*-graphs of vertices of the generalized transitive tournament polytope

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Abstract

A nonnegative matrix $T = (t_{ij})_{i,j=1}^n$ is a generalized transitive tournament matrix (GTT matrix) if $t_{ii} = 0$, $t_{ij} = 1 - t_{ji}$ for $i \neq j$, and $1 \leq t_{ij} + t_{jk} + t_{ki} \leq 2$ for $i, j, k$ pairwise distinct. The problem we are interested in is the characterization of the set of vertices of the polytope $\{\text{GTT}\}_n$ of all GTT matrices of order $n$. In 1992, Brualdi and Hwang introduced the *-graph associated to each $T \in \{\text{GTT}\}_n$. We characterize the comparability graphs of $n$ vertices which are the *-graphs of some vertex of $\{\text{GTT}\}_n$. As an application of the theoretical work we conclude that no comparability graph of at most 6 vertices and with at least one edge is the *-graph of a vertex. In order to obtain the set of all vertices of $\{\text{GTT}\}_6$ it only remains to analyse two noncomparability graphs.

1. Introduction

A matrix will be said to be a $(\lambda_1, \lambda_2, \ldots, \lambda_k)$-matrix when the set of all its different entries coincides with the set $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$. A nonnegative matrix $T = (t_{ij})_{i,j=1}^n$ is said to be a generalized transitive tournament matrix, abbreviated GTT matrix, if

- $t_{ii} = 0$ for all $i \in \{1, 2, \ldots, n\}$,
- $t_{ij} + t_{ji} = 1$ for all $i \neq j \in \{1, 2, \ldots, n\}$,
- $1 \leq t_{ij} + t_{jk} + t_{ki} \leq 2$ for all $i \neq j \neq k \in \{1, 2, \ldots, n\}$.

A GTT matrix which is also a $(0,1)$-matrix is said to be a transitive tournament (TT) matrix. The problem in which we are interested is to determine the vertices of the polytope $\{\text{GTT}\}_n$ of all GTT matrices of order $n$. It is clear that each TT matrix of

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order \( n \) is a vertex of \( \text{GTT}_n \). We shortly expose the main results dealing with the problem:

In 1979, Cruse [4] obtained the first example of a vertex of \( \text{GTT}_6 \) which is not a TT matrix. In 1980, Dridi [5] proved that the TT matrices are exactly the vertices of \( \text{GTT}_n \) if and only if \( n \leq 5 \). In 1992, Brualdi and Hwang [3] introduced the *-graph associated to each \( T \in \text{GTT}_n \), and they gave several necessary conditions for the *-graph of any vertex of \( \text{GTT}_n \). In [1] we obtained a complete characterization of the \((0, \frac{1}{2}, 1)\)-matrices which are vertices of \( \text{GTT}_n \). In [8] Nutov and Penn obtained for each \( k \geq 3 \) a \((0, \frac{1}{k}, \frac{2}{k}, \ldots, (k-1)/k, 1)\)-matrix which is a vertex of \( \text{GTT}_{2k+2} \), these are the first known vertices with some entry different from 0, \( \frac{1}{2} \), or 1. In [2] we introduced the graph \( \Gamma_T \) associated to each \( T \in \text{GTT}_n \) which generalize the *-graph of \( T \). \( \Gamma_T \) was employed to develop a computable criterion for determine whether any given TT matrix is or is not a vertex of \( \text{GTT}_n \).

In this paper we associate to any graph \( \gamma \) of \( n \) vertices a new graph \( \Gamma_\gamma \). For any matrix \( T \in \text{GTT}_n \) with *-graph equal to \( \gamma \) consider the graph \( \Gamma_T \). If \( \gamma \) is a comparability graph then \( \Gamma_T \) will be a spanning subgraph of \( \Gamma_\gamma \) of a certain special class. In order to determine if a comparability graph \( \gamma \) is the *-graph of some vertex of \( \text{GTT}_n \) we only need to analyse that special class of subgraphs of \( \Gamma_\gamma \). As an application of our theoretical work we conclude that no comparability graph of at most 6 vertices and with at least one edge is the *-graph of a vertex. In order to obtain the set of all vertices of \( \text{GTT}_6 \), it only remains to analyse two noncomparability graphs.

2. Graphs

We will work only with graphs having neither loops nor multiple edges. Let \( G_n \) be the set which is composed of the undirected graphs with vertex-set \( V_n = \{1, 2, \ldots, n\} \). Given \( \gamma \in G_n \), the edge-set of \( \gamma \) will be denoted by \( E(\gamma) \). We will write \( \gamma = (V_n, E(\gamma)) \) or \( (V(\gamma), E(\gamma)) \). The edge with endpoints \( a, b \in V(\gamma) \) will be denoted \( \{a, b\} \) or \( \{b, a\} \). Two vertices \( a \) and \( b \) are said to be adjacent if \( E(\gamma) \) contains the edge \( \{a, b\} \). An edge \( \{a, b\} \) admits two orientations: \( a \rightarrow b \) or \( b \rightarrow a \). The complement \( \bar{\gamma} \) of \( \gamma \) is the graph of \( G_n \) in which \( \{a, b\} \in E(\bar{\gamma}) \) if and only if \( \{a, b\} \notin E(\gamma) \).

We introduce the three types of graphs which are important in this work:

1. In [3] was introduced the *-graph \( \gamma_T = (V_n, E(\gamma_T)) \in G_n \) associated to \( T = (t_{ij})_{i,j=1}^n \in \text{GTT}_n \) with edge-set
   \[
   E(\gamma_T) = \{ \{i, j\} \mid 1 \leq i < j \leq n; \ 0 < t_{ij} < 1 \}.
   \]

2. In [2] was introduced the graph \( \Gamma_T = (V(\Gamma_T), E(\Gamma_T)) \) associated to \( T = (t_{ij})_{i,j=1}^n \in \text{GTT}_n \) with vertex-set
   \[
   V(\Gamma_T) = \{ v_{ij} \mid 1 \leq i, j \leq n; \ 0 < t_{ij} < 1 \}\]
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and edge-set

\[ E(\Gamma_T) = \{ e_{ijk} \mid 1 \leq i, j, k \leq n; \ 0 < t_{ij}, t_{jk} < 1; \ t_{ik} = 1 - t_{ij}; \ t_{ik} \in \{0, 1\} \} \]

the endpoints of \( e_{ijk} \) are \( v_{ij} \) and \( v_{jk} \). Note that for each \( \{i, j\} \in E(\gamma_T) \) we have two vertices \( v_{ij}, v_{ji} \in V(\Gamma_T) \) which are joined by \( e_{iji} = e_{jij} \in E(\Gamma_T) \).

3. We introduce a new graph. For each \( \gamma = (V_n, E(\gamma)) \in G_n \) we define the graph \( \Gamma_{\gamma} = (V(\Gamma_{\gamma}), E(\Gamma_{\gamma})) \) with vertex-set

\[ V(\Gamma_{\gamma}) = \{ v_{ij} \mid 1 \leq i, j \leq n; \ \{i, j\} \in E(\gamma) \} \]

and edge-set

\[ E(\Gamma_{\gamma}) = \{ e_{ijk} \mid 1 \leq i, j, k \leq n; \ \{i, j\}, \{ j, k\} \in E(\gamma); \ \{i, k\} \notin E(\gamma) \} \]

the endpoints of \( e_{ijk} \) are \( v_{ij} \) and \( v_{jk} \). Note that for each \( \{i, j\} \in E(\gamma) \) we have two vertices \( v_{ij}, v_{ji} \in V(\Gamma_{\gamma}) \) which are joined by \( e_{iji} = e_{jij} \in E(\Gamma_{\gamma}) \).

In order to attack the problem of characterizing the vertices of the polytope \( \{\text{GTT}\}_n \), it was essential to introduce the *-graph associated to a GTT matrix. Moreover, in most of the results on this problem it is important if the *-graphs admit a transitive orientation or not. Recall that a graph \( \gamma \in G_n \) is said to be a comparability graph or to admit a transitive orientation if it is possible to orient each edge of \( \gamma \) so that the resulting digraph satisfies the transitive law

\[ a \rightarrow b, \ b \rightarrow c \quad \text{implies} \quad a \rightarrow c. \]

For our purposes there exists a useful characterization of comparability graphs in terms of the graphs \( \Gamma_{\gamma} \); but before that it is necessary to introduce some well-known definitions.

Any finite sequence of edges of \( \gamma \in G_n \)

\[ \{a_1, a_2\}, \{a_2, a_3\}, \ldots, \{a_{s-1}, a_s\}, \{a_s, a_{s+1}\} \]

is said to be a path of \( \gamma \) of length \( s \) and we use the notation \([a_1, a_2, \ldots, a_{s+1}]\) for this path. If \( a_1 = a_{s+1} \), then it is said to be a cycle of \( \gamma \) of length \( s \) and it is denoted \((a_1, a_2, \ldots, a_s)\). A triangular chord of a cycle \((a_1, a_2, \ldots, a_s)\) of \( \gamma \) is one of the edges \([a_i, a_{i+2}]\) with \( i \in \{1, \ldots, s-2\} \), or \([a_{s-1}, a_1]\), or \([a_s, a_2]\).

In the next theorem, condition (ii) is the usual characterization of comparability graphs due to Gilmore and Hoffman [6]. Condition (iii) is the interpretation of condition (ii) in terms of \( \Gamma_{\gamma} \).

**Theorem 1.** Given \( \gamma \in G_n \), the following statements are equivalent:

(i) \( \gamma \) is a comparability graph;
(ii) any cycle of \( \gamma \) of odd length has a triangular chord;
(iii) \( \Gamma_{\gamma} \) is a bipartite graph.
Proof. In order to prove the equivalence of (ii) and (iii) it is sufficient to point out that the cycles of $\Gamma_\gamma$ are in a bijective correspondence with the cycles of $\gamma$ without triangular chords. □

3. Subgraphs and the main result

Consider any graph $\gamma = (V, E)$. A spanning subgraph of $\gamma$ is any graph $(V, F)$ such that $F \subseteq E$. Given $W \subseteq V$, the subgraph of $\gamma$ induced by $W$ is the graph $(W, E_W)$, where

$$E_W \equiv \{ \{i, j\} \in E \mid i, j \in W \}.$$ 

For $h = 1, \ldots, r$ let $(W_h, E_{W_h})$ be the subgraph of $\gamma$ induced by $W_h \subseteq V$. Suppose that $W_i \cap W_j = \emptyset$ for $1 \leq i \neq j \leq r$, then we define

$$\bigoplus_{1 \leq h \leq r} (W_h, E_{W_h}) \equiv \left( \bigcup_{1 \leq h \leq r} W_h \right) \cup \left( \bigcup_{1 \leq h \leq r} E_{W_h} \right).$$

If $W_1 \cup \ldots \cup W_r = V$ then $(W_1, E_{W_1}) \oplus \cdots \oplus (W_r, E_{W_r})$ is said to be the spanning subgraph of $\gamma$ induced by the partition $\{W_1, \ldots, W_r\}$ of $V$.

Let $\gamma = (V, E(\gamma)) \in G_n$ and $\Gamma_\gamma = (V(\Gamma_\gamma), E(\Gamma_\gamma))$. We are interested in a class of spanning subgraphs of $\Gamma_\gamma$ induced by special partitions of $V(\Gamma_\gamma)$. Recall that each edge $\{i, j\} \in E(\gamma)$ is associated with exactly two vertices $v_{ij}, v_{ji} \in V(\Gamma_\gamma)$. Let $\{E_1(\gamma), \ldots, E_r(\gamma)\}$ be a partition of $E(\gamma)$, and define for each $h = 1, \ldots, r$ the set

$$V_h(\Gamma_\gamma) \equiv \{v_{ij} \in V(\Gamma_\gamma) \mid \{i, j\} \in E_h(\gamma)\}.$$

Clearly, $\{V_1(\Gamma_\gamma), \ldots, V_r(\Gamma_\gamma)\}$ is a partition of $V(\Gamma_\gamma)$, and we will say that it is associated to the partition $\{E_1(\gamma), \ldots, E_r(\gamma)\}$ of $E(\gamma)$.

If $\Gamma$ is a spanning subgraph of $\Gamma_\gamma$ induced by a partition of $V(\Gamma_\gamma)$ which is associated to a partition of $E(\gamma)$, then we will employ the notation $\Gamma \leq \Gamma_\gamma$.

The next result is fundamental to this work.

Theorem 2. Let $\gamma \in G_n$ be a comparability graph and let $T = (t_{ij})_{i,j=1}^n$ be a GTM-matrix such that the *-graph $\gamma_T$ of $T$ is equal to $\gamma$. Then $\Gamma_T \leq \Gamma_\gamma$.

Proof. (1) $\Gamma_T$ is a spanning subgraph of $\Gamma_\gamma$:

First, it is clear that $V(\Gamma_T) = V(\Gamma_\gamma)$ since $\gamma_T = \gamma$ implies

$$0 < t_{ij} < 1 \iff \{i, j\} \in E(\gamma).$$

On the other hand, $E(\Gamma_T) \subseteq E(\Gamma_\gamma)$ because $e_{ijk} \in E(\Gamma_T)$ implies that there exists some $a$ such that $0 < a < 1$ with $t_{ij} = a$, $t_{jk} = 1 - a$, and $t_{ik} \in \{0, 1\}$. As $\gamma_T = \gamma$ then $\{i, j\}, \{j, k\} \in E(\gamma)$ and $\{i, k\} \notin E(\gamma)$, i.e., $e_{ijk} \notin E(\Gamma_\gamma)$.

(2) $\Gamma_T$ is a spanning subgraph of $\Gamma_\gamma$ induced by a partition of $V(\Gamma_\gamma)$: Let $\Gamma_1, \ldots, \Gamma_r$ be the connected components of the graph $\Gamma_T$. We will see that $\Gamma_T = \Gamma_1 \oplus \cdots \oplus \Gamma_r$. 

i.e., that $\Gamma_T$ is the spanning subgraph of $\Gamma_\gamma$ induced by the partition \( \{ V(\Gamma_1), \ldots, V(\Gamma_r) \} \) of $V(\Gamma_\gamma)$. Therefore, we must show that

$$\text{if } v_{ij}, v_{jk} \in V(\Gamma_h) \text{ and } e_{ijk} \in E(\Gamma_\gamma) \implies e_{ijk} \in E(\Gamma_h).$$

(i) $e_{ijk} \in E(\Gamma_\gamma)$ implies, by Theorem 1, that any path of $\Gamma_\gamma$ with extremes $v_{ij}$ and $v_{jk}$ has odd length.

(ii) As $v_{ij}$ and $v_{jk}$ belong to the same connected component of $\Gamma_T$ then it there exists a path $C$ of edges of $E(\Gamma_h)$ with extremes $v_{ij}$ and $v_{jk}$. By (i), the path $C$ has odd length. Let

\[ C = [v_{ij}, v_{j_1}, v_{j_2}, \ldots, v_{j_{2p-l}}, v_{jk}]. \]

(iii) From the definition of $\Gamma_T$ it follows that $\exists a$ with $0 < a < 1$ such that

$$t_{ij} = a \implies t_{j_1} = 1 - a \implies \ldots \implies t_{j_{2p-l}} = a \implies t_{jk} = 1 - a.$$

(iv) On the other hand, $e_{ijk} \in E(\Gamma_\gamma)$ implies that

$$\{i, k\} \not\subseteq E(\gamma) \implies \{i, k\} \not\subseteq E(\gamma_T) \implies t_{jk} \in \{0, 1\}.$$

From (iii) and (iv) we conclude that $e_{ijk} \in E(\Gamma_T)$.

(3) The partition $\{ V(\Gamma_1), \ldots, V(\Gamma_r) \}$ of $V(\Gamma_\gamma)$ is associated to a partition of $E(\gamma)$: We must show that

$$\text{if } \{i, j\} \in E(\gamma) \implies \exists h \text{ such that } v_{ij}, v_{ji} \in V(\Gamma_h).$$

Suppose $\{i, j\} \in E(\gamma)$. As $\gamma_T = \gamma$ then $\{i, j\} \in E(\gamma_T)$. Hence, there exists some $a$ with $0 < a < 1$ such that $t_{ij} = a$ and $t_{ji} = 1 - a$. As $t_{ii} = 0$ then $e_{ij} \in E(\Gamma_T)$ with endpoints $v_{ij}$ and $v_{ji}$. That is, $v_{ij}$ and $v_{ji}$ lies in the same component of $\Gamma_T$. \(\square\)

It is important to point out that Theorem 2 is not true when $\gamma$ is not a comparability graph.

4. Characterization

We recall that our objective is to characterize the vertices of $\{\Gamma_T\}_n$ whose *-graph is equal to a given graph $\gamma \in G_n$. Theorem 2 permits to attack computationally the problem when $\gamma$ is a comparability graph. Namely, if $\gamma \in G_n$ is a comparability graph then the study of those subgraphs $\Gamma \subseteq \Gamma_\gamma$ (that is, the spanning subgraphs of $\Gamma_\gamma$ which are induced by partitions of $V(\Gamma_\gamma)$ which are associated to partitions of $E(\gamma)$) will give us the desired characterization.

In this section we will consider fixed a comparability graph $\gamma \in G_n$ and a subgraph $\Gamma \subseteq \Gamma_\gamma$ with $r$ connected components $\Gamma_1, \ldots, \Gamma_r$. As $\Gamma_\gamma$ is a bipartite graph, the set $V(\Gamma_\gamma)$ can be divided into white and black vertices in such a way that two vertices of the same color are never adjacent vertices. The same is valid for $V(\Gamma)$ since $\Gamma$ is a spanning subgraph of $\Gamma_\gamma$. 
We define the set
\[ \Phi_\gamma = \{(i,j,k) \mid \{i,j\}, \{j,k\}, \{k,i\} \in E(\gamma)\} \]
composed by all triangles of \( \gamma \). Consider \( r \) unknowns \( e^{(1)}, \ldots, e^{(r)} \) and define for each vertex \( v_{ij} \in V(\Gamma) \)
\[ e_{ij} = e^{(h)} \quad \text{if} \quad v_{ij} \in \Gamma_h \text{ and white}, \]
\[ e_{ij} = -e^{(h)} \quad \text{if} \quad v_{ij} \in \Gamma_h \text{ and black}. \]

For each set
\[ \Phi = \{(i_1,j_1,k_1),(i_2,j_2,k_2),\ldots,(i_r,j_r,k_r)\} \subseteq \Phi_\gamma \]
composed by \( r \) triangles of \( \gamma \) we define the linear system \( S[\Gamma, \Phi] \):
\[ e_{i_1j_1} + e_{j_1k_1} + e_{k_1i_1} = 0, \]
\[ e_{i_2j_2} + e_{j_2k_2} + e_{k_2i_2} = 0, \]
\[ \ldots \]
\[ e_{i_rj_r} + e_{j_rk_r} + e_{k_ri_r} = 0 \]
on the unknowns \( e^{(1)}, e^{(2)}, \ldots, e^{(r)} \). Let
\[ \Theta_\Gamma = \{ \Phi \subseteq \Phi_\gamma \mid |\Phi| = r; \ S[\Gamma, \Phi] \text{ has only the trivial solution}\}. \]

Note. Observe that if the number of connected components of \( \Gamma \), \( |\Gamma| \), is greater than the number of triangles of \( \Phi_\gamma \), \( |\Phi_\gamma| \), then \( \Theta_\Gamma = \emptyset \).

On the other hand, we define
\[ M[\Gamma] = \{ T \in \{\text{GTT}\}_n \mid \Gamma_T = \Gamma \}, \]
and for each \( \Phi \in \Theta_\Gamma \)
\[ M[\Gamma, \Phi] = \{ T \in M[\Gamma] \mid t_{ij} + t_{jk} + t_{ki} = 1 \text{ or } 2 \ \forall (i,j,k) \in \Phi \}. \]

We now introduce some notations. Let \( V[\{\text{GTT}\}_n] \) be the set of vertices of \( \{\text{GTT}\}_n \). Let
\[ V[\gamma] = \{ T \in V[\{\text{GTT}\}_n] \mid \gamma_T = \gamma \}, \]
and
\[ V[\Gamma] = \{ T \in V[\gamma] \mid \Gamma_T = \Gamma \}. \]

The next result justifies the introduction of all these definitions.

**Theorem 3.** Let \( \gamma \in G_n \) be a comparability graph, then
\[ V[\gamma] = \bigcup_{\Gamma \leq \Gamma'} V[\Gamma] = \bigcup_{\Gamma \leq \Gamma'} \bigcup_{\Phi \in \Theta_\Gamma} M[\Gamma, \Phi]. \]
Proof. The first equality is an immediate consequence of Theorem 2. The second equality is Corollary 5 in [2] for the case \( \gamma \in G_n \) is a comparability graph. \( \Box \)

Note. In order to calculate \( V[\gamma] \) when \( \gamma \in G_n \) is a comparability graph it is necessary to be able to calculate \( M[\Gamma, \Phi] \). We point out that in a great many a cases we will have \( \Theta_\Gamma = \emptyset \). When this happens for all \( \Gamma \leq \gamma \), then it is not needed to follow since we conclude directly that \( V[\gamma] = \emptyset \). In fact, for \( n = 6 \) this is exactly what will happen in the most part of the cases as we will see.

5. Calculating \( M[\Gamma, \Phi] \)

Consider a comparability graph \( \gamma \in G_n \), a subgraph \( \Gamma \leq \gamma \) with \( r \) connected components \( \Gamma_1, \ldots, \Gamma_r \) and a set

\[ \Phi = \{ (i_1, j_1, k_1), (i_2, j_2, k_2), \ldots, (i_r, j_r, k_r) \} \in \Theta_\Gamma. \]

Let \( \tilde{\gamma} \in G_n \) be the complement of \( \gamma \) with

\[ E(\tilde{\gamma}) = \{ \{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_u, b_u\} \}. \]

Fixed \( \delta = (\delta_1, \ldots, \delta_r) \in \{1, 2\}^r \) and \( \lambda = (\lambda_1, \ldots, \lambda_u) \in \{0, 1\}^u \) we can define a unique matrix \( T[\Gamma, \Phi, \delta, \lambda] = (t_{ij})_{i,j=1}^n \) that satisfies:

1. \( t_{ii} = 0 \) for \( i = 1, \ldots, n \),
2. \( t_{u+h} = \lambda_h \) for \( h = 1, \ldots, u \),
3. \( t_{u+h} = 1 - \lambda_h \) for \( h = 1, \ldots, u \),
4. \( t_{ij} = a_h \) for \( v_{ij} \in \Gamma_h \) and white,
5. \( t_{ij} = 1 - a_h \) for \( v_{ij} \in \Gamma_h \) and black,
6. \( t_{j,k} + t_{j,k} + t_{j,k} = \delta_h \) for \( h = 1, \ldots, r \).

Since \( S[\Gamma, \Phi] \) has a unique solution the linear system \( S[\Gamma, \Phi, \delta] \) on the unknowns \( a_1, \ldots, a_r \) defined in (6) under conditions (4) and (5) has a unique solution \( (a_1, \ldots, a_r) \).

It implies the existence and uniqueness of \( T[\Gamma, \Phi, \delta, \lambda] \).

We define the set

\[ \tilde{M}[\Gamma, \Phi] = \{ T[\Gamma, \Phi, \delta, \lambda] | \delta \in \{1, 2\}^r, \lambda \in \{0, 1\}^u \}. \]

It is important to point out that if \( T \in M[\Gamma, \Phi] \) then \( T = T[\Gamma, \Phi, \delta, \lambda] \) for some \( \delta \in \{1, 2\}^r \) and \( \lambda \in \{0, 1\}^u \) (see the definition of \( M[\Gamma, \Phi] \)). Therefore,

\[ M[\Gamma, \Phi] = \{ T \in \tilde{M}[\Gamma, \Phi] | T \in \{GT\}_n, \Gamma_T = \Gamma \}. \]

As \( \tilde{M}[\Gamma, \Phi] \) contains exactly \( 2^{r+u} \) matrices, the problem of determine the set \( M[\Gamma, \Phi] \) is easily computable.
6. Algorithm and $V[\{\text{GTT}\}_6]$

We have shown that with the following algorithm we can find for a comparability graph $\gamma \in G_6$ all elements of $V[\gamma]$, that is, all vertices of $\{\text{GTT}\}_6$ with *-graph equal to $\gamma$:

1. For each partition of $E(\gamma)$ of at most $|\Phi|$ elements construct the spanning subgraph $\Gamma \leq \Gamma_\gamma$ it induces.
2. For each subset $\Phi \subseteq \Phi_\gamma$ with $|\Phi| = |\Gamma|$ solve the linear system $S[\Gamma, \Phi]$ and construct the set $\Theta_\Gamma$.
3. For each $\Phi \in \Theta_\Gamma$ calculate $M[\Gamma, \Phi]$.

We have implemented this algorithm. In [3], Brualdi and Hwang state several necessary conditions that fulfils the *-graph of a vertex of $\{\text{GTT}\}_n$. We have applied the algorithm to every comparability graph of $G_6$ which fulfils Brualdi and Hwang's conditions and we conclude that

**Lemma 4.** No comparability graph of 6 vertices with at least one edge is the *-graph of a vertex of $\{\text{GTT}\}_6$.

**Note.** Between all the graphs of $G_6$ for which it was necessary to apply the algorithm, only for the graph $\gamma \in G_6$ with edge-set

$$E(\gamma) = \{\{1,2\}, \{1,4\}, \{1,6\}, \{2,3\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\}, \{4,6\}, \{5,6\}\},$$

the subgraph $\Gamma \leq \Gamma_\gamma$ induced by the partition of $E(\gamma)$ given by

$$E_1(\gamma) = \{\{1,2\}, \{1,6\}, \{2,3\}, \{3,4\}, \{4,6\}\},$$

$$E_2(\gamma) = \{\{2,5\}, \{5,6\}\},$$

$$E_3(\gamma) = \{\{1,4\}, \{4,5\}\},$$

and

$$E_4(\gamma) = \{\{3,5\}\},$$

and the set of triangles of $\gamma$,

$$\Phi = \{\{(1,4,6),(2,3,5),(3,4,5),(4,5,6)\},$$

the system $S[\Gamma, \Phi]$ has as unique solution the trivial case. Nevertheless, for this case no matrix of type $T[\Gamma, \Phi, \delta, \lambda]$ is a matrix of $M[\Gamma, \Phi]$.

In order to reach the complete classification of the set of vertices of $\{\text{GTT}\}_6$ it only remains to determine if the graphs $\gamma_1$ and $\gamma_2$ with edge-sets

$$E(\gamma_1) = \{\{1,2\}, \{2,3\}, \{2,4\}, \{3,4\}, \{3,5\}, \{4,6\}, \{5,6\}\},$$

$$E(\gamma_2) = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,5\}, \{3,6\}, \{4,5\}, \{4,6\}, \{5,6\}\}$$

are the *-graphs of some vertex of \{\text{GTT}\}_6 which is not a \((0, \frac{1}{2}, 1)\)-matrix (see [1]). Note that neither \(\gamma_1\) nor \(\gamma_2\) are comparability graphs.

References

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