Total domination number of grid graphs

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Abstract

We use the link between the existence of tilings in Manhattan metric with \{1\}-bowls and minimum total dominating sets of Cartesian products of paths and cycles. From the existence of such a tiling, we deduce the asymptotical values of the total domination numbers of these graphs and we deduce the total domination numbers of some Cartesian products of cycles. Finally, we investigate the problem of total domination numbers for some Cartesian products of two paths. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let \( G = (V,E) \) be a simple graph (i.e. without loops nor multiple edges), \( V \) will denote its vertex set and \( E \) its edge set. For a subset \( S \) of \( V \), we denote by \( N(S) = \bigcup_{v \in S} N(v) \) where \( N(v) \) is the neighborhood of the vertex \( v \).

We say that a set \( D \) of vertices in a graph \( G \) is a dominating set, if \( N(D) \cup D = V \). The domination number \( \gamma(G) \) of a graph \( G \) is the smallest cardinality of a dominating set. A total dominating set \( D \) is a subset of \( V \) such that \( N(D) = V \). The total domination number \( \gamma_t(G) \) of a graph \( G \) is the smallest cardinality of a total dominating set.

The Cartesian product of two graphs \( G \) and \( H \) is the graph denoted by \( G \square H \), with \( V(G \square H) = V(G) \times V(H) \) (where \( \times \) denotes the Cartesian product of sets) and \((u,u') \) and \((v,v') \) \( \in E(G \square H) \) if and only if \( u = v \) and \((u',v') \in E(H) \) or \( u' = v' \) and \((u,v) \in E(G) \).

We denote by \( G_{k_1,...,k_d} = \square_{i=1}^d A_i \), where \( A_i \) is either a path or a cycle of length \( k_i \). We will call the integer \( d \), the dimension of \( G_{k_1,...,k_d} \). If each \( A_i \) is a cycle then we will call \( G_{k_1,...,k_d} \) a torus graph. If each \( A_i \) is a path then we will call \( G_{k_1,...,k_d} \) a grid graph. If at least one \( A_i \) is a cycle and at least one \( A_j \) \((j \neq i)\) is a path then we will call \( G_{k_1,...,k_d} \) a cylinder graph.

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The domination number of the Cartesian product of two paths $P_k \square P_n$ have been intensively investigated. Johnson [9] has attributed the (unpublished) proof of NP-completeness of the decision version problem of the domination problem (i.e., given a graph $G$ and an integer $m$, is there a dominating set of $G$ of size $m$ or less?) for arbitrary grid graphs to F.T. Leighton. Nevertheless, the complexity of determining the value of $\gamma(P_k \square P_n)$ remains unknown. Until now, only a few cases were settled when $k \leq 16$ [8,2,4]. Total dominating set problem also has been shown NP-complete for bipartite graphs [12].

The domination number of some torus graphs was studied in [10]. In [6], the authors show the link between the existence of perfect Lee codes and minimum dominating sets of $G_{k_1, \ldots, k_d}$. From the existence of such a code they deduce the asymptotical values of the domination number of these graphs. Here, we use a similar technique to obtain the asymptotical values of the total domination number of grid graphs. Moreover, we give the values of the total domination number for some torus graphs. Our technique is based on the existence of tiling with some objects in discrete geometry [7].

The $\{1\}$-bowl in $Z^d$ centered on a point $X$, is the set of points $Y$ at Manhattan distance 1 from $X$. In our graph language, a $\{1\}$-bowl is the neighborhood of $X$ in the infinite grid graph of dimension $d$. Thus, a tiling of $Z^d$ with $\{1\}$-bowls is a total dominating set of the $d$-dimensional infinite grid graph. In [7], the authors exhibit tilings of $Z^d$ with $\{1\}$-bowls.

2. Asymptotical values

2.1. Tilings of $Z^d$ with $\{1\}$-bowls

In this section, we describe the tilings of $Z^d$ $(d \geq 2)$ given in [7]. The reader can find in [5] the first results concerning the existence of perfect Lee-codes which are periodic tilings with bowls. In a graph theoretical language a periodic tiling with $\{1\}$-bowl is a total dominating set of a torus graph (or shortly torus).

Consider a bicoloring (‘black’ and ‘white’) of $Z^d$ defined by a point of $Z^d$ is ‘black’ if and only if the sum of its coordinates is even. Let $\mathcal{B}$ (respectively, $\mathcal{W}$) be the set of ‘black’ (resp. ‘white’) points. If $d \geq 3$ then the set $\mathcal{F}$ of $\{1\}$-bowls centered on the following points forms a tiling of $\mathcal{W}$ [7]:

$$X = \left( x_1, \ldots, x_{d-2}, 2a + \varepsilon, 2a + \varepsilon + 2db + \sum_{i=1}^{d-2} (2i + 1)x_i \right),$$

where $x_1, \ldots, x_{d-2}, a, b$ are in $Z$ and

$$E = \begin{cases} 0 & \text{if } \left\lfloor \frac{\sum_{i=1}^{d-2} x_i}{2} \right\rfloor = 0 \pmod{2}, \\ 1 & \text{if } \left\lfloor \frac{\sum_{i=1}^{d-2} x_i}{2} \right\rfloor = 1 \pmod{2}. \end{cases} \quad (1)$$
Similarly, to obtain a tiling of $\mathcal{B}$ with $\{1\}$-bowls; let $W$ be any ‘white’ point of $\mathbb{Z}^d$. We consider the set

$$\mathcal{F}' = \{ X \text{ such that } X = Y + W \text{ for every } Y \text{ such that } Y \in \mathcal{F} \}. \quad (2)$$

Now, to obtain a tiling of $\mathbb{Z}^d$ when $d \geq 3$, it is sufficient to combine the tiling $\mathcal{F}$ of $\mathcal{B}$ and the tiling $\mathcal{F}'$ of $W$.

As observed in [7], this tiling is periodic on the torus $\mathcal{T}_{4d} = [4d, \ldots, 4d]$ ($d$ times). This means that this tiling defines a total dominating set of the graph $\bigcup_{i=1}^{d} C_{4d}$.

Moreover, when $d \geq 4$ is even, the authors give a periodic tiling on the torus $\mathcal{T}_{2d} = [2d, \ldots, 2d]$ ($d$ times). This tiling is given by the sets $\mathcal{F}$ and $\mathcal{F}'$ of $\{1\}$-bowls centered on the points $X$

$$X = \left( x_1, \ldots, x_{d-2}, 2a + \varepsilon, 2a + \varepsilon + \sum_{i=1}^{d-2} (2i + 1)x_i \right),$$

where $x_1, \ldots, x_{d-2}, a$ are in $\mathbb{Z}$ and

$$\varepsilon = \begin{cases} 
0 & \text{if } \left\lfloor \frac{\sum_{i=1}^{d-2} x_i}{2} \right\rfloor = 0 \text{ (mod 2)}, \\
1 & \text{if } \left\lfloor \frac{\sum_{i=1}^{d-2} x_i}{2} \right\rfloor = 1 \text{ (mod 2)},
\end{cases} \quad (3)$$

$$\mathcal{F}' = \{ X \text{ such that } X = Y + W \text{ for every } Y \text{ such that } Y \in \mathcal{F} \}.$$

For $d = 2$, the authors give a tiling which works for more general bowls. Here, we only give the description of such a tiling for $\{1\}$-bowl (see Fig. 1).
Consider the vectors in $\mathbb{Z}^2$ defined by $U = (2, 2)$ and $V = (3, -1)$. Let
\[
\mathcal{F} = \{ X \text{ such that } X = \alpha U + \beta V \text{ for all } \alpha, \beta \in \mathbb{Z} \}. \tag{4}
\]
Now, to obtain the tiling of $\mathbb{Z}^2$ we conclude similarly at the previous cases.

From (1)–(4), and since a vertex of the $d$-dimensional torus graph has $2d$ neighbors, we deduce the total domination number for some torus graphs.

**Corollary 2.1.** For any integer $d, k_1, \ldots, k_d$ such that $d \geq 2$ and such that $\forall i, k_i \equiv 0 \pmod{4d}$, we have
\[
\gamma_t(G_{k_1, \ldots, k_d}) = \frac{\prod_{i=1}^d k_i}{2^d}. \tag{5}
\]
Moreover, if $d$ is even then for any integers $k_1, \ldots, k_d$ such that $\forall i, k_i \equiv 0 \pmod{2d}$, this equality still holds.

### 2.2. Asymptotical results

As in [6], we deduce from the tilings given in the previous section, the asymptotical values for $\gamma_t(G_{k_1, \ldots, k_d})$.

A subset $B$ of the $d$-dimensional grid $\mathbb{Z}^d$ is a box of size $k_1 \cdot \ldots \cdot k_d$ if $B$ is isomorphic to $[1, k_1] \times \cdots \times [1, k_d]$.

**Lemma 2.2.** The size of an $\{1\}$-bowl of $\mathbb{Z}^d$ is $2d$. If there exists a periodic tiling $T$ of $\mathbb{Z}^d$ with $\{1\}$-bowls, then for every integers $k_1, \ldots, k_d$, there exists a box of size $k_1 \cdot \ldots \cdot k_d$ with at most $\lfloor (k_1 \cdot \ldots \cdot k_d/2d) \rfloor$ vertices of $T$.

**Proof.** If the tiling $T$ is $q$-periodic then in a box of size $q \cdot \ldots \cdot q$ (times) there is exactly $q^d/2d$ vertices of it. Thus, a box of size $qk_1 \cdot \ldots \cdot qk_d$ contains exactly $q^d(k_1 \cdot \ldots \cdot k_d/2d)$ vertices of $T$. This box is a disjoint union of $q^d$ boxes of size $k_1 \cdot \ldots \cdot k_d$. Then at least one of these boxes has required property. \hfill \Box

Let $B = [a_1, b_1] \times \cdots \times [a_d, b_d]$ be a box of the $d$-dimensional grid $\mathbb{Z}^d$, and let $X = (x_1, \ldots, x_d)$ be a vertex of $\mathbb{Z}^d$ out of $B$. We define $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_d)$ the projection of $x$ over $B$ by
\[
\bar{x}_i = \begin{cases} 
  a_i + 1 & \text{if } x_i < a_i, \\
  x_i & \text{if } a_i \leq x_i \leq b_i, \\
  b_i - 1 & \text{if } x_i > b_i.
\end{cases}
\]

**Theorem 2.3.** For $k_1, \ldots, k_d$ positive integers, we have
\[
\gamma_t(G_{k_1, \ldots, k_d}) \leq \frac{\prod_{i=1}^d (k_i + 2)}{2d}.
\]
Proof. Let $T$ be the $q$-periodic tiling of $\mathbb{Z}^d$, defined in Section 2.1 with $q = 4d$ if $d$ is odd and $q = 2d$ if $d$ is even. By Lemma 2.2, we can choose a box $B = [a_1, b_1] \times \cdots \times [a_d, b_d]$ of size $(k_1 + 2) \cdots (k_d + 2)$ such that $$|B \cap T| \leq \left\lfloor \prod_{i=1}^d (k_i + 2) \cdot 2^d \right\rfloor.$$ The box $B' = [a_1 + 1, b_1 - 1] \times \cdots \times [a_d + 1, b_d - 1]$ is isomorphic to $\square_{i=1}^d P_{k_i}$. A vertex of $B'$ which belongs to the $\{1\}$-bowl centered on a vertex $v$ of $B - B'$, belongs also to the $\{1\}$-bowl centered on $\bar{v}$. We build a total dominating set of $B'$, by taking the vertices of $B' \cap T$ and by projecting the vertices of $B - B'$, over $B'$. Now to complete the proof of Theorem 2.3, it is sufficient to observe that a total dominating set of $\square_{i=1}^d P_{k_i}$ is also a total dominating set of $G_{k_1, \ldots, k_d}$.

Corollary 2.4.

$$\lim_{k_1, \ldots, k_d \to \infty} \frac{\gamma_t(G_{k_1, \ldots, k_d})}{k_1 \cdots k_d} = \frac{1}{2d}.$$ 

Proof. The size of an $\{1\}$-bowl on the torus graph $\square_{i=1}^d C_{k_i}$ is $2d$. Moreover, a total dominating set of $G_{k_1, \ldots, k_d}$ is also a total dominating set of $\square_{i=1}^d C_{k_i}$. Thus, we have $$\gamma_t(G_{k_1, \ldots, k_d}) \geq \gamma_t(\square_{i=1}^d C_{k_i}) \geq \frac{k_1 \cdots k_d}{2d}.$$ So the lower and upper bounds have the same asymptotic value. The result follows.

As observed in [6], using projection is slightly rough, hence the bound of Theorem 2.3 can be improved.

3. Total domination of two-dimensional grid graphs

3.1. Small values of $k$

In this section, we give the values of $\gamma_t(P_1 \square P_n)$ when $k \leq 4$.

Since $P_1 \square P_n \simeq P_n$, we have:

Proposition 3.1. For any $n \geq 2$, we have $\gamma_t(P_1 \square P_n) = \lfloor (n + 2) / 4 \rfloor + \lfloor (n + 3) / 4 \rfloor$.

Proposition 3.2. For any $n \geq 2$, we have $\gamma_t(P_2 \square P_n) = 2 \lfloor (n + 2) / 3 \rfloor$.

Proof. Let $D$ be a total dominating set of $P_2 \square P_n$. Let $f(l)$ be the number of vertices of $D$ in the first $l$ columns of $P_2 \square P_n$, for any $2 \leq l \leq n$. We claim that $f(l+3) \geq f(l)+2$. Indeed, let $C_j, C_{j+1}$ and $C_{j+2}$ be three consecutive columns of $P_2 \square P_n$. To dominate the vertices $x_{1,j+1}$ and $x_{2,j+1}$, we need one vertex among $\{x_{1,j}, x_{1,j+2}, x_{2,j+1}\}$ and one more vertex among $\{x_{1,j+1}, x_{2,j}, x_{2,j+2}\}$.
Since $f(2), f(3) \geq 2$ and $f(4) \geq 4$, we obtain

$$\forall n \geq 2, \quad |D| = f(n) \geq 2 \left\lceil \frac{n + 2}{3} \right\rceil.$$  

Now, to describe our total dominating set $D$, we consider blocks $B \cong P_2 \square P_3$ and $D \cap B = \{x_{1,1}, x_{2,1}\}$. If $n \equiv 0 \pmod{3}$ then $P_2 \square P_n$ can be partitioned with blocks $B$. If $n \equiv 1 \pmod{3}$ then $P_2 \square P_n$ can be partitioned with blocks $B$, plus a block $B'$ isomorphic to $P_2 \square P_4$ and $D \cap B' = \{x_{1,2}, x_{2,2}, x_{1,3}, x_{2,3}\}$. If $n \equiv 2 \pmod{3}$ then $P_2 \square P_n$ can be partitioned with blocks $B$, plus a block $B'$ isomorphic to $P_2 \square P_2$ and $D \cap B' = \{x_{1,2}, x_{2,2}\}$.

In any case, we have

$$|D| = 2 \left\lceil \frac{n + 2}{3} \right\rceil \quad \Box$$

**Proposition 3.3.** For any $n \geq 3$, we have $\gamma_t(P_3 \square P_n) = n$.

**Proof.** Let $D$ be a minimum total dominating set of $P_3 \square P_n$. First, observe that we may assume that there does not exist a column $C_i$ of $P_3 \square P_n$ such that $|C_i \cap D| = 3$. Indeed, in the opposite case, if $n > i \geq 2$ then we consider the set $D' = \big(D - \{x_{1,i}, x_{3,i}\}\big) \cup \{x_{2,i-1}, x_{2,i+1}\}$. If $i = n$ then $D' = \big(D - \{x_{1,i}, x_{3,i}\}\big) \cup \{x_{2,i-1}\}$, else $D' = \big(D - \{x_{1,i}, x_{3,i}\}\big) \cup \{x_{2,i+1}\}$. In all cases $D'$ is a total dominating set of $P_3 \square P_n$ with $|D'| \leq |D|$. Now, let $i$ be the smallest index such that $D \cap C_i = \emptyset$. Note that $i > 1$, for otherwise we must have $|D \cap C_2| = 3$, which contradicts the first assumption. We claim that

We may assume that $|D \cap C_j| = 1 \quad \forall j < i$.  \hspace{1cm} (5)

Indeed, in the opposite case, we consider the total dominating set $D' = \big(D - \bigcup_{k<i} (C_k \cap D)\big) \bigcup \{x_{2,k}\}$ which verifies $|D'| \leq |D|$

By (5), it is easy to see that $\{x_{2,i-1}\} = D \cap C_i$. Hence, to dominate the vertices $x_{1,i}$ and $x_{3,i}$ $D$ must contain the vertices $x_{1,i+1}$ and $x_{3,i+1}$. Moreover, since $|D \cap C_{i+1}| < 3$, to dominate the vertices $x_{1,i+1}$ and $x_{3,i+1}$, $D$ must contain the vertices $x_{1,i+2}$ and $x_{3,i+2}$. Now, we will consider the total dominating set $D' = \big(D - \{x_{1,i+1}, x_{3,i+1}, x_{1,i+2}, x_{3,i+2}\}\big) \bigcup \{x_{2,i}, x_{2,i+1}, x_{2,i+2}, x_{2,i+3}\}$ if $i \geq 3$ else $D' = \big(D - \{x_{1,i+1}, x_{3,i+1}, x_{1,i+2}, x_{3,i+2}\}\big) \bigcup \{x_{2,i}, x_{2,i+1}, x_{2,i+2}\}$. In any case $|D| \geq |D'| \geq n$.

To complete the proof, it is sufficient to observe that the set $\bigcup_{i=1}^n \{x_{2,i}\}$ is a total dominating set of $P_3 \square P_n$. \Box

We also obtain the exact values of $\gamma_t(P_4 \square P_n)$, we give here the result without proof.

**Proposition 3.4.** For any $n \geq 4$, we have

$$\gamma_t(P_4 \square P_n) = \begin{cases} \frac{6n + 8}{5} & \text{if } n \equiv 1, 2, 4 \pmod{5}, \\ \frac{6n + 8}{5} + 1 & \text{if } n \equiv 0, 3 \pmod{5}. \end{cases}$$
3.2. General bounds

As mentioned in Section 2, using projection is slightly rough, so, in this section, we improve the bound of Theorem 2.3 for the two-dimensional grid graphs.

**Theorem 3.5.** If \( k \) and \( n \) are two integers greater than 16 then

\[
\frac{3kn + 2(k + n)}{12} - 1 \leq \gamma_t(P_k \square P_n) \leq \left\lfloor \frac{(k + 2)(n + 2)}{4} \right\rfloor - 4.
\]

**Proof.** In fact, to prove the upper bound, we examine what happens when we use projection near the ‘corner’ (a vertex of degree two) of the grid.

Let \( T \) be the 4-periodic tiling of \( \mathbb{Z}^2 \), defined in (4) with the ‘white’ point \( W = (1, 0) \). By Lemma 2.2, we can choose a box \( B = [a_1, b_1] \times [a_2, b_2] \) of size \( (k + 2)(n + 2) \) such that

\[ |B \cap T| \leq \left\lfloor \frac{(k + 2)(n + 2)}{4} \right\rfloor. \]

Without loss of generality, we may assume that \( a_1 = a_2 = 0 \), \( b_1 = k + 1 \) and \( b_2 = n + 1 \). Thus, the box \( B' = [1, k] \times [1, n] \) is isomorphic to \( P_k \square P_n \). We denote by \( x_{i,j} \) with \( 0 \leq i \leq n + 1 \) and \( 0 \leq j \leq k + 1 \), the vertex of \( P_{k+2} \square P_{n+2} \) corresponding to the point of coordinates \((j, i)\) in \( B \). Let \( D = B \cap T \). Near a corner, we modify the projection defined in Section 2, in order to obtain a set \( D' \) with one less vertex on each corner than \( D \) and such that \( B' \subseteq N(D') \). We consider the following four cases:

**Case 1:** The vertex \( x_{1,k} \) is dominated by the vertex \( x_{1,k-1} \) (see Fig. 2).
Observe that by definition of our tiling $T$, exactly one of the two vertices $x_{0,k-1}$ (if $k$ is odd) or $x_{2,k-1}$ (if $k$ is even) belongs to $D$.

If $x_{0,k-1}$ belongs to $D$ then by definition of our tiling, the vertex $x_{2,k+1}$ belongs also to $D$. Moreover $\tilde{x}_{0,k-1} = \tilde{x}_{2,k+1}$, then the set $D' = (D \cup \{ \tilde{x}_{0,k-1} \}) - \{ x_{0,k-1}, x_{2,k+1} \}$ verifies $|D'| = |D| - 1$, and $B' \subseteq N(D')$.

Assume that $x_{2,k-1}$ belongs to $D$. By definition of our tiling, the vertices $x_{0,k-3}, x_{3,k+1}$ and $x_{4,k+1}$ belong to $D$. In this case, we define $D' = (D \cup \{ x_{3,k}, x_{4,k}, x_{1,k-2} \}) - \{ x_{0,k-3}, x_{2,k-1}, x_{3,k+1}, x_{4,k+1} \}$.

Case 2: The vertex $x_{1,k}$ is dominated by the vertex $x_{1,k+1}$.

Observe that by definition of our tiling $T$, exactly one of the two vertices $x_{0,k+1}$ (if $k$ is odd) or $x_{2,k+1}$ (if $k$ is even) belongs to $D$.

If $x_{0,k+1}$ belongs to $D$ then $D' = D - \{ x_{0,k+1} \}$ satisfies $B' \subseteq N(D')$.

Assume that $x_{2,k+1}$ belongs to $D$. By definition of our tiling, the vertex $x_{0,k-1}$ belongs to $D$. We have $\tilde{x}_{0,k-1} = \tilde{x}_{2,k+1}$, then the set $D' = (D \cup \{ \tilde{x}_{0,k-1} \}) - \{ x_{0,k-1}, x_{2,k+2} \}$ verify $|D'| = |D| - 1$, and $B' \subseteq N(D')$.

Case 3: The vertex $x_{1,k}$ is dominated by the vertex $x_{0,k}$.

If $x_{1,k} \notin D$, then by definition of our tiling, $D$ must contain the vertices $x_{0,k-3}, x_{1,k-3}, x_{2,k-1}, x_{3,k-1}, x_{4,k+1}$ and $x_{5,k+1}$. In this case, observe that the set $D' = (D - \{ x_{0,k-3}, x_{0,k}, x_{2,k-1}, x_{4,k+1}, x_{5,k+1} \}) \cup \{ x_{1,k-2}, x_{2,k}, x_{3,k}, x_{4,k} \}$ satisfies $B' \subseteq N(D')$ and $|D'| = |D| - 1$.

If $x_{1,k} \in D$, then by definition of our tiling, $D$ must contain the vertices $x_{0,k-3}, x_{1,k-3}, x_{2,k-3}, x_{3,k-1}, x_{4,k-1}, x_{5,k+1}$ and $x_{6,k+1}$. In this case, observe that the set $D' = (D - \{ x_{0,k-3}, x_{0,k}, x_{2,k-3}, x_{3,k-1}, x_{4,k-1}, x_{5,k+1}, x_{6,k+1} \}) \cup \{ x_{1,k-4}, x_{2,k-2}, x_{3,k-3}, x_{4,k}, x_{5,k} \}$ satisfies $B' \subseteq N(D')$ and $|D'| = |D| - 1$.

Case 4: The vertex $x_{1,k}$ is dominated by the vertex $x_{2,k}$.

If $x_{1,k} \in D$, then by definition of our tiling, $D$ must contain the vertices $x_{0,k-5}, x_{1,k-5}, x_{2,k-3}, x_{3,k-3}, x_{4,k-1}, x_{5,k}, x_{6,k+1}$ and $x_{7,k+1}$. In this case, observe that the set $D' = (D - \{ x_{0,k-5}, x_{2,k-3}, x_{3,k-3}, x_{4,k-1}, x_{5,k}, x_{6,k+1}, x_{7,k+1} \}) \cup \{ x_{1,k-4}, x_{2,k-4}, x_{3,k-2}, x_{4,k-2}, x_{5,k}, x_{6,k} \}$ satisfies $B' \subseteq N(D')$ and $|D'| = |D| - 1$.

If $x_{1,k} \notin D$, then by definition of our tiling, $D$ must contain the vertices $x_{0,k-7}, x_{2,k-5}, x_{3,k-3}, x_{4,k-1}, x_{5,k-1}, x_{6,k-1}, x_{7,k-1}$ and $x_{8,k+1}$. In this case, observe that the set $D' = (D - \{ x_{0,k-7}, x_{2,k-5}, x_{3,k-3}, x_{4,k-1}, x_{5,k-1}, x_{6,k-1}, x_{7,k-1}, x_{8,k+1} \}) \cup \{ x_{1,k-6}, x_{2,k-4}, x_{3,k-4}, x_{4,k-2}, x_{5,k-2}, x_{6,k}, x_{7,k} \}$ satisfies $B' \subseteq N(D')$ and $|D'| = |D| - 1$.

By symmetry and since $k, n \geq 16$, we can complete the proof of the upper bound.

For proving the lower bound, we use a similar technique than one given in [3]. Let $D$ be the subset of vertices of a minimum total dominating set of $P_k \boxplus P_n$ which are in rows labelled 1, 2, $k - 1$ or $k$, and columns labelled 1, 2, $n - 1$ or $n$. A vertex is called deficient if

1. it is in a row labelled 1 or $k$, or in a column labelled 1 or $n$, and it belongs to $D$, or
2. its neighborhood intersects at least two vertices of $D$. 

We obtain a lower bound on the number of deficient vertices in $D'$, the subset of vertices of $D$ which are in the first two rows. Let $f(l)$ be the number of deficient vertices of $D'$ which are in the first $l$ columns. Clearly $f(l+1) \geq f(l)$ for $l = 1, \ldots, n-1$. We claim that

$$f(l + 6) \geq f(l) + 2 \quad \text{for all } l = 1, \ldots, n - 6.$$  \hspace{1cm} (6)

Indeed, let $C_j$, $C_{j+1}$, $C_{j+2}$, $C_{j+3}$, $C_{j+4}$ and $C_{j+5}$ be six consecutive columns of $P_k \square P_n$. If there exists two vertices in $D'$ in the first row then these vertices are deficient and so the assertion holds. So, we may assume that there exists at most one vertex in $D' \cap (C_j \cup C_{j+1} \cup C_{j+2} \cup C_{j+3} \cup C_{j+4} \cup C_{j+5})$ on the first row.

To dominate the vertex $x_{1,j+1}$, assume first that $x_{1,j}$ or $x_{1,j+2}$ belongs to $D'$. In this case, to dominate the vertices $x_{1,j+2}$, $x_{1,j+3}$ and $x_{1,j+4}$, $D'$ must contain the vertices $x_{2,j+2}$, $x_{2,j+3}$ and $x_{2,j+4}$. So, the vertex $x_{2,j+3}$ is deficient.

Now, because of domination of $x_{1,j+4}$, we may assume that $x_{2,j+2}$ and $x_{2,j+4}$ belong to $D'$. Also, because of domination of $x_{1,j+2}$ and $x_{1,j+3}$, we may assume by symmetry that $x_{2,j+2}$ belongs to $D'$, and so the vertex $x_{2,j+3}$ is deficient. Now to dominate $x_{1,j+3}$, $D'$ must contain one more vertex in $\{x_{1,j+2}, x_{1,j+4}, x_{2,j+3}\}$. If $D$ contains a vertex $x$ among $\{x_{1,j+2}, x_{1,j+4}\}$ then $x$ is deficient. Otherwise, $D$ must contain $x_{2,j+3}$; but, in this case, $x_{2,j+3}$ is deficient.

Using (6) and $f(1) \geq 0$, $f(2)$, $f(3)$, $f(4)$, $f(5) \geq 1$, we obtain

$$f(n) \geq \left\lceil \frac{n}{6} \right\rceil + \frac{n + 4}{6}.$$  

Similarly, there are at least $\lfloor n/6 \rfloor + \lfloor (n+4)/6 \rfloor$ deficient vertices in the last two rows, and there are at least $\lfloor k/6 \rfloor + \lfloor (k+4)/6 \rfloor$ deficient vertices in the last and the first two columns. Next, let $D_1$ (respectively, $D_2$, $D_3$ and $D_4$) be the subset of vertices of $D$ which are in the first (resp. last, last and first) two rows (resp. columns, rows and columns) except the last (resp. last, first and first) two columns (resp. rows, columns and rows). Thus, by symmetry, there are at least $2(\lfloor n - 2/6 \rfloor + \lceil n + 2/6 \rceil)$ in $D_1$ and in $D_3$, and at least $2(\lceil (k - 2)/6 \rceil + \lfloor (k+2)/6 \rfloor)$ in $D_2$ and in $D_4$, deficient. Therefore, there are at least

$$2 \left( \left\lfloor \frac{n - 2}{6} \right\rfloor + \left\lceil \frac{n + 2}{6} \right\rceil \right) + 2 \left( \left\lfloor \frac{k - 2}{6} \right\rfloor + \left\lceil \frac{k + 2}{6} \right\rceil \right) \geq \frac{2(n + k)}{3} - 7$$

deficient in $D$.

Moreover, by definition of deficience, it is easy to check that for any grid graph $G$, the total number of vertices dominated by $\gamma_t(G)$ vertices, when $m$ of which are deficient, is at most $4\gamma_t(G) - m$. Hence,

$$|D| > \frac{3kn + 2(k + n)}{12} - 2. \quad \square$$

It seems to be interesting to develop more general techniques than these used in the proof of Theorem 3.5, in order to give ‘good’ bounds of total domination numbers of grid graphs for higher dimensions than 2.
4. Concluding remarks on the algorithmic aspect

Using the algebraic approach developed by Klavžar and Žerovnik in [11,13], we may obtain a constant time algorithm to compute the total domination number of fasciagraphs and rotagraphs which are classes of graphs containing $P_k \square P_k$ and $C_k \square C_n$. This means in time which depends only on the size and structure of a monograph and is independent of the number of monographs. For example, for $P_k \square P_n$ for fixed $k$, the constant depends only on $k$.

The similarity of total domination with domination, suggests the question:

**Question 4.1.** Is it NP-complete to determine, for a given grid graph $G$ and an integer $m$, if there is a total dominating set of $G$ of size $m$ or less?

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References


For further reading