A Novel Computer Virus Model and Its Stability

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Abstract—Computer virus is a malicious code which can cause great damage. A SEIR model is proposed which considered that the newly entered computer in the network has been infected by other ways. It is concerned with the constant immigration, which includes susceptible, exposed and infectious, the threshold and equilibrium of this model are investigated. The locally and globally asymptotical stable results of virus-free and viral equilibrium were being proved. Finally, some numerical examples are given to demonstrate the analytical results.

Index Terms—Computer Virus; Equilibrium; Locally Asymptotically; Simulation

I. INTRODUCTION

Computer virus is a malicious code which including virus, Trojan horses, worm, logic bomb and so on. It is a program that can copy itself and attack other computers. And they are residing by erasing data, damage files, or modifying the normal operation. In recent years, some models of the computer virus is proposed [1-3, 5-12]. The action of computer virus throughout a network can be studied by using epidemiological models for disease propagation [4]. Based on the Kermack and McKendrick SIR classical epidemic model, dynamical models for malicious objects propagation were proposed [5-11]. These dynamical modeling of the spread process of computer virus is an effective approach to the understanding of behavior of computer viruses. On this basis, some effective measures can be posed to prevent infection.

The computer virus has latent ability. An infected computer which is newly entered in the network is all infectious. In this paper, a novel model of computer virus is proposed which considering the constant immigration and computer virus flux into exposed class E is given by aN and flux into infectious class I is given by bN. The susceptible computer which flux into the susceptible class S is given by \((1-aN-bN)\). If \(0 < a + b < 1\), the model has only one viral equilibrium and it is globally asymptotically stable; If \(a + b = 0\), the model has only one virus-free equilibrium if \(R_0 < 1\), and if \(R_0 > 1\) the system has another viral equilibrium and they are globally asymptotically stable; If \(a + b = 1\), the model has only one viral equilibrium and it is globally asymptotically. In this paper, the dynamics of this model has been fully studied through some numerical examples.

This paper is organized as follows. Section 2 formulates a novel computer virus model. Section 3, investigates the stability of the equilibriums. Section 4 some numerical examples are given to present the effectiveness of the theoretic results. Finally, Section 5 summarizes this work.

II. MODEL FORMULATIONS

At any time, a computer is classified as internal and external depending on weather it is connected to internet or not. At that time all to the internet computers are further categorized into four classes: (1) Susceptible computers, that is, uninfected computers and new computers which connected to network; (2) Exposed computers, that is, infected but not yet broken-out; (3) Infectious computers; (4) Recovered computers, that is, virus-free computer having immunity. Let \(S(t), E(t), I(t), R(t)\) denote their corresponding numbers at time \(t\), without ambiguity. \(S(t), E(t), I(t), R(t)\) will be abbreviated as \(S, E, I, R\) respectively. Our assumption on the dynamical transfer of the computers is depicted in Fig. 1, and the model is formulated as following system of differential equations:

\[
\begin{align*}
S' &= (1-a-b)N - \beta SI - dS, \\
E' &= aN + \beta SI - (r + d)E, \\
I' &= bN + rE - (\eta + d)I, \\
R' &= \eta I - dR.
\end{align*}
\]

\(N(t) = S(t) + E(t) + I(t) + R(t).\) (2)

We may see that first three equations in (1) are independent of the fourth equation, and therefore, the
fourth equation can be omitted without loss of generality. Hence, system (1) can be rewritten as
\[
\begin{align*}
S' &= (1-a-b)N - \beta SI - dS, \\
E' &= aN + \beta SI - (r+d)E, \\
I' &= bN + rE - (\eta + d)I.
\end{align*}
\]
where \(N\) denotes the rate at which external computers are connected to the network; \(a\) denotes the rate of the new computer which has been infected computer virus but hasn’t been broken; \(b\) denotes the rate of the new computer which has been infected computer virus and it is has been broken; \(\beta\) denotes the rate at which, when having a connection to one infected computer, one susceptible computer can become exposed; \(r\) denotes the rate of which, the exposed computer is broken and come into the infectious state; \(\eta\) denotes the rate recovery rate of infected computers are cured; \(\mu\) denotes the rate at which one computer is removed from the network. All the parameters are nonnegative.

When having connection to one exposed computer, one susceptible computer can become exposed, and its infectious rate is \(\beta SI\); \(aN\) is the number of the new enter the network has been infected but hasn’t broken; \(bN\) is the number of the new enter the network has been infected;

Moreover, all feasible solutions of the system (3) are bounded and enter the region \(D\), where
\[
D = \{ (S,E,I) \in R^3 | S \geq 0, E \geq 0, I \geq 0, S + E + I \leq \frac{N}{d} \}.
\]
Refer to [12], we analysis the threshold \(R_0\) of the system (3).

Let \(x = (E,I,S)^T\), then, the system (3) can be written as
\[
x' = F(x) - V(x), \quad F(x) = \begin{pmatrix} \beta SI \\ 0 \\ 0 \end{pmatrix}, \\
V(x) = \begin{pmatrix} (r+d)E - aN \\ -rE + (\eta + d)I - bN \\ 0 \end{pmatrix}.
\]
The Jacobian matrices of \(F(x)\) and \(V(x)\) at the \(P^0(1-a-b)N/d, aN, bN\) are
\[
F = DF(P^0) = \begin{pmatrix} 0 & \beta N \\ 0 & 0 \end{pmatrix}, \\
V = \begin{pmatrix} r + d & 0 \\ -r & \eta + d \end{pmatrix}.
\]
then
\[
V^{-1} = \frac{1}{(\eta + d)(r+d)} \begin{pmatrix} \eta + d & 0 \\ r & r + d \end{pmatrix}.
\]
Thus
\[
R_0 = \frac{\beta r}{d(\eta + d)(r+d)}.
\]
Especially, when \(a + b = 0\), obviously \(a = b = 0\), the system (3) can be revised as
\[
\begin{align*}
S' &= N - \beta SI - dS, \\
E' &= \beta SI - (r+d)E, \\
I' &= rE - (\eta + d)I.
\end{align*}
\]
And there
\[
R_0 = \frac{\beta r N}{d(\eta + d)(r+d)}.
\]
For the system (4), if \(R_0 \leq 1\), there always exists the virus-free equilibrium is \(P^0_1(S_1^0,0,0) = P^0(\frac{N}{d},0,0)\); if \(R_0 > 1\), there also exists an virus-free equilibrium is \(P^0_1(\frac{N}{d},0,0)\) and a viral equilibrium is
\[
\begin{align*}
P^*_1 &= \frac{(r+d)(\eta + d)}{\beta r}, (\eta + d)I^*_1, \\
& \text{there}
\end{align*}
\]
When \(a + b = 1\), the system (3) can be revised as
\[
\begin{align*}
S' &= -\beta SI - dS, \\
E' &= aN + \beta SI - (r+d)E, \\
I' &= bN + rE - (\eta + d)I.
\end{align*}
\]
For the system (5), there has only one viral equilibrium
\[
P^*_1(0,aN/\frac{(r+d)}{(r+d)(\eta + d)}), \quad bN(r+d) + aNr.
\]
When \(0 < a + b < 1\), the system (2.3) has only a viral equilibrium
\[
P^*_1(S_1^*, \frac{aN + \beta S_1^*I_1}{(r+d)}, \frac{[a + br + bd]N}{(r + d)(\eta + d)(r+d) - \beta r S_1^*},)
\]
Let
\[
G(S_1^*) = (1-a-b)N - \beta S_1^*I_1 - dS_1^*.
\]
when $S_t \in (0, \frac{(1-a-b)N}{d})$, discuss the roots of equation $G(S_t) = 0$.

For system (3), we have

$$G'(S_t) = -d - \beta I_t^*$$

$$= -d - \beta \frac{[ar + br + bd]N}{(\eta + d)(r + d) - \beta r S_t^*}$$

$$= -d - \beta \frac{[ar + br + bd]N, \beta r S_t^*}{[(\eta + d)(r + d) - \beta r S_t^*]} < 0,$$

thus we get $G(S_t)$ monotone decreasing in $(0, \frac{(1-a-b)N}{d})$.

Case (1): If $R_0 = \frac{\beta r (1-a-b)N}{d(\eta + d)(r + d)} < 1$, from (6), we get

$$G(0) = (1-a-b)N > 0,$$

$$G\left(\frac{(1-a-b)N}{d}\right) = -\beta S_t^* I_t^*,$$

there

$$I_t^* = \frac{[ar + br + bd]N}{(\eta + d)(r + d) - \beta r S_t^*} < 0,$$

thus

$$G\left(\frac{(1-a-b)N}{d}\right) > 0.$$

$G(S_t) = 0$ has no real root in $(0, \frac{(1-a-b)N}{d})$ if $R_0 < 1$.

Case (2): If $R_0 = \frac{\beta r (1-a-b)N}{d(\eta + d)(r + d)} > 1$,

$$I_t^* = \frac{[ar + br + bd]N}{(\eta + d)(r + d) - \beta r S_t^*} > 0,$$

thus

$$G(0) = (1-a-b)N > 0,$$

$$G\left(\frac{(1-a-b)N}{d}\right) < 0.$$

For that $G(S_t)$ monotone decreasing in $(0, \frac{(1-a-b)N}{d})$, the equation $G(S_t) = 0$ has only one positive root if $R_0 > 1$.

From the case, the system (3) has only one viral equilibrium $P_1(S_t^*, E_t^*, I_t^*) = \frac{aN + \beta S_t^* I_t^*}{(r + d)} \frac{[ar + br + bd]N}{(\eta + d)(r + d) - \beta r S_t^*}$ if $R_0 > 1$.

III. STABILITY OF THE EQUILIBRIUMS

Theorem 3.1.1. $P_1^0$ is locally asymptotically stable if $R_0 < 1$. Whereas $P_1^0$ is unstable if $R_0 > 1$.

Proof. The Jacobin matrix of system (2.4) about $P_1^0$ is given by

$$J_1^0 = \begin{bmatrix} -d & 0 & -\beta S_t^0 \\ 0 & -(r + d) & \beta S_t^0 \\ 0 & r & -(\eta + d) \end{bmatrix},$$

which equals to

$$f(\lambda) = (\lambda + d)(\lambda^2 + a_1 \lambda + a_2 - \beta S_t^0 r) = 0, \quad (7)$$

where

$$a_1 = (r + d) + (\eta + d),$$

$$a_2 = (r + d)(\eta + d).$$

then, Eq. (7) has negative real roots roots

$$\lambda_1 = -d < 0,$$

$$\lambda_2 = -a_1 - \sqrt{a_1^2 - 4a_2(1 - R_0)} < 0,$$

$$\lambda_3 = -a_1 - \sqrt{a_1^2 - 4a_2(1 - R_0)}.$$

when $R_0 \leq 1$, $\lambda_1 < 0$ and When $R_0 > 1$, $\lambda_1 > 0$.

Then there are no positive real roots of (7), thus $P_1^0$ is a locally asymptotically stable equilibrium if $R_0 \leq 1$, and $P_1^0$ is unstable if $R_0 > 1$.

The proof is completed.

Theorem 3.1.2. $P_1^0$ is globally asymptotically stable with respect to $D$ if $R_0 < 1$.

Proof. Let $L = \frac{r}{(\eta + d)} E' + I'$, obviously $L > 0$.

Thus

$$L' = \frac{r}{(\eta + d)} E' + I'$$

$$= \frac{r}{(r + d)} \frac{[\beta S_t (r + d)E] + rE - (\eta + d)I}{d(r + d)}$$

$$= \frac{\beta N r - (r + d)(\eta + d)}{(\eta + d)(R_0 - 1)}$$

$$< 0.$$ The proof is completed.

Theorem 3.1.3. $P_1^0$ is globally asymptotically stable.

Proof. Let $L = S$, obviously $L > 0$.

Then

$$L' = -\beta S_t - dS < 0.$$ The proof is completed.

Theorem 3.1.4 The viral equilibrium $P_1^*(S_t^*, E_t^*, I_t^*)$ and $P_1^*(S_t^*, E_t^*, I_t^*)$ is locally asymptotically stable.

Proof. Denote $P_1^*(S_t^*, E_t^*, I_t^*)$ and $P_1^*(S_t^*, E_t^*, I_t^*)$ as $P^*$.

The Jacobin matrix of the system (3) about $P^*$ is given by
J' = \begin{bmatrix}
-\beta I^r - d & 0 & -\beta S^s \\
\beta I^r & -(r + d) & \beta S^s \\
0 & r & -(\eta + d)
\end{bmatrix},
which equals is
\[ f(\lambda) = \lambda^3 + \lambda^2 A_2 + \lambda A_1 + A_0 = 0, \quad (8) \]
where
\[
A_0 = 1,
A_1 = ([\beta I^r + d] + [(r + d) + (\eta + d)]),
A_2 = ([r + d] + (\eta + d))([\beta I^r + d] + [(r + d)(\eta + d) - \beta S^r] r),
A_3 = (r + d)(\eta + d)\beta I^r + d[(r + d)(\eta + d) - \beta S^r r] > 0,
\]
then
\[ A_1 A_2 - A_3 > 0. \]
According to the Hurwitz criterion, all roots of Eq. (8) have negative real parts. Thus the equilibrium \( P^*_1 \) is locally asymptotically stable.

Case (2): When \( 0 < a + b < 1 \) and \( R_0 > 1 \),
\[
A_0 = 1,
A_1 = ([\beta I^r + d] + [(r + d) + (\eta + d)]),
A_2 = ([r + d] + (\eta + d))([\beta I^r + d] + [(r + d)(\eta + d) - \beta S^r] r) > 0,
A_3 = (r + d)(\eta + d)\beta I^r + d[(r + d)(\eta + d) - \beta S^r r] > 0,
\]
then
\[ A_1 A_2 - A_3 > 0. \]
According to the Hurwitz criterion, all roots of Eq. (8) have negative real parts. Thus the equilibrium \( P^*_1 \) is locally asymptotically stable.

The proof is completed.

Now, let us examine the global stability of \( P^* \) with respect to the \( D \) by means of a classical geometric approach [13]. For our proposed, the following obvious result will be useful.

**Lemma 3.1.** If \( R_0 > 1 \), the system (3) is uniformly persistent, i.e., there exists \( c_i > 0 \) (independent of initial conditions), such that
\[
\liminf S(t) > c_i, \liminf E(t) > c_i, \liminf I(t) > c_i.
\]

**Remark 1.** We have proved that the virus-free equilibrium \( P^*_0 \) is unstable if \( R^*_0 > 1 \) when \( a + b = 0 \) and the system (3) has no virus-free equilibrium when \( 0 < a + b < 1 \). Furthermore, the instability of \( P^*_0 \), together with \( P^*_1 \), \( P^*_2 \) in \( \partial D \) (\( \partial D \) denotes the boundary of \( D \)), imply the uniform persistence of the satiate variable. The uniform persistence of system (3) is to show the existence of a compact absorbing set in \( \text{int} D \) (\( \text{int} D \) denotes the interior of \( D \)), which is a necessary condition for proving the global stability by using the geometric approach.

**Theorem 3.1.4.** \( P^* \) is globally asymptotically stable if \( R^*_0 > 1 \).

**Proof.** The second compound matrix of the Jacobin matrix \( J' \) can be calculated as follows [14-16]:
\[
J'_{22} = \begin{bmatrix}
-(\beta I^r + d) & 0 & \beta S^s \\
\beta I^r & -(r + d) - (\eta + d) & 0 \\
0 & \beta I^r & -(r + d) - (\eta + d)
\end{bmatrix}
\]
Set \( P \) as the following diagonal matrix
\[
P(x) = (1, \frac{E^r}{T}, \frac{E^s}{T}).
\]
Denote
\[
P_x P^{-1} = \text{diag}(0, \frac{E^r}{T}, \frac{I^r}{T}, \frac{I^s}{I}).
\]
Therefore, the matrix \( B = P_x P^{-1} + AJ_{22}A^{-1} \) can be written in the following block form:
\[
B = \begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{bmatrix},
\]
with
\[
B_{11} = (-\beta I^r - d) - (r + d),
B_{12} = \beta S^s \frac{I^r}{T},
B_{21} = \left(\frac{E^r}{E} - \frac{I^r}{T}\right)(1,1),
B_{22} = \left[\begin{array}
E^r - \frac{I^r}{T} & -\beta I^r - d - (r + d) \\
\beta I^r & \frac{E^r}{E} - \frac{I^r}{T} - (r + d) - (\eta + d)
\end{array}\right],
\]
thus
\[
g_1 = \mu(B_{11}) + |B_{12}|, \quad g_2 = \mu(B_{22}) + |B_{21}|, \quad (9)
\]
there
\[
\mu_1(B_{11}) = (-\beta I^r - d) - (r + d),
\mu_1(B_{22}) = \max\left\{\frac{E^r}{I} - \frac{I^r}{T} - (\beta I^r - d) - (r + d) + \beta I^r, \frac{E^r}{E} - \frac{I^r}{I} - (r + d) - (\eta + d)\right\}
\]
\[
\mu_2(B_{22}) = \frac{E^r}{E} - \frac{I^r}{T} - (r + d) - (\eta + d).
\]
The vector norm \( \| \| \) in \( R^3 \) is chose as
\[
\|a, v, w\| = \max\{|a|, |v|, |w|\}.
\]
The Lozinski measure \( \mu(B) \) with respected to \( \| \| \) as follows (17)
\[
\mu(B) \leq \sup\{g_1, g_2\},
\]
there
\[
g_1 = (-\beta I^r - d) - (r + d) + \left|\frac{I^r}{E} \beta S^s \right|(1,1),
g_2 = \frac{E^r}{E} - \frac{I^r}{I} - (\eta + d) - d + \left|\frac{E^r}{E} r(0,0)\right|, \quad (10)
\]
Figure 2. Dynamical behavior of system (2.4) with $a+b=0$. Time series of $S(t), E(t), I(t)$ with $R_0 < 1$.

Figure 3. Dynamical behavior of system (2.4) with $a+b=0$. Time series of $S(t), E(t), I(t)$ with $R_0 > 1$.

Figure 4. Dynamical behavior of system (5) with $a+b=1$. Time series of $S(t), E(t), I(t)$.
Figure 5. Dynamical behavior of system (5) with $0 < a + b < 1$, Time series of $S(t), E(t), I(t)$ if $a = 0.3, b = 0.3, \beta = 0.0001, N = 100, r = 0.8, d = 0.01, n = 0.8$ and then $R_0 = 4.9875 > 1$.

Figure 6. Dynamical behavior of system (5) with $0 < a + b < 1$, Time series of $S(t), E(t), I(t)$ if $a = 0.1, b = 0.1, \beta = 0.0001, N = 100, r = 0.8, d = 0.001, n = 0.8$ and then $R_0 = 9.9750 > 1$.

Figure 7. Dynamical behavior of system (5) with $0 < a + b < 1$, Time series of $S(t), E(t), I(t)$ if $a = 0.1, b = 0.1, \beta = 0.0001, N = 100, r = 0.6, d = 0.001, n = 0.6$ and then $R_0 = 13.2890 > 1$. 
From the system (2.3), we find that
\[
\frac{I}{E} \beta S^* = \frac{E'}{E} - \frac{aN}{E} + (r + d),
\]
\[
\frac{I'}{T} = \frac{bN}{T} + \frac{rE}{I} - (\eta + d).
\] (11)
Thus
\[
g_1 = \frac{E'}{E} - \frac{aN}{E} - (\beta t' + d),
\]
\[
g_2 = \frac{bN}{T} - \frac{rE}{I} - d.
\] (12)
Relations (9)-(12) imply
\[
\mu(B) \leq \frac{E'}{E} - d,
\]
thus
\[
\frac{1}{I} \int_{\tau^0}^{t} \mu(B) d\tau \leq \frac{1}{I} \int_{\tau^0}^{t} \left( \frac{E'}{E} - d \right) d\tau = \frac{1}{I} \ln \left( \frac{E(t)}{E(0)} \right) - \frac{d}{2}. (13)
\]
If \( R_0 > 1 \), then the virus-free equilibrium is unstable by the Theorem 3.1.2. Moreover, the behavior of the local near \( D_0 \) as describe in Theorem 3.1.2 implies that the system (3) is uniformly persistent in \( D \), i.e. there exists a constant \( c_1 > 0 \) and \( T > 0 \), such that \( t > T \) implies
\[
\liminf_{t \to \infty} S(t) > c_1,
\]
\[
\liminf_{t \to \infty} E(t) > c_1,
\]
\[
\liminf_{t \to \infty} I(t) > c_1.
\]
And
\[
\liminf_{t \to \infty} [1 - S(t) - E(t) - I(t)] > c_1.
\]
For all \( ((S(0), E(0), I(0)) \in D \) [18, 19]
\[
\bar{q} = \limsup_{t \to \infty} \sup_{x \in \Omega} \frac{1}{I} \int_{\tau^0}^{t} \mu(B) d\tau \leq \frac{-d}{2} < 0. \quad (14)
\]
The claimed result follows by combining Theorem 3.1.1, Lemma 3.1 and the negativity of \( \bar{q} \).

The proof is completed.

IV. SIMULATION

Now, we would analysis the locally asymptotically stable of the equilibriums through some numerical examples.

Case 1: For the system (4), if \( R_0 < 1 \) , there exists the virus-free equilibrium \( P_1^0 \). Let \( a = 0, b = 0, \beta = 0.0001, N = 10, r = 0.8, d = 0.01, n = 0.8 \), and then \( R_0 = 0.0122 < 1 \). Fig. 2 shows the solution of system (4) if \( R_0 < 1 \).

Case 2: For the system (4), if \( R_0 > 1 \), there exists the viral equilibrium \( P_1^* \). Let \( a = 0, b = 0, \beta = 0.001, N = 10, r = 0.5, d = 0.01, n = 0.5 \), and then \( R_0 = 19.92 > 1 \). Fig. 3 shows the solution of system (4) if \( R_0 > 1 \).

Case 3: For the system (5), there exists the viral equilibrium \( P_2^* \). Let \( a = 0.5, b = 0.5, \beta = 0.01, r = 0.05, d = 0.001, n = 0.1 \). Fig. 4 shows the solution of system (5) if \( a+b = 0 \).

Case 4: For the system (3), there exists the viral equilibrium \( P_3^* \). Fig. 5-Fig. 7 shows the solution of system (3) if \( 0 < a+b < 1 \). From the Fig. 2, We can see that the virus-free equilibrium \( P_1^0 \) of system (4) is locally asymptotically stable if \( R_0 < 1 \).

From the Fig. 3, We can see that the viral equilibrium \( P_1^* \) of system (4) is locally asymptotically stable \( R_0 > 1 \).

From the Fig. 4, We can see that the viral equilibrium \( P_2^* \) of the system (5) is locally asymptotically stable.

From the Fig. 5- Fig. 7, We can see that the viral equilibrium \( P_3^* \) of the system (3) is locally asymptotically stable.

For the system (3), from the Fig. 5 and Fig. 6, we can see that the number of computer virus is not affected by the parameters \( a \) and \( b \). If we’re keeping \( a \) and \( b \) the same, from the Fig. 6 and Fig. 7, we can see that the number of computer virus actually decreases as we reduce the \( R_0 \).

V. CONCLUSION

We considering the fact, the computers which is newly entered in the network computer has been infected by other ways such as removable media, a novel computer virus model is proposed. Denotes \( a \) is the rate of the new computer which has been infected computer virus but hasn’t been broken; \( b \) denotes the rate of the new computer which has been infected computer virus and it is has been broken. If \( 0 < a+b < 1 \), the system (3) has only one viral equilibrium and it is locally asymptotically stable; If \( a+b = 0 \), the system (3) can rewritten as system (4) and it has only one virus-free equilibrium if \( R_0 < 1 \), and if \( R_0 > 1 \) the system has a viral equilibrium and which are locally asymptotically stable; If \( a+b = 1 \), the System (3) can be rewritten as system (5) and it has only one viral equilibrium which is locally asymptotically. In this paper, the dynamics of this model has been fully studied through some numerical examples.

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