality (11).

\[
W_e A_e + A_e^T W_e + \sum_{p=1}^{M} A_{pe}^T W_e A_{pe} - W_e S_{ie} W_e + \tilde{Q}_1 \geq 0. \tag{11}
\]

The strategy pair
\[
u_1^*(t, y_1) = F_{ie}^* y_1(t), \tag{12a}
\]
\[
v^*(t) = V_{1e}^{-1} E_{1e}^T P_e \tilde{x}(t), \tag{12b}
\]
\[
d\tilde{x}(t) = \left[ A_e - U_{1e} P_e \right] \tilde{x}(t) dt + \sum_{p=1}^{M} A_{pe} \tilde{x}(t) dw_p(t), \tag{12c}
\]
\(\tilde{x}(0) = x^0\) is in saddle-point equilibrium. That is, if these conditions hold then inequality (11) related to cost function \(J(u_1, v, x(0))\) is satisfied. Moreover, \(J(u_1^*, v^*, x(0)) = x^T(0) P_e x(0)\).

**Proof:** Since this theorem can be proved by tracing the technique that has been established in [17], it is omitted.

It should be noted that the assumptions of Theorem 1 results in a sufficient condition. Furthermore, although \(v^*(t)\) means worst case input-disturbance, a general disturbance model is unknown for the practical systems.

The soft-constrained stochastic Nash games via static output feedback strategy are given below.

**Theorem 2:** Assume that for all \(u_i(t), i = 1, \ldots, N\) and \(v(t)\), the closed-loop system is AMSS. Suppose that \(N\) real symmetric matrices \(P_{ie} \geq 0\) and \(N\) real symmetric matrices \(W_{ie}\) exist such that

\[
P_{ie} A_e + A_e^T P_{ie} + \sum_{p=1}^{M} A_{pe}^T P_{ie} A_{pe} - P_{ie} S_{ie} P_{ie} \\
+ P_{ie} M_{ie} P_{ie} + Q_i = 0, \tag{13a}
\]
\[
W_{ie} A_e^T + \sum_{p=1}^{M} A_{pe}^T W_{ie} A_{pe} - W_{ie} S_{ie} W_{ie} \\
+ Q_i \geq 0, \tag{13b}
\]
where \(i = 1, \ldots, N\), \(A_e := A_e - \sum_{j=1, j \neq i}^{N} S_{je} P_{je}, S_{ie} := B_{ie} R_{11}^{-1} B_{ie}^T, M_{ie} := E_c V_{1e}^{-1} E_{1e}^T\), Define the set \((F_{ie}^*, \ldots, F_{Ne}^*)\) by

\[
u_i^*(t) = F_{ie}^* y_1(t), \tag{14a}
\]
\[
F_{ie}^* C_1 = -R_{11}^{-1} B_{ie}^T P_{ie}, \quad i = 1, \ldots, N. \tag{14b}
\]

Then, \((F_{ie}^*, \ldots, F_{Ne}^*) \in \mathcal{F}_N\), and this strategy set denotes the soft-constrained stochastic Nash equilibrium. Furthermore, \(J_i(F_{ie}^* x, \ldots, F_{Ne}^* x, x(0)) = x^T(0) P_{ie} x(0)\).

**Proof:** Now, let us consider the following problem in which the cost function (15) is minimal at \(F_{ie} = F_{ie}^*\).

\[
\phi(F_{ie}) := \sup_{v \in L^2_{u}(0, \infty)} E \int_0^\infty \left( \|x(t)\|^2_{T_2} - \|v(t)\|^2_{V_{1e}} \right) dt, \tag{15}
\]
where \(T_2 := \tilde{Q}_1 + C_1^T F_{ie}^* R_{ii} F_{ie}^* C_1^T\) and \(x(t)\) follows from

\[
dx(t) = \left[ A_e - \sum_{j=1, j \neq i}^{N} S_{je} P_{je} + B_{ie} F_{ie}^* C_1 \right] x(t) \\
+ E_c v(t) dt + \sum_{p=1}^{M} A_{pe}(x(t) dw_p(t), \quad x(0) = x^0. \tag{16}
\]

Note that the function \(\phi\) coincides with function \(J\) in Theorem 1. Applying Theorem 1 to this minimization problem as \(P_{ie} \rightarrow P_e, A_e \rightarrow \sum_{j=1, j \neq i}^{N} S_{je} P_{je} \rightarrow A_e, B_{ie} \rightarrow B_{ie}, \tilde{Q}_i \rightarrow \tilde{Q}_1\) and \(R_{ii} \rightarrow R_{11}, V_{1e} \rightarrow V_{1e}\) yields the fact that the function \(\phi\) is minimal at

\[
F_{ie}^* C_1 = -R_{11}^{-1} B_{ie}^T P_{ie} \Rightarrow F_{ie}^* C_1 = -R_{11}^{-1} B_{ie}^T P_{ie}. \tag{17}
\]

Moreover, the minimal value is \(x^T(0) P_{ie} x(0)\).

It should be noted that if \(Q_i \geq 0\) for all \(i = 1, \ldots, N\), matrix inequality (13b) is trivially satisfied by \(W_{ie} = 0\) [17]. In the case that one of these matrices is not positive semidefinite, one might consider \(W_{ie} = -P_{ie}\) as a solution to (13b).
\[
\det \begin{bmatrix}
\hat{A}_s^T \oplus \hat{A}_s^T + \sum_{p=1}^{M} A_{s00}^T \otimes A_{s00}^T & -(L_2 \bar{P}_{100}) \oplus (L_2 \bar{P}_{100}) & \cdots & -(L_N \bar{P}_{100}) \oplus (L_N \bar{P}_{100}) \\
-(L_1 \bar{P}_{200}) \oplus (L_1 \bar{P}_{200}) & \hat{A}_s^T \oplus \hat{A}_s^T + \sum_{p=1}^{M} A_{s00}^T \otimes A_{s00}^T & \cdots & -(L_N \bar{P}_{200}) \oplus (L_N \bar{P}_{200}) \\
\vdots & \vdots & \ddots & \vdots \\
-(L_1 \bar{P}_{N00}) \oplus (L_1 \bar{P}_{N00}) & -(L_2 \bar{P}_{N00}) \oplus (L_2 \bar{P}_{N00}) & \cdots & \hat{A}_s^T \oplus \hat{A}_s^T + \sum_{p=1}^{M} A_{s00}^T \otimes A_{s00}^T 
\end{bmatrix} \neq 0, \quad (20)
\]

where \( L_i := U_{s_i} + M_{00} \).

### III. ASYMPTOTIC STRUCTURE

In order to obtain the approximate Nash strategies for the CSMAREs (13a), asymptotic structure is investigated.

First, we suppose that the following structure of solution (13a) is satisfied [16].

\[
P_i := \begin{bmatrix} P_{i00} & P_{i01} \Pi_e & P_{i02} & \cdots & P_{i0M} \\ P_{i10} & P_{i11} & \Pi_e & \cdots & P_{i1M} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{iN0} & P_{i(N-1)1} & \Pi_e & \cdots & P_{i(N-1)M} \\ P_{i(N+1)0} & P_{i(N+1)1} & \Pi_e & \cdots & P_{iN0} \end{bmatrix}, \quad P_{000} := P_{000}^T,
\]

\[
P_{0f0} := \begin{bmatrix} P_{0f0}^T \\ P_{0f1}^T \\ \vdots \\ P_{0fM}^T \end{bmatrix}, \quad \Pi_e P_{0f} := P_{0f}^T \Pi_e,
\]

\[
P_{0f} := \begin{bmatrix} P_{0f0} & P_{0f1} \Pi_e & P_{0f2} & \cdots & P_{0fM} \\ P_{0f1} & \Pi_e & \cdots & \Pi_e & P_{0fM} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \Pi_e & \cdots & P_{0fM} \\ \vdots & \vdots & \cdots & \vdots & \Pi_e & \cdots & P_{0fM} \\ P_{0fM} & \Pi_e & \cdots & \Pi_e & \cdots & \Pi_e \end{bmatrix}.
\]

The following zeroth-order equations of the CSMAREs (13a) are given as \( \nu := \sqrt{e_1^2 + e_2^2 + \cdots + e_{2N}^2 + \mu^2} \to 0^+ \).

\[
\hat{P}_{100} \hat{A}_s + \hat{A}_s^T \hat{P}_{000} + \sum_{p=1}^{M} A_{s00}^T \hat{P}_{000} A_{s00} + \hat{P}_{000} U_{s_i} \hat{P}_{000} + Q_{s_i} = 0, \quad (18a)
\]

\[
A_{s00}^T P_{i0j} + P_{i0j} A_{s00} - P_{i0j} U_{s_i} P_{i0j} + C_{s00}^T Q_{s00} C_{s00} = 0, \quad (18b)
\]

\[
P_{i0k} = 0, \quad k > i, \quad \hat{P}_{ij} = 0, \quad i \neq j, \quad (18c)
\]

\[
\begin{bmatrix} P_{110} & P_{120} & \cdots & P_{1N0} \\ P_{101} & P_{102} & \cdots & P_{10N} \\ \vdots & \vdots & \ddots & \vdots \\ P_{N10} & P_{N20} & \cdots & P_{NN0} \end{bmatrix}
\]

\[
\begin{bmatrix} \hat{P}_{111} & -I_{N-1} & \cdots & -I_{N-1} \\ \hat{P}_{120} & \hat{P}_{220} & \cdots & \hat{P}_{2N0} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{P}_{N10} & \hat{P}_{N20} & \cdots & \hat{P}_{NN0} \end{bmatrix}
\]

\[
\begin{bmatrix} \hat{P}_{111} & -I_{N-1} & \cdots & -I_{N-1} \\ \hat{P}_{120} & \hat{P}_{220} & \cdots & \hat{P}_{2N0} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{P}_{N10} & \hat{P}_{N20} & \cdots & \hat{P}_{NN0} \end{bmatrix}
\]

where \( \hat{A}_s \) is a stable matrix with \( \hat{A}_s := A_s - \sum_{j=1}^{N} U_{s_j} \bar{P}_{j00} - \sum_{j=1, j \neq i}^{N} M_{00} \bar{P}_{j00}, \quad M_{00} := E_{00} V_{ii}^{-1} E_{00}, \quad U_{i00} := B_{0i} R_{i}^{-1} B_{i}^T - E_{0i} V_{ii}^{-1} E_{0i}, \quad U_{i0i} := B_{0i} R_{i}^{-1} B_{i}^T - E_{0i} V_{ii}^{-1} E_{0i}, \)

\[
E_{0i} V_{ii}^{-1} E_{0i}, \quad U_{i0i} := B_{0i} R_{i}^{-1} E_{0i} - E_{0i} V_{ii}^{-1} E_{0i}, \quad U_{i0i} := B_{0i} R_{i}^{-1} B_{i}^T - E_{0i} V_{ii}^{-1} E_{0i},
\]

Before establishing the asymptotic structure of the reduced-order solution, we introduce the following assumption.

**Assumption 1:** The CSAREs (18a) has solution \( \bar{P}_{000} = 1, \ldots, N. \) This means that the solution \( x_0(t) \) is of the closed-loop stochastic system

\[
dx_0(t) = \left[ A_s - \sum_{j=1}^{N} U_{s_j} \bar{P}_{j00} \right] x_0(t) dt + \sum_{p=1}^{M} \bar{A}_{p0} x_0(t) dw_p(t) \quad (19)
\]

is AMSS.

It may be noted that the stochastic stabilizability is necessary condition for the existence of the stabilizing solution of CSAREs.

The following theorem shows the relation between the solutions \( \bar{P}_i \) and the zeroth-order solutions \( \bar{P}_{i0k}, i = 1, \ldots, N, \) \( k \geq l, \quad l \leq N. \)

**Theorem 3:** Under Assumption 1, suppose that the condition (20) that is given at the top of this page holds. Then, the CSMAREs (13a) possess the power series expansion at \( \| \nu \| = 0. \) That is, the following form is satisfied.

\[
P_i = \bar{P}_{i0} + O(\| \nu \|)
\]

\[
\begin{bmatrix} \bar{P}_{i00} & 0 & 0 & \cdots & 0 \\ \bar{P}_{i10} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{P}_{i(N-1)0} & 0 & 0 & \cdots & 0 \\ \bar{P}_{i(N+1)0} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{P}_{iN0} & 0 & 0 & \cdots & 0 \end{bmatrix}
\]

\[
\begin{bmatrix} \bar{P}_{i00} & 0 & 0 & \cdots & 0 \\ \bar{P}_{i10} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{P}_{i(N-1)0} & 0 & 0 & \cdots & 0 \\ \bar{P}_{i(N+1)0} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{P}_{iN0} & 0 & 0 & \cdots & 0 \end{bmatrix}
\]

Proof: First, zeroth-order solutions for the asymptotic structure of CSMAREs (13a) are established. Under As-
Sumption 1, the following equality holds.
\[
A_{ii} - U_{iii} - \bar{Q}_{iii}^{-1} - AT_{ii} = \begin{bmatrix} I_{n_i} & 0 \\ P_{ii} & I_{n_i} \end{bmatrix} \begin{bmatrix} A_{ii} - U_{ii} \\ 0 \\ -P_{ii} & I_{n_i} \end{bmatrix} \begin{bmatrix} I_{n_i} & 0 \\ 0 & -A_{ii}^{T} \end{bmatrix},
\]
(22)
where \(\hat{A}_{ii} := A_{ii} - U_{ii}P_{ii}\). Since \(T_{ii} \) is nonsingular, \(\hat{A}_{ii}\) is also nonsingular. This means that \(T_{ii}^{+}\) can be expressed explicitly in terms of \(A_{ii}^{+}\). Therefore, using the above result, the formulations (18) are obtained. These transformations can be done by the lengthy, but direct algebraic manipulated assumptions [16], which are omitted here.

It is noteworthy that the local uniqueness is newly shown compared with the existing results [6], [7], [16]. Moreover, it may be noted that the formulas under the equation (18) have been used to simplify the expressions for the first time to the stochastic case.

IV. PARAMETER INDEPENDENT
SOFT-CONSTRAINED NASH STRATEGY

Using the result (21), the N-order approximate soft-constrained stochastic Nash strategy is given.
\[
\bar{u}_{i}(t) := \bar{F}_{i}\bar{y}_{i}(t), \quad \bar{F}_{i}C_{i} = -R_{ii}^{-1}B_{i}^{T}\bar{P}_{i}, \quad i = 1, \ldots, N,
\]
(23)
Before introducing the theorem, the following assumption is imposed [4].

Assumption 2: Define \(\bar{S}_{ie} := B_{ie}R_{ii}^{-1}B_{i}^{T}. \) There exists a small \(\bar{\sigma}\) such that for all \(|\nu| \in (0, \bar{\sigma})\) the following facts hold.
i) \(A_{e} - \sum_{j=1}^{N} \bar{S}_{je}\bar{P}_{j}, A_{1e}, \ldots, A_{Me} \mid \bar{Q}_{i} \) is exactly observable.
i) \(A_{e} - \sum_{j=1}^{N} \bar{S}_{je}\bar{P}_{j}, A_{1e}, \ldots, A_{Me} \) is stable.

Theorem 4: Under Assumptions 1 and 2, the use of the approximate stochastic Nash strategy (23) results in \(\bar{J}_{i}(\bar{u}_{1}, \ldots, \bar{u}_{N})\) satisfying
\[
\bar{J}_{i}(\bar{u}_{1}, \ldots, \bar{u}_{N}) = \bar{J}_{i}(u_{1}^{*}, \ldots, u_{N}^{*}) + O(|\nu|),
\]
(24)
where \(\bar{J}_{i}(u_{1}^{*}, \ldots, u_{N}^{*})\) are the optimal equilibrium values of the cost functions (3).

Proof: When \(\bar{u}_{i}(t) := \bar{F}_{i}\bar{y}_{i}(t)\) is applied, by using the existing result of [17], the equilibrium value of the cost performances are
\[
\bar{J}_{i}(\bar{u}_{1}, \ldots, \bar{u}_{N}) = x^{T}(0)X_{ie}x(0).
\]
(25)
where \(X_{ie}\) is the positive semidefinite solution of the following multmodeling stochastic algebraic Lyapunov equation (MSALE)
\[
X_{ie} = \left( A_{e} - \sum_{j=1}^{N} \bar{S}_{je}\bar{P}_{j} \right) + \left( A_{e} - \sum_{j=1}^{N} \bar{S}_{je}\bar{P}_{j} \right)^{T} X_{ie} + \sum_{p=1}^{M} A_{pe}^{T}X_{ie}A_{pe} + \bar{Q}_{i} + \bar{P}_{i}\bar{S}_{i}\bar{P}_{i}, X_{ie}M_{ie}X_{ie} = 0,
\]
(26)
where \(\bar{S}_{i} := B_{i}R_{ii}^{-1}B_{i}^{T} \).

Subtracting (13a) from (26) and using the relation \(\bar{P}_{ie} = P_{ie} = O(|\nu|)\), it is easy to verify that \(V_{ie} = X_{ie} - P_{ie}\) satisfies the following MSALE.
\[
V_{ie} = \left( A_{e} - \sum_{j=1}^{N} \bar{S}_{je}\bar{P}_{j} \right) + \left( A_{e} - \sum_{j=1}^{N} \bar{S}_{je}\bar{P}_{j} \right)^{T} V_{ie} + \sum_{p=1}^{M} A_{pe}^{T}V_{ie}A_{pe} = O(|\nu|).
\]
(27)
Thus, under Assumption 2, it is easy to verify that \(V_{ie} = O(|\nu|)\) because \(A_{e} - \sum_{j=1}^{N} \bar{S}_{je}\bar{P}_{j} \) is stable by using the standard stochastic Lyapunov theorem [4] for sufficiently small \(|\nu|\). Consequently, the equality (21) holds.

Although \(\varepsilon_{1}\) is unknown, it is possible to design the approximate stochastic Nash strategy which achieves the \(O(|\nu|)\) approximation for the equilibrium value of the cost functional.

V. NUMERICAL EXAMPLE FOR
MEGAWATT-FREQUENCY CONTROL

In order to demonstrate the efficiency of the soft-constrained stochastic Nash games for SMS, we present results for the practical multimachine power systems. The state variable model of the megawatt-frequency control problem was developed in [15]. In this example, since all states are available, the state feedback strategy can be implemented as a special case.

The system matrices are given by the top of the next page. It is assumed that time constant of the governors represent the small singular perturbations. Hence, small parameters are \(\varepsilon_{1} = 0.030\) and \(\varepsilon_{2} = 0.029\). Moreover, it should be noted that \(\mu = 0\).

It should be noted that the deterministic disturbance \(v(t) := [0.1, 0.1]^{T}\) and the state dependent noise related to the load frequency constant [15] are both considered compared with the existing results [16], [17]. We suppose that the error of the load frequency constant is within 5% of the nominal value. Therefore, the proposed design method is very useful because the resulting strategy can be implemented to more practical SMS.

Now, let us evaluate the costs using the parameter independent approximate soft-constrained stochastic Nash strategies (23). The values of the optimal cost performance and the proposed strategies (23) for various \(\varepsilon_{i}\), \(i = 1, 2\) are given in Table 1, where \(\phi_{i} := (\bar{J}_{i}(\bar{u}_{1}^{*}, \bar{u}_{2}^{*}) - J_{i}(u_{1}^{*}, u_{2}^{*})) / |\nu|\). It is easy to verify that for each parameters \(\varepsilon_{i}, |\bar{J}_{i}(\bar{u}_{1}^{*}, \bar{u}_{2}^{*}) - J_{i}(u_{1}^{*}, u_{2}^{*})| = O(|\nu|)\) because of \(\phi_{i} < \infty\). Therefore, the formula (24) is correct. Moreover, we can obtain the parameter independent strategy even though the stochastic noise exist in the systems as compared with the existing result [16].

VI. CONCLUSION

In this paper, static output feedback soft-constrained stochastic Nash games for the SMS have been studied. We
have proposed the new method to find the static output feedback Nash strategy set. It should be noted that the proposed design methodology is quite different from the existing methods in the cases of the deterministic systems [9], [16] and the state feedback strategies [17]. As a result, we can find the Nash strategy even if the system is governed by Itô’s differential equation. Moreover, since the parameter independent strategy can be implemented using the local output measurements, the method can be applied to practice more realistically. As another important feature, it has been newly shown that the resulting strategies achieve $O(\|\nu\|)$ approximation of the optimal cost performance. Finally, through the solution of the practical megawatt-frequency control example, we have proposed the new design method can obtain reliable results against both stochastic and deterministic uncertainties.

REFERENCES