

## SKEW-SYMMETRIC VECTOR FIELDS ON A CR-SUBMANIFOLD OF A PARA-KÄHLERIAN MANIFOLD

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We deal with a CR-submanifold  $M$  of a para-Kählerian manifold  $\widetilde{M}$ , which carries a  $J$ -skew-symmetric vector field  $X$ . It is shown that  $X$  defines a global Hamiltonian of the symplectic form  $\Omega$  on  $M^\top$  and  $JX$  is a relative infinitesimal automorphism of  $\Omega$ . Other geometric properties are given.

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**1. Introduction.** CR-submanifolds  $M$  of some pseudo-Riemannian manifolds  $\widetilde{M}$  have been first investigated by Rosca [10], and also studied in [2, 3, 11].

If  $\widetilde{M}$  is a para-Kählerian manifold, it has been proved that any coisotropic submanifold  $M$  of  $\widetilde{M}$  is a CR-submanifold (such CR-submanifolds have been denominated CICR-submanifolds [6]).

In this note, one considers a foliate CICR-submanifold  $M$  of a para-Kählerian manifold  $\widetilde{M}(J, \widetilde{\Omega}, \widetilde{g})$ . It is proved that the necessary and sufficient condition in order that the leaf  $M^\top$  of the horizontal distribution  $D^\top$  on  $M$  carries a  $J$ -skew-symmetric vector field  $X$ , that is,  $\nabla X = X \wedge JX$ , is that the vertical distribution  $D^\perp$  on  $M$  is autoparallel.

In this case,  $M$  may be viewed as the local Riemannian product  $M = M^\top \times M^\perp$ , where  $M^\top$  is an invariant totally geodesic submanifold of  $M$  and  $M^\perp$  is an isotropic totally geodesic submanifold.

Furthermore, if  $\Omega$  is the symplectic form of  $M^\top$ , it is shown that  $X$  is a global Hamiltonian of  $\Omega$  and  $JX$  is a relative infinitesimal automorphism of  $\Omega$  (a similar discussion can be made for proper CR-submanifolds of a Kählerian manifold).

**2. Preliminaries.** Let  $\widetilde{M}(J, \widetilde{\Omega}, \widetilde{g})$  be a  $2m$ -dimensional para-Kählerian manifold, where, as is well known [7], the triple  $(J, \widetilde{\Omega}, \widetilde{g})$  of tensor fields is the *paracomplex operator*, the *symplectic form*, and the *para-Hermitian metric tensor field*, respectively.

If  $\widetilde{\nabla}$  is the Levi-Civita connection on  $\widetilde{M}$ , these manifolds satisfy

$$J^2 = Id, \quad d\widetilde{\Omega} = 0, \quad (\widetilde{\nabla}J)\widetilde{Z} = 0, \quad \widetilde{Z} \in \Gamma T\widetilde{M}. \quad (2.1)$$

Let  $x : M \rightarrow \widetilde{M}$  be the immersion of an  $l$ -codimensional submanifold  $M$ ,  $l < m$ , in  $\widetilde{M}$  and let  $T_p^\perp M$  and  $T_p M$  be the normal space and the tangent space at each point  $p \in M$ .

If  $J(T_p^\perp M) \subset T_p M$ , then  $M$  is said to be a *coisotropic* submanifold of  $\widetilde{M}$  (see [2]). If  $\widetilde{W} = \text{vect}\{h_a, h_{a^*}; a = 1, \dots, m, a^* = a + m\}$  is a real Witt vector basis on  $\widetilde{M}$ , one has

$$\tilde{g}(h_a, h_b) = \tilde{g}(h_{a^*}, h_{b^*}) = \delta_{ab}. \tag{2.2}$$

Next, if  $\widetilde{W}^* = \{\omega^a, \omega^{a^*}\}$  denotes the associated cobasis of  $\widetilde{W}$ , then  $\tilde{g}$  and  $\tilde{\Omega}$  are expressed by

$$\tilde{g} = 2 \sum \omega^a \otimes \omega^{a^*}, \tag{2.3}$$

$$\tilde{\Omega} = \sum \omega^a \wedge \omega^{a^*}. \tag{2.4}$$

We recall also that  $\widetilde{W}$  may split as

$$\widetilde{W} = \tilde{S} + \tilde{S}^*, \tag{2.5}$$

where the pairing  $(\tilde{S}, \tilde{S}^*)$  defines an involutive automorphism of square 1, that is,

$$Jh_a = h_{a^*}, \quad Jh_{a^*} = h_a, \tag{2.6}$$

and the local connection forms  $\tilde{\theta}_B^A \in \Lambda^1 \widetilde{M}$ ,  $A, B \in \{1, 2, \dots, 2m\}$  satisfy

$$\tilde{\theta}_b^{a^*} = 0, \quad \tilde{\theta}_{b^*}^a = 0, \quad \tilde{\theta}_b^a + \tilde{\theta}_{a^*}^{b^*} = 0. \tag{2.7}$$

It has been proved in [10] that any coisotropic submanifold  $M$  of a para-Kählerian manifold  $\widetilde{M}$  is a CR-submanifold of  $\widetilde{M}$  and such a submanifold has been called a CICR-submanifold [6].

Let  $D^\top : p \rightarrow D_p^\top = T_p M \setminus J(T_p^\perp M)$  and  $D^\perp : p \rightarrow D_p^\perp = J(T_p^\perp M) \subset T_p M$  be the two complementary differentiable distributions on  $M$ . One has

$$JD_p^\top = D_p^\top, \quad JD_p^\perp = T_p^\perp M, \tag{2.8}$$

and  $D^\top$  (resp.,  $D^\perp$ ) is called the *horizontal* (resp., *vertical*) *distribution* on  $M$ .

As in the Kählerian case, the vertical distribution  $D^\perp$  is always involutive.

If  $M$  is defined by the Pfaffian system

$$\omega^r = 0, \quad r = 2m + 1 - l, \dots, 2m, \tag{2.9}$$

then one has

$$\begin{aligned} D_p^\top &= \{h_i, h_{i^*}, i = 1, \dots, m - l, i^* = i + m\}, \\ D_p^\perp &= \{h_r, r = m - l + 1, \dots, m\}. \end{aligned} \tag{2.10}$$

Further denote by

$$\varphi^\perp = \omega^{m-l+1} \wedge \dots \wedge \omega^m \tag{2.11}$$

the simple unit form which corresponds to  $D^\perp$ .

Then, in order that the distribution  $D^\top$  be also involutive, it is necessary and sufficient that  $\varphi^\perp$  be a conformal integral invariant of  $D^\top$ , that is,

$$\mathcal{L}_{D^\top} \varphi^\perp = f \varphi^\perp \tag{2.12}$$

for a certain scalar function  $f$ .

By a standard calculation, one derives that the above equation implies

$$\theta_i^r = 0, \tag{2.13}$$

and in this case, one may write

$$d\varphi^\perp = -\left(\sum \theta_r^r\right) \wedge \varphi^\perp, \tag{2.14}$$

that is,  $\varphi^\perp$  is exterior recurrent.

In this case, as is known [2, 10],  $M$  is a foliated CR-submanifold of  $\tilde{M}$ .

We will investigate now the case when the leaf  $M^\top$  of  $D^\top$  carries a  $J$ -skew-symmetric vector field  $X$ , that is,

$$\nabla X = X \wedge JX. \tag{2.15}$$

One may express  $\nabla X$  as

$$\nabla X = (JX)^b \otimes X - X^b \otimes JX, \tag{2.16}$$

where

$$X = X^i h_i + X^{i*} h_{i*} = X^i \omega^{i*} + X^{i*} \omega^i. \tag{2.17}$$

Recalling Cartan structure equations [4],

$$\begin{aligned} \nabla h &= \theta \otimes e \in A^1(M, TM), \\ d\omega &= -\theta \wedge \omega, \\ d\theta &= -\theta \wedge \theta + \Theta. \end{aligned} \tag{2.18}$$

In the above equations,  $\theta$ , respectively  $\Theta$ , are the local connection forms in the bundle  $W$ , respectively the curvature forms on  $M$ .

Then making use of Cartan structure equations, one finds by a standard calculation that (2.16) implies that the vertical distribution  $D^\perp$  is autoparallel, that is,  $\nabla_{Z'} Z'' \in D^\perp$ , for all  $Z', Z'' \in D^\perp$ , which, in terms of connection forms, is expressed by

$$\theta_r^i = 0. \tag{2.19}$$

We agree to call  $\theta_r^i$  and  $\theta_i^r$  the *mixed connection forms*.

Taking account of (2.13) and (2.19), one derives from (2.16)

$$dX^b = 2(JX)^b \wedge X^b, \tag{2.20}$$

which agrees with the general equation of skew-symmetric killing vector fields [5, 8].

Next, by (2.1), one has

$$\nabla JX = (JX)^b \otimes JX - X^b \otimes X, \tag{2.21}$$

which shows that  $JX$  is a gradient vector field.

Hence, we may state the following theorem.

**THEOREM 2.1.** *Let  $x : M \rightarrow \widetilde{M}$  be an improper immersion of a CR-submanifold in a para-Kählerian manifold  $\widetilde{M}(J, \widetilde{\Omega}, \widetilde{g})$  and let  $D^\top$  (resp.,  $D^\perp$ ) be the horizontal distribution (resp., the vertical distribution) on  $M$ . If  $M$  is a foliate CR-submanifold, then the necessary and sufficient condition in order that the leaf  $M^\top$  of  $D^\top$  carries a  $J$ -skew-symmetric vector field  $X$  is that  $D^\perp$  is an autoparallel foliation. In this case, the CR-submanifold  $M$  under consideration may be viewed as the local Riemannian product  $M = M^\top \times M^\perp$ , where  $M^\top$  is an invariant totally geodesic submanifold of  $M$  and  $M^\perp$  is an isotropic totally geodesic submanifold. In addition, in this case,  $JX$  is a gradient vector field.*

**3. Properties.** In this section, we will point out some additional properties of  $X$  involving the symplectic form  $\Omega$  of  $M^\top$  and the exterior covariant differential  $d^\nabla$  of  $\nabla X$ . Operating on (2.16) and (2.21), one derives by a short calculation

$$\begin{aligned} d^\nabla(\nabla X) &= \nabla^2 X = 2(X^b \wedge (JX)^b) \otimes JX, \\ d^\nabla(\nabla JX) &= \nabla^2 JX = 2(X^b \wedge (JX)^b) \otimes X, \end{aligned} \tag{3.1}$$

which gives

$$\begin{aligned} \nabla^2(X + JX) &= 2(X^b \wedge (JX)^b) \otimes (X + JX), \\ \nabla^2(X - JX) &= -2(X^b \wedge (JX)^b) \otimes (X - JX). \end{aligned} \tag{3.2}$$

Therefore, we agree to define  $X + JX$  and  $X - JX$  as *2-covariant recurrent vector fields*. It should also be noticed that by reference to the general formula

$$\nabla_V(X_1 \wedge \cdots \wedge X_p) = \sum (X_1 \wedge \cdots \wedge \nabla_V X_j \wedge \cdots \wedge X_p), \quad V \in \Gamma TM, \tag{3.3}$$

one finds by (2.15) and (2.21)

$$\nabla_V(X \wedge JX) = 2g(V, JX)(X \wedge JX). \tag{3.4}$$

This shows that the covariant derivative of  $X \wedge JX$  with respect to any vector field  $V$  is proportional to  $X \wedge JX$ .

On the other hand, by the general formula

$$\nabla^2 V(Z, Z') = R(Z, Z')V, \tag{3.5}$$

where  $R$  denotes the curvature tensor field and  $V, Z, Z'$  are vector fields, one has (see also [9])

$$\mathcal{R}(Z, V) = \text{Tr}R(\cdot, Z)V, \quad (3.6)$$

where  $\mathcal{R}$  is the Ricci tensor field of  $\nabla$ .

Since in the case under consideration one must take in (3.6) the para-Hermitian trace, then setting in (3.6)  $Z = V = X$ , one finds

$$\mathcal{R}(X, X) = 0, \quad (3.7)$$

that is, the Ricci curvature of  $X$  vanishes.

Denote by  $\tilde{\Omega}$  the symplectic form of  $\tilde{M}$ , then  $\Omega = \tilde{\Omega}|_{M^\top}$  is a symplectic form of rank equal to the dimension of  $M^\top$ , that is, in our case,  $2(m-l)$ .

Then, if  ${}^bZ : Z \rightarrow -i_Z\Omega$  is the symplectic isomorphism, by a short calculation and on behalf of (2.4), one gets

$${}^bX = -(JX)^\flat, \quad (3.8)$$

and since  $JX$  is a gradient vector field, we conclude according to a known definition (see also [1]) that  $X$  is a *global Hamiltonian* of  $\Omega$ .

In a similar manner, one finds

$${}^b(JX) = X^\flat, \quad (3.9)$$

and by (2.20), it follows that

$$d(\mathcal{L}_{JX}\Omega) = 0, \quad (3.10)$$

which shows that  $JX$  is a relative infinitesimal automorphism of  $\Omega$  [1].

We state the following theorem.

**THEOREM 3.1.** *Let  $M$  be a CR-submanifold of a para-Kählerian manifold  $\tilde{M}$  and let  $\Omega$  be the symplectic form on  $M^\top$ . If  $M$  carries a  $J$ -skew-symmetric vector field  $X$ , then the following properties hold:*

- (i)  $X$  is a *global Hamiltonian* of  $\Omega$  and  $JX$  is a *relative infinitesimal automorphism* of  $\Omega$ ;
- (ii) the *Ricci tensor field*  $\mathcal{R}(X, X)$  vanishes;
- (iii) the *vector fields*  $X + JX$  and  $X - JX$  are *2-covariant recurrent*.

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