\( \mathcal{H}_2 \) Guaranteed Cost Computation of Discretized Uncertain Continuous-time Systems∗

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 abst ract—This paper proposes a new discretization technique with constant sampling time for time-invariant systems with uncertain parameters belonging to a polytopic domain. The aim is to provide an equivalent discrete-time representation of the continuous-time system whose \( \mathcal{H}_2 \) guaranteed cost is an upper bound for the \( \mathcal{H}_2 \) worst case norm of the original system. The resulting discrete-time model is described in terms of homogeneous polynomial matrices obtained by Taylor series expansion of degree \( \ell \). The discretization residual error, associated to the chosen approximation degree, is represented by additive norm-bounded uncertain terms. As a second contribution, new linear matrix inequality (LMI) relaxations for the computation of \( \mathcal{H}_2 \) guaranteed costs for discrete-time systems with polynomial dependence on the uncertain parameter and additive norm-bounded uncertainties are proposed. A numerical experiment shows that the \( \mathcal{H}_2 \) costs of the discretized system become tighter to the continuous-time ones as the order in the Taylor series expansion, the degrees in the Lyapunov function and the Pólya’s relaxation level increase.

I. INTRODUCTION

Any realistic strategy of modeling or control must take into account the uncertainties in the physical processes arising, for instance, from hidden and unmodeled dynamics, parameter variation, sensor noises, external perturbations and actuator constraints [1], [2]. Owing to the presence of uncertainties, the evaluation of \( \mathcal{H}_\infty \) and \( \mathcal{H}_2 \) norms of uncertain time-invariant linear systems is one of the most important problems in control systems [3]–[5]. The \( \mathcal{H}_2 \) norm, usually related to energy, was extensively studied in the 1960’s in the linear quadratic Gaussian optimal control problem. Until today, the \( \mathcal{H}_2 \) norm is widely recognized as one of the most representative costs used to derive a feedback control law [6], [7].

Additionally, with the advance of computers and network architecture, digital controllers [8] have been largely employed due to their higher reliability, lower cost, greater flexibility and better performance when compared to analog devices [2]. Thus, many researches have been made in system modeling and control design using discrete-time based methodologies. As most of the signals of interest, like command inputs, actuator outputs, sensor readings, are generally continuous-time, discretization procedures are required. As far as the authors known, there are few methods that treat the discretization of uncertain systems in the literature, as for instance [2], [9]. However, those methods are only approximations and the exact discretization of uncertain systems is still an open problem, mainly due to the difficulty in dealing with the exponential of uncertain matrices. Some of the existing discretization methods use a first order Taylor approximation [10]–[13], which generates a polytope with affine dependence on the parameters, leading, usually, to inaccurate discrete-time models for large values of sampling time. In all cases, there is no guarantee that the design of a digital controller will stabilize or provide the prescribed level of \( \mathcal{H}_2 \) disturbance attenuation for the continuous-time closed-loop system.

The aim of this paper is to propose a systematic procedure to obtain a discretized model from an uncertain continuous-time polytopic system. Additionally to the approach presented in [14], the main characteristic of the discrete-time model is to present an \( \mathcal{H}_2 \) guaranteed cost that is an upper bound to the \( \mathcal{H}_2 \) worst case norm of the original system. This result can be viewed as a first step for the implementation of digital \( \mathcal{H}_2 \) controllers in the context of Networked Control Systems (NCS). To obtain the discretized model, an extension of the Taylor series expansion of an arbitrary degree \( \ell \) is considered. A constant sampling period is assumed, ignoring the inter-sampling information [15]. The resulting discrete-time model is composed by homogeneous polynomially parameter-dependent matrices of degree \( \ell \) with parameters lying in the unit simplex plus additive norm-bounded uncertain terms. The bounds, related to the residual error of the approximation, depends on the degree \( \ell \) of the series expansion, on the sampling period and on the original continuous-time polytopic domain. Moreover, linear matrix inequality (LMI) conditions are proposed for the computation of \( \mathcal{H}_2 \) guaranteed costs for discrete-time linear systems with matrices depending polynomially on the uncertain parameters plus norm-bounded uncertain terms. The conditions are solved in terms of homogeneous polynomial matrices of arbitrary degree and Pólya’s relaxations. The \( \mathcal{H}_2 \) bounds computed for the discrete-time model become closer to the \( \mathcal{H}_2 \) guaranteed costs of the original uncertain continuous-time system as the degree \( \ell \) of the approximation increases. A numerical example illustrates the advantages of the proposed discretization technique to provide more precise approximations. For a given discretization level, the conservativeness of the obtained results for \( \mathcal{H}_2 \) guaranteed cost computation can be reduced by increasing the degrees

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of Lyapunov functions and the level of Pólya’s relaxations, at the price of a larger computational effort.

The remainder of the paper is structured as follows. Section II presents the notation and preliminary results related to the discretization procedure. Sections III and IV introduce the main results for, respectively, the discretization technique and the $\mathcal{H}_2$ guaranteed cost computation. Section V provides a numerical example. Section VI summarizes the paper and discusses future works.

II. Preliminaries

Consider the continuous-time uncertain linear system

$$\begin{align*}
\dot{x}(t) &= E(\alpha)x(t) + F(\alpha)w(t) \\
z(t) &= G(\alpha)x(t)
\end{align*}$$

(1)

where $x(t) \in \mathbb{R}^{n_x}$ is the system state, $w(t) \in \mathbb{R}^{n_w}$ is the exogenous perturbation and $z(t) \in \mathbb{R}^{n_z}$ is the output. Matrices $E(\alpha) \in \mathbb{R}^{n_x \times n_x}$, $F(\alpha) \in \mathbb{R}^{n_z \times n_w}$ and $G(\alpha) \in \mathbb{R}^{n_z \times n_x}$ are uncertain and belong to a polytopic domain, i.e., they can be written as a convex combination of the $N$ known vertices

$$(E,F,G)(\alpha) = \sum_{i=1}^{N} \alpha_i(E_i,F_i,G_i)$$

(2)

and $\alpha$ is a time-invariant parameter vector belonging to the unit simplex, given by

$$\Lambda_N = \left\{ (\zeta_1, \ldots, \zeta_N) \in \mathbb{R}^N : \sum_{i=1}^{N} \zeta_i = 1, \ z_i \geq 0, \ i = 1, \ldots, N \right\}.$$ 

Assume that system (1) is stable and the perturbation is such that $w(t) = e_i\delta(t)$ where $e_i \in \mathbb{R}^{n_w}$, $i = 1, \ldots, n_w$, are vectors with the $i$-th entry equal to 1 and all the others zeroed. Therefore, denoting by $z_i(t)$ the corresponding output trajectory, the $\mathcal{H}_2$ norm for the continuous-time system (1) can be expressed for a fixed $\alpha \in \Lambda_N$ as

$$\|H(s)\|_2^2 = \sum_{i=1}^{n_w} \int_0^\infty z_i(t)z_i(t)dt = \left\| G(\alpha)(sI - E(\alpha))^{-1}F(\alpha) \right\|^2_2.$$ 

(3)

To prove the main results, the following lemma is needed.

**Lemma 1** Given a scalar $\lambda > 0$ and matrices $M$ and $N$ of compatible dimensions, then

$$MN + N'M' \leq \lambda MM' + \lambda^{-1}NN'$$

since

$$\left( \sqrt{\lambda}M' - \frac{1}{\sqrt{\lambda}}N \right)' \left( \sqrt{\lambda}M' - \frac{1}{\sqrt{\lambda}}N \right) \geq 0.$$ 


III. DISCRETIZATION OF UNCERTAIN SYSTEMS

This section presents a new discretization procedure, based on Taylor series expansion, for a polytopic uncertain continuous-time system sampled with a constant period. The resulting discrete-time system has the same input matrix as the continuous-time model, $B(\alpha) = F(\alpha)$, and state space matrices $A(\alpha)$ and $C(\alpha)$ are represented by homogeneous polynomials of degree $\ell$ on $\alpha \in \Lambda_N$ plus a norm-bounded term. The additional terms, related to the residue of the approximation, depend on the degree $\ell$ of the series expansion, on the sampling period and on the uncertain parameter as well. Note that, differently from [17], that applies only to precisely known systems and needs to determine the output matrix $C$, all the development is made in terms of the product $C(\alpha)'C(\alpha)$. As a matter of fact, the determination of $C(\alpha)$ remains a challenging problem.

Therefore, the matrices of (5) can be written as

$$\begin{align*}
A(\alpha) &= A_t(\alpha) + \Delta A_t(\alpha), \\
B(\alpha) &= F(\alpha), \\
C(\alpha)'C(\alpha) &= C_t(\alpha) + \Delta C_t(\alpha)
\end{align*}$$

(7)

Following the lines presented in [16], [17], for a given $\alpha \in \Lambda_N$, this formulation allows to obtain the $\mathcal{H}_2$ norm of (1) by computing the $\mathcal{H}_2$ norm of (4), as shown below.

$$\|H(z)\|_2^2 = \left\| C(\alpha)(zI - A(\alpha))^{-1}B(\alpha) \right\|^2_2 = \text{Tr} \left( B(\alpha)' \sum_{k=0}^{\infty} (A(\alpha))^{-k}C(\alpha)'C(\alpha)A(\alpha)^kB(\alpha) \right)$$

$$= \text{Tr} \left( F(\alpha)' \sum_{k=0}^{\infty} e^{(\alpha)'kT} \right)$$

$$\times \left[ \int_{kT}^{(k+1)T} e^{(\alpha)'(t-kT)}G(\alpha)'G(\alpha)e^{(\alpha)'(t-kT)}dt \right] e^{(\alpha)'T}F(\alpha)$$

$$= \text{Tr} \left( \sum_{k=0}^{\infty} \int_{kT}^{(k+1)T} F(\alpha)'e^{(\alpha)'(t-kT)}G(\alpha)'G(\alpha)e^{(\alpha)'(t-kT)}F(\alpha)dt \right)$$

$$= \text{Tr} \left( \int_{0}^{\infty} F(\alpha)'e^{(\alpha)'t}G(\alpha)'G(\alpha)e^{(\alpha)'t}F(\alpha)dt \right)$$

$$= \left\| G(\alpha)(sI - E(\alpha))^{-1}F(\alpha) \right\|^2_2 = \|H(s)\|_2^2.$$ 

(6)
with
\[ A_\ell(\alpha) = \sum_{j=0}^{\ell} \frac{E(\alpha)^j}{j!} T^j, \]  
(8)

\[ C_\ell(\alpha) = \sum_{j=0}^{\ell} \frac{T^{r+q-1}}{s! \  q! \ (s + q - 1)} \times (E(\alpha))^{r-1} G(\alpha) G(\alpha)(E(\alpha))^{q-1} \]  
(9)

and
\[ \Delta A_\ell(\alpha) = e^{E(\alpha)T} - A_\ell(\alpha) \]  
(10)

\[ \Delta C_\ell(\alpha) = \int_0^T e^{E(\alpha)s} G(\alpha) G(\alpha)e^{E(\alpha)ds} \]  
(11)

where \( \Delta A_\ell(\alpha) \) and \( \Delta C_\ell(\alpha) \) are the residues of the \( \ell \)-order Taylor series expansion, and \( T \) is the sampling time.

Since \( E(\alpha) \in \mathbb{R}^{n_1 \times n_2} \) and, in the matrix case, products in multinomial series are non commutative, one has
\[
E(\alpha)^q = \left( \sum_{i=1}^{N} \alpha_i E_i \right)^q = \sum_{p \in \mathcal{P}(q)} \prod_{i=1}^{q} \alpha_{p_i} E_{p_i} \]  
= \sum_{p \in \mathcal{P}(q)} \alpha_{p_1} E_{p_1} \cdots \alpha_{p_q} E_{p_q} \]  
= \sum_{p \in \mathcal{P}(q)} \alpha_{p} E_{p} = E_{p_1} \cdots E_{p_q} \]  
= \sum_{k \in \mathcal{K}(q)} \alpha_{k} \sum_{p \in \mathcal{R}(k)} E_{p} \]  
(12)

where \( \alpha_{k} = \alpha_{k_1} \alpha_{k_2} \cdots \alpha_{k_N} \), \( k = (k_1 k_2 \cdots k_N) \), \( \alpha_{p} = (\alpha_{p_1}, \alpha_{p_2}, \ldots, \alpha_{p_q}) \), \( p = (p_1 p_2 \cdots p_q) \), for \( q \in \mathbb{N} \), \( \mathcal{K}(q) \) is the set of \( N \)-tuples obtained as all possible combinations of non-negative integers \( k_i \), \( i = 1, \ldots, N \), such that \( k_1 + k_2 + \cdots + k_N = q \), that is
\[
\mathcal{K}(q) \triangleq \left\{ k = (k_1 k_2 \cdots k_N) \in \mathbb{N}^N : \sum_{j=1}^{N} k_j = q, \ k_j \geq 0 \right\},
\]
\( \mathcal{P}(q) \) is the set of \( q \)-tuples obtained as all possible combinations of non-negative integers \( p_i \), \( i = 1, \ldots, q \), such that \( p_i \in \{1, \ldots, N\} \), that is
\[
\mathcal{P}(q) \triangleq \left\{ p = (p_1 \cdots p_q) \in \mathbb{N}^q : p_i \in \{1, \ldots, N\}, \ i = 1, \ldots, q \right\}
\]
and \( \mathcal{R}(k) \), \( k \in \mathcal{K}(q) \), is the subset of all \( q \)-tuples \( p \in \mathcal{P}(q) \) such that the elements \( j \) of \( p \) have multiplicity \( k_j \), for \( j = 1, \ldots, N \), that is
\[
\mathcal{R}(k) \triangleq \left\{ p = (p_1 \cdots p_q) \in \mathbb{N}^q : m_p(j) = k_j, \ j = 1, \ldots, N \right\}
\]
where \( m_p(j) \) denotes the multiplicity of the element \( j \) in \( p \).

To illustrate the definitions, consider \( q = 4 \) and \( N = 2 \). In this case, the set \( \mathcal{K}(q) \) is given by
\[
\mathcal{K}(4) = \{ (40), (31), (22), (13), (04) \}
\]
and, for instance, choosing \( k = 31 \), \( \alpha_{k} = \alpha_{1}^{3} \alpha_{2} \), the set \( \mathcal{R}(k) \) is
\[
\mathcal{R}(31) = \{ (1112), (1121), (1211), (2111) \}.
\]
Finally, one has,
\[
\sum_{p \in \mathcal{R}(q)} E_p = E_1^r E_2 + E_1 E_2^r E_1 + E_1 E_2 E_1^r + E_2 E_1^r.
\]
(15)

By definition, for \( N \)-tuples \( k \) and \( k' \), one writes \( k \geq k' \) if \( k_i \geq k'_i \), \( i = 1, \ldots, N \). Operations of summation \( k + k' \) and subtraction \( k - k' \) (whenever \( k' \leq k \)) are defined componentwise. Consider, also, the \( N \)-tuple \( \mathbf{e}_i \in \mathbb{R}^N \) with the \( i \)-th component fixed as 1 and the others equal to zero.

Using the definitions presented above, one can write (8) as
\[
A_\ell(\alpha) = I + TE(\alpha) + \frac{T^2}{2} E(\alpha)^2 + \cdots + \frac{T^\ell}{\ell!} E(\alpha)^\ell
\]
\[
= \left( \sum_{i=1}^{N} \alpha_i \right) I + T \left( \sum_{i=1}^{N} \alpha_i \right) E(\alpha) + \cdots + \frac{T^\ell}{\ell!} E(\alpha)^\ell
\]
\[
= \sum_{k \in \mathcal{K}(\ell)} \alpha^{k} \left( \frac{T^{\ell}}{\ell!} \sum_{p \in \mathcal{R}(k)} E_p \right)
\]
\[
\triangleq \sum_{k \in \mathcal{K}(\ell)} \alpha^{k} A_k
\]
(16)

and matrix (9) can also be written as
\[
C_\ell(\alpha) = \sum_{k \in \mathcal{K}(\ell)} \alpha^{k} \left( T \sum_{k' \in \mathcal{K}(\ell-1)} \sum_{p \in \mathcal{R}(k-k')} \frac{(2\ell-2)!}{k!} G_i G_j \right)
\]
\[
+ \cdots + \frac{T^{\ell+q-1}}{s! \  q! \ (s + q - 1)} \times \sum_{k \in \mathcal{K}(\ell-1)} \sum_{p \in \mathcal{R}(k-k')} \frac{(2\ell-q-s)!}{k!} E_i G_j E_p
\]
\[
+ \cdots + \frac{T^{2\ell-1}}{(2\ell-1)!} \sum_{k \in \mathcal{K}(\ell-1)} \sum_{p \in \mathcal{R}(k-k')} \frac{(2\ell-q-s)!}{k!} G_i E_j E_p
\]
\[
= \sum_{k \in \mathcal{K}(\ell)} \alpha^{k} C_k
\]
where \( k! = k_1! k_2! \cdots k_N! \) and \( A_k \) and \( C_k \) are the coefficients of the discretized system polynomial matrices \( A_\ell(\alpha) \) and \( C_\ell(\alpha) \).

This methodology is an approximation of degree \( \ell \) from the discretized model described by (5). For a precisely known system, this discrete-time representation using a sufficiently large degree of approximation can recover the \( \mathcal{H}_2 \) performance index, as presented in [17]. In order to estimate how
important are the neglected terms, bounds $\delta_A$ and $\delta_C$ of the uncertainties given by (10) and (11) can be computed as follows
\[
\delta_A = \sup_{\alpha \in \Lambda_{\alpha}} ||\Delta A(\alpha)||, \\
\delta_C = \sup_{\alpha \in \Lambda_{\alpha}} ||\Delta C(\alpha)||
\]
(17)
by performing, for instance, a search in a grid of values of $\alpha \in \Lambda_{\alpha}$. Note that this strategy does not produce upper bounds to the discretization errors, but the estimates can be tight enough to the actual values if a fine grid is used.

IV. $\mathcal{H}_2$ NORM COMPUTATION

This section presents a new LMI based condition for $\mathcal{H}_2$ guaranteed cost computation of polynomially parameter-dependent discrete-time systems with additive norm-bounded uncertainties.

Theorem 1 If there exist symmetric positive-definite matrices $P_k \in \mathbb{R}^{n_k \times n_k}$, $k \in \mathcal{K}(g)$, with $g \in \mathbb{N}$, positive scalar variables $\lambda_A$ and $\lambda_C$, such that the following LMIs hold
\[
\sum_{k' \in \mathcal{K}(d)} k! \sum_{k \geq k'} \sum_{k \in \mathcal{K}(g)} \sum_{k \geq k'+k} \text{Tr} \left( B_k P_{k-k'} B_k^T \right) - \frac{g!}{(k-k'-k)!} \mu^2 < 0, \quad \forall k \in \mathcal{K}(g+2+d)
\]
(18)
\[
M_k + \sum_{k' \in \mathcal{K}(w-g)} M_{k-k'} + \sum_{k \in \mathcal{K}(w-2g)} M_{k-k} > 0 \quad \forall k \in \mathcal{K}(w)
\]
(19)
where
\[
M_k = \frac{w!}{k!} \begin{bmatrix} -\left( \lambda_A \delta_A^2 + \frac{1}{4} \lambda_C \delta_C^2 \right) & I & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & \lambda_A I & * & * \\ I & 0 & 0 & \lambda_C I & \\ \end{bmatrix}
\]
(20)
\[
M_{k-k'} = \frac{(w-g)!}{k!} \begin{bmatrix} P_{k-k'} & * & * & * \\ 0 & P_{k-k'} & * & * \\ 0 & 0 & P_{k-k'} & * \\ 0 & 0 & 0 & 0 \\ \end{bmatrix}
\]
(21)
\[
M_{k-k} = \frac{(w-2g)!}{k!} \begin{bmatrix} -C_{k-k} & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \\ \end{bmatrix}
\]
(22)
\[
M_{k-k-k} = \frac{(w-g)!}{k!} \begin{bmatrix} 0 & * & * & * \\ P_k A_{k-k-k} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \end{bmatrix}
\]
(23)
with discretized model of approximation level $\ell \in \mathbb{N}$ and $w = \max (2\ell + d, \ell + g + d)$, then $\mu$ is an $\mathcal{H}_2$ guaranteed cost for system (4) and, consequently, for system (1).

Proof: Firstly, note that $(\sum_{i=1}^{N} \alpha_i)^d = 1$ for any $d \in \mathbb{N}$, then (18) multiplied by $\alpha^k$ and summed up for all $k \in \mathcal{K}(g+2+d)$ can be written as
\[
\text{Tr} \left( B(\alpha) P(\alpha) B(\alpha) \right) < \mu^2
\]
(24)
which is the trace condition for the $\mathcal{H}_2$ guaranteed cost computation for the discrete-time system given by (4). Similarly, (19) multiplied by $\alpha^k$ and summed up for all $k \in \mathcal{K}(w)$ can be rewritten as
\[
\begin{bmatrix} \Xi(\alpha) \\ P(\alpha)^T A(\alpha) & P(\alpha)^T \lambda_C I & * & * \\ 0 & P(\alpha) & \lambda_A I & * \\ I & 0 & 0 & \lambda_C I \end{bmatrix} > 0,
\]
(25)
with $\Xi(\alpha) = P(\alpha) - C(\alpha) - \frac{1}{4} \lambda_C \delta_C^2 I - \lambda_A \delta_A^2 I$. Inequality (25) is equivalent by Schur’s complement, to
\[
\begin{bmatrix} P(\alpha) - C(\alpha) & -\frac{\lambda_A}{2} \delta_A^2 I & * & * \\ P(\alpha)^T A(\alpha) & P(\alpha)^T \lambda_C I & * & * \\ 0 & P(\alpha) & \lambda_A I & * \\ I & 0 & 0 & \lambda_C I \end{bmatrix} < 0
\]
\[
-\frac{\lambda_C}{2} \delta_C^2 I \leq \lambda \delta_C^2 I,
\]
\[
T_c - \lambda_C^{-1} N_c N_c - \frac{\lambda_C}{2} \delta_C^2 I \geq \mathbb{M}_1 > 0.
\]
(26)
Denoting the left-hand side of the above inequality by $\mathbb{M}_1$ and knowing that $\Delta C(\alpha)' \Delta C(\alpha) \leq \delta_C^2 I$, one has
\[
T_c - \lambda_C^{-1} N_c N_c - \frac{\lambda_C}{2} \delta_C^2 I \geq \mathbb{M}_1 > 0.
\]
(27)
Then, using Lemma 1, one has
\[
T_c - M_c N_c - N_c M_c' \geq T_c - \lambda_C M_c M_c' - \lambda_C^{-1} N_c N_c \geq \mathbb{M}_1 > 0
\]
or
\[
\begin{bmatrix} P(\alpha) - (C(\alpha) + \Delta C(\alpha)) & -\frac{\lambda_A}{2} \delta_A^2 I & * & * \\ P(\alpha)^T A(\alpha) & P(\alpha)^T \lambda_C I & * & * \\ 0 & P(\alpha) & \lambda_A I & * \\ I & 0 & 0 & \lambda_C I \end{bmatrix} > 0
\]
(28)
Similarly, applying Schur’s complement
\[
\begin{bmatrix} P(\alpha) - (C(\alpha) + \Delta C(\alpha)) & * \\ P(\alpha)^T A(\alpha) & P(\alpha) \end{bmatrix} > 0.
\]
(29)
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1The symbol $*$ represents symmetric blocks.
denoting the left side of the above inequality by $M_2$, and using the fact that $\Delta A_i(\alpha)' \Delta A_i(\alpha) \leq \delta_A^2 I$, one has
\[ T_a - \lambda_A^{-1} N_a' N_a - \lambda_A \begin{bmatrix} \Delta A_i(\alpha)' \Delta A_i(\alpha) & 0 \\ 0 & 0 \end{bmatrix} \geq M_2 > 0. \]

The latter inequality, by Lemma 1, can be rewritten as
\[ T_a - M_a N_a - N_a' M_a' - \lambda_A^{-1} N_a' N_a \geq M_2 > 0 \]
or
\[ \begin{bmatrix} P(\alpha) - (C_i(\alpha) + \Delta C_i(\alpha)) & * \\ P(\alpha) (A_i(\alpha) + \Delta A_i(\alpha)) & P(\alpha) \end{bmatrix} > 0, \]
which is, by Schur’s complement, equivalent to the gramian condition for discrete-time system (4)
\[ A(\alpha) P(\alpha) A(\alpha)' - P(\alpha) + C(\alpha)' C(\alpha) < 0, \]
with $A(\alpha)$ and $C(\alpha)' C(\alpha)$ given in (7). Since $\mu$ is an upper bound to the $\mathcal{H}_2$ norm of discrete-time system (4) and, for any fixed $\alpha \in \Lambda$, equality (6) holds, $\mu$ is also a guaranteed cost for the continuous-time system (1).

It is important to emphasize that the minimization of $\mu^2$ yields the smallest $\mathcal{H}_2$ guaranteed cost such that Theorem 1 holds, but the conditions are only sufficient. Such limitation comes from the fact that Theorem 1 uses upper bounds $\delta_A$ and $\delta_C$ instead of the actual errors $\Delta A_i(\alpha)$ and $\Delta C_i(\alpha)$, respectively. Moreover, some additional conservatism is introduced by use of Lemma 1. Nevertheless, with the increase of $g$ and $d$, Theorem 1 provides sharper estimations for the $\mathcal{H}_2$ guaranteed costs.

V. NUMERICAL EXAMPLE

A numerical comparison between the approach proposed in this paper and other methods in the literature for discretization and computation of $\mathcal{H}_2$ norm of linear uncertain systems is presented next. All the routines were implemented in Matlab, version 7.10 (R2010a) using Yalmip [18] and SeDuMi [19], in an AMD Phenom (TM) II X6 1090T (3.2GHz), 4GB RAM, Windows 7.

Example 1 Consider the continuous-time uncertain system (1) whose vertices are:
\[ E_1 = \begin{bmatrix} -0.9 & 0.2 \\ -0.5 & -1.9 \end{bmatrix}, \quad E_2 = \begin{bmatrix} -1.1 & -0.1 \\ 0.5 & -2.8 \end{bmatrix}, \]
\[ F_1 = \begin{bmatrix} 1.0 & 0.0 \end{bmatrix}', \quad F_2 = \begin{bmatrix} 2.0 & 0.0 \end{bmatrix}', \]
\[ G_1 = \begin{bmatrix} 1.0 & 0.0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 1.0 & 0.0 \end{bmatrix}. \]
The aim is to show that the proposed discretization method provides discretized systems with $\mathcal{H}_2$ guaranteed costs closer to the worst case $\mathcal{H}_2$ norm of the corresponding uncertain continuous-time system, when compared to traditional methods that discretize the vertices using a first order Euler approximation. In the latter case, the matrix vertices of the discretized system with sampling time $T$ are given by
\[ A_i = I + E_i T, \quad B_i = F_i T, \quad C_i = G_i, \quad \forall i = 1, \ldots, N. \]

The $\mathcal{H}_2$ worst case norm for the continuous-time system in this example is 1.3406, which can be calculated, for instance, using a continuous-time condition from [3, Theorem 6] or a polytope partition combined with Delaunay triangulation [20], [21]. Using a first order Euler approximation with $T = 0.1s$ and computing an $\mathcal{H}_2$ guaranteed cost by means of a discrete-time LMI condition from [3, Theorem 6] with sufficiently large degrees $g$ of Lyapunov matrices and $d$ of Pólya’s Theorem, the attained value is $\mu = 0.4363$. It is important to emphasize that this value is not, in fact, an upper bound for the $\mathcal{H}_2$ norm of the continuous-time system.

On the other hand, employing the discretization technique and Theorem 1 proposed in this paper, the obtained guaranteed costs are always upper bounds for the $\mathcal{H}_2$ norm of continuous-time system, as can be seen in Table I. The table also presents the $\mathcal{H}_2$ guaranteed cost values for the discretized system corresponding to the Taylor series approximation level $\ell$, degrees $g$ of the Lyapunov matrices and $d$ of Pólya’s Theorem. It can be noted that the use of higher degrees of Taylor series approximation ($\ell$) allows to obtain more accurate discrete-time representations (4) since the bounds ($\delta_A$ and $\delta_C$) to the discretization errors (10) and (11) decrease. Moreover, the increase on $g$ and $d$ in Theorem 1 provides less conservative results, which means that the gap between $\mu$ and the $\mathcal{H}_2$ worst case norm of the continuous-time system reduces.

<table>
<thead>
<tr>
<th>$\ell$</th>
<th>$\delta_A$, $\delta_C$</th>
<th>$g$</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\delta_A = 3.6 \times 10^{-2}$</td>
<td>0</td>
<td>2.0123</td>
</tr>
<tr>
<td></td>
<td>$\delta_C = 1.1 \times 10^{-2}$</td>
<td>1</td>
<td>1.7584</td>
</tr>
<tr>
<td>2</td>
<td>$\delta_A = 3.5 \times 10^{-3}$</td>
<td>0</td>
<td>1.4944</td>
</tr>
<tr>
<td></td>
<td>$\delta_C = 3.5 \times 10^{-4}$</td>
<td>1</td>
<td>1.3637</td>
</tr>
<tr>
<td>3</td>
<td>$\delta_A = 2.5 \times 10^{-4}$</td>
<td>0</td>
<td>1.4662</td>
</tr>
<tr>
<td></td>
<td>$\delta_C = 1.0 \times 10^{-5}$</td>
<td>1</td>
<td>1.3423</td>
</tr>
<tr>
<td>4</td>
<td>$\delta_A = 1.4 \times 10^{-5}$</td>
<td>0</td>
<td>1.4643</td>
</tr>
<tr>
<td></td>
<td>$\delta_C = 3.0 \times 10^{-7}$</td>
<td>1</td>
<td>1.3408</td>
</tr>
<tr>
<td>5</td>
<td>$\delta_A = 6.4 \times 10^{-7}$</td>
<td>0</td>
<td>1.4642</td>
</tr>
<tr>
<td></td>
<td>$\delta_C = 1.3 \times 10^{-8}$</td>
<td>1</td>
<td>1.3407</td>
</tr>
</tbody>
</table>

Figure 1 shows a comparison between the exact continuous-time system $\mathcal{H}_2$ worst case norm computed with [21] and the $\mathcal{H}_2$ guaranteed cost calculated by Theorem 1 with $g = 4$, $d = 0$ and different levels of approximation, $\ell = 1, \ldots, 4$. Note that the sampling time directly influences the computation of $\mu$ through the conditions of Theorem 1, since the discretization errors increase with the growth of the sampling period, requiring larger values of $\ell$ to reach sharper results.
VI. CONCLUSION AND FUTURE WORKS

This paper proposed a new discretization procedure for uncertain time-invariant linear systems based on Taylor series expansion. As main characteristic, the discrete-time representation provides an $H_2$ guaranteed cost that accurately approximates the worst case $H_2$ norm of the original continuous-time uncertain system, if a sufficiently large degree of approximation $\ell$ is used. The obtained discrete-time system is described in terms of homogeneous polynomial matrices with parameters lying in the unit simplex plus an additive norm bounded uncertainty representing the discretization residual error.

LMI conditions for the computation of $H_2$ guaranteed costs for this class of discrete-time systems in terms of homogeneous polynomially parameter-dependent solutions of arbitrary degree $g$ and Pólya’s relaxation level $d$ were also presented. Differently from the traditional methods in the literature for discretization of uncertain systems, which use a first order Taylor approximation, the results obtained by the proposed conditions always provide an upper bound to the actual worst case $H_2$ norm of the original uncertain continuous-time system, as illustrated in the numerical experiment. Additionally, the example also illustrated that, for a given discretization level $\ell$, the increase in the degrees $g$ and $d$ allows to attain less conservative results.

As future research, the authors are investigating conditions to cope with the design of robust $H_2$ digital state-feedback controllers for continuous-time uncertain systems, in the context of Networked Control Systems.