The First Law of Isolated Horizons via Noether Theorem

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Abstract

A general recipe proposed elsewhere to define, via Noether theorem, the variation of energy for a natural field theory is applied to Einstein-Maxwell theory. The electromagnetic field is analysed in the geometric framework of natural bundles. Einstein-Maxwell theory turns then out to be natural rather than gauge-natural. As a consequence of this assumption a correction term à la Regge-Teitelboim is needed to define the variation of energy, also for the pure electromagnetic part of the Einstein-Maxwell Lagrangian. Integrability conditions for the variational equation which defines the variation of energy are analysed in relation with boundary conditions on physical data. As an application the first law of thermodynamics for rigidly rotating horizons is obtained.

1 Introduction

In a previous paper [28] a recipe to define, via Noether theorem, the variation of Noether conserved quantities in natural field theories was proposed. The main result of the theory there developed was to describe, for vacuum General Relativity, the quasilocal stress-energy content of a spatially bounded region of spacetime with non-orthogonal boundaries.
The purpose of the present paper is to generalise the recipe of [28] to Einstein-Maxwell theory in order to calculate the Noether quasilocal conserved quantities, still remaining in the general case of spacetime regions with non-orthogonal boundaries. As a remarkable application we define a first principle of thermodynamics for spacetimes admitting an *isolated horizon* (in the sense of [2]).

Using a geometric approach to field theories in the framework of natural theories, it is possible to associate to each infinitesimal symmetry ξ on the base manifold M (spacetime of dimension m) a conserved quantity $Q(\xi)$. This Noether charge is obtained by integrating the Noether current on a compact $(m-1)$ submanifold Σ of M with boundary $\partial\Sigma$ and it can be decomposed into a bulk term $Q_{bulk}(\xi)$ and a surface term $Q_{surface}(\xi)$. The bulk term is obtained by integrating on Σ the *reduced Noether current*, which is a $(m-1)$-form vanishing on shell (i.e. along solutions). The surface term is the integral over $\partial\Sigma$ of the *superpotential* related to the symmetry generated by ξ which in turn, is a $(m-2)$-form algorithmically calculated starting from the Lagrangian of the theory [22]. To obtain physically expected values, in analogy with the original prescription suggested by Regge-Teitelboim in [43] (when dealing with the analysis of Hamiltonian boundary

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terms), we have to correct the Noether conserved quantities by suitably adding boundary terms to $Q(\xi)$; this approach was extensively developed for natural and gauge-natural theories in a global and geometric framework, called *covariant ADM formalism.* [23].
Following the same idea which leads to the covariant ADM formalism, we can proceed one step further by defining the *variation* of the corrected conserved quantities through the addition of a further suitable boundary term τ (which is still present in the covariant ADM definition of the variation of the conserved quantity), such that $\delta Q(\xi)$ turns out to be a pure (off-shell) bulk term. As a particular case, if we consider a Cauchy surface Σ in spacetime and a vector field ξ transverse to Σ the variation of the Hamiltonian can be defined as the variation of the Noether charge associated to ξ. In this situation the vector ξ is identified with the generator of time flow, meaning that the parameter "t" generating the flow of ξ itself is identified with time. This technique allows us to obtain the Hamiltonian equations of motion and to define the symplectic structure of the theory together with its phase space. Variation of energy is simply defined as the on-shell value of the variation of the Hamiltonian. A problem which arises is then to formally integrate, if possible, the variation of the conserved quantities to obtain the explicit expression for the Hamiltonian and for the energy. This problem is related to the choice of suitable boundary conditions as well as to the choice of a background solution. The chosen background can be interpreted as a reference solution in the one-parameter family of solutions satisfying the same boundary conditions (namely the one-parameter family along which variations are performed).
We remark that the final formula for the variation of the Hamiltonian is independent on divergence terms which might be added to the Lagrangian. This is a mathematical and physical advantage since we do no longer have to take care about surface terms in the action functional. The expression of the variation of the Hamiltonian is unique for each element of the equivalence class $[L]$ formed by all the Lagrangians differing one from the other only for divergence terms. Moreover, the variation of the Hamiltonian reproduces the correct equations of motion in the whole phase space of the theory and not only in a phase space restricted by boundary conditions such as, for example, a suitable fall-off requirement on solutions at spatial infinity.
In this framework, in fact, the symplectic structure of the field theory is uniquely defined by the bulk term, generated by the so called *reduced symplectic current.* There is no contribution to the symplectic structure from the boundary terms, which are instead pushed directly into the very definition of the variation of the Hamiltonian.
The variation of energy, as previously remarked, is defined as the on-shell variation of the Hamiltonian. In this framework different definitions of energy for the same system do in fact arise in correspondence to different definitions of the variations of the Hamiltonian (and consequently of the variations of energy). Different variations are related to the different vertical vector fields which generate different one-parameter families of solutions starting from the same solution. Different one-parameter families correspond to different boundary conditions the solutions have to satisfy. The variational equation which defines the variation of energy can be (or can be not) integrated, depending on the boundary conditions imposed, when integrable different kinds of energy are defined.
This viewpoint is in full accordance with the classical treatment of thermodynamical systems. When we impose different boundary conditions on the same thermodynamical system, in fact we perform each time a different choice between the intensive or extensive variables for the system (in a symplectic framework we can say that we are choosing the "*control parameters*") and correspondingly we expect to find different energies for the system.

In vacuum General Relativity, as shown in [28], the definition of quasilocal energy via Noether theorem leads to the same results previously obtained by using the Trace-K action functional [10], [11] or by using a Hamiltonian analysis as in [7], [37]. The Hilbert Lagrangian is sensitive to the formalism presented here. In presence of non-orthogonal boundaries, in fact, the generalized Regge-Teitelboim correction term τ (we add to the definition of δH) is fundamental for the definition of the variation of energy of the gravitational field. In this way a definition of quasilocal internal energy for spatially bounded gravitating systems can be obtained where the reference background is properly taken into account. This is obtained by imposing Dirichlet boundary conditions, i.e. the metric at the boundary of the world tube is kept fixed.
In this paper we apply our general recipe [28] to the case of Einstein-Maxwell theory. Electromagnetism is here treated as a natural theory. In natural theories each vector field tangent to the spacetime manifold can be naturally lifted to a vector field tangent to the configuration bundle and moreover this lifted vector field is a symmetry for the Lagrangian [22]. We remark that in the natural lift of spacetime vector fields no gauge freedom remains undetermined (quite different is the case in gauge theories!). In this well defined geometric framework the Lie derivative of the fields (i.e. the electromagnetic potential) is uniquely defined, has a correct mathematical interpretation and it is thence possible to define Noether conserved quantities via Noether theorem.
Maxwell theory, opposite to other gauge theories, can be treated as a natural theory by using a suitable representation of the gauge group $U(1)$. The configuration bundle turns out to be a *natural* $U(1)$ bundle; this in turn implies that the configuration bundle is a trivial $U(1)$ bundle, meaning that no magnetic charge is allowed: see [21].
In this particular framework the electromagnetic potential is a geometric object of order 2 and it is possible to apply the formula which defines the correct variation of energy if we introduce also an electromagnetic correction term τ (related to the representation chosen) in the definition of the variation of the Hamiltonian. This allows to define correctly the energy for the system. Moreover we obtain the symplectic structure and the phase space of the theory, with results in full accordance with [13].
To perform the integration of the variational equation which defines the variation of energy for the electromagnetic field we choose two different sets of boundary conditions, corresponding respectively, to an adiabatic system and to an electrically isolated system [37]. In both these cases we obtain a priori a suitable energy contribution to the Einstein-Maxwell Lagragian due to the pure electromagnetic field. In the first case this contribution vanishes, this meaning that the contribution to the energy comes out only from the pure gravitational part of the Noether charge and it keeps track of the electromagnetic field only through the solutions of Einstein-Maxwell field equations. In the second case the electromagnetic contribution to energy turns out instead to be proportional to the product of the electric charge and the electrostatic potential (integrated over the boundary), a result which is physically reliable if compared with Classical Electrodynamics [33].

Gluing together the results obtained for the gravitational field and for the electromagnetic field, we can calculate the energy content of a spatially bounded region of spacetime in the framework of Einstein-Maxwell theory and it is then possible to apply the theory developed so far to the case of rigidly rotating horizons (see [3], [7]).
The classical definition of black hole thermodynamics, based on the definition of entropy for a Killing horizon, deals in fact with quite unphysical models, as it applies only to static or quasi-static spacetimes (which means small perturbations from a static situation and thus no radiation is

admitted nearby the horizon). Moreover, to define the concept of event horizon for non-stationary spacetimes, we need to know the whole history of the spacetime and this is in contrast with the concept of physical observer.

A generalization of these concepts to more physical situations has then been proposed by Ashtekar and his coworkers in [2], [3] (and references therein), where they introduce the notion of isolated horizon as a 3-dimensional null hypersurface Δ embedded into spacetime. Cross sections of isolated horizons are, roughly speaking, non-expanding surfaces, isolated from the outside and with a null flux of matter and radiation through or outside them. To define the variation of energy and the conserved quantities of isolated horizons in the Noether framework we do not need that the spacetime has a global Killing vector field but it is sufficient to assume the existence of a local Killing vector field for the 3-metric of the horizon; this geometric requirement is fulfilled by isolated horizons. Indeed the horizon Killing vector field ensures that isolated horizon are (quasi) locally in equilibrium, but they are allowed to admit nearby radiation.

The Noether formalism developed to define quasilocal conserved quantities for Einstein-Maxwell theory naturally applies to isolated horizons and in particular we can define the area A_Δ, the angular momentum J_Δ and the charge Q_Δ of the horizon through integrals on the cross sections Δ_t of Δ and they are conserved on the whole horizon. From the definition of the variation of energy we can obtain a first principle for rigidly rotating horizons, which are defined as (weakly) isolated horizons with an internal symmetry, generated by a vector field tangent to the cross sections. This requirement together with the definition of (weakly) isolated horizons, ensures that when we evaluate the variation of energy on a cross section Δ_t, a first law of black holes thermodynamics is defined under the form:

$$\delta E_\Delta = \frac{k_{(l)}}{k}\delta A_\Delta + \Phi_{(l)}\delta Q_\Delta + \Omega_{(l)}\delta J_\Delta \tag{1}$$

where $k_{(l)}$, $\Phi_{(l)}$, $\Omega_{(l)}$ are parameters of the horizon related respectivly to its temperature, its electrostatic potential and its angular velocity.

This paper is divided into six Sections. In Section 2 we review the definition of the Hamiltonian structure of a field theory and the definition of the variation of energy, via Noether theorem. The formalism used is a pure geometric approach, based on the definition of Lagrangian field theories on jet-bundles [36]. In Section 3 we apply the formalism to the case of General Relativity and we find an explicit expression for the variation of energy of spatially bounded gravitating systems. We introduce here all concepts and notation that are useful for a $(3+1)$ approach to field theories. In Section 4 we apply our definitions to the electromagnetic field and we analyse the definition of variation of energy for Einstein-Maxwell theory. The general theory is applied to calculate the energy for two particular sets of boundary conditions, corresponding to different physical situations. In Section 5 we finally introduce the boundary and geometric conditions which define an isolated and a rigidly rotating horizon. The direct evaluation of the formula defining the variation of energy on such surfaces is nothing but the first law of thermodynamics for rigidly rotating horizons.

2 Geometric framework

The definition of the Hamiltonian for a field theory, as well as the definition of energy as its on shell value, can be based on Noether's theorem. It has been shown in [28] that the formula obtained via this definition reproduces, in applications, the same results obtained in the ADM Hamiltonian formalism [7], [37] or obtained using a Hamilton-Jacobi analysis of the Trace-K action functional

[10], [11]. The advantages arising from Noether formalism are related to the fact that the Noether (covariantly) conserved quantities are independent on the addition of boundary terms to the Lagrangian and consequently these latter terms do not influence the definition of energy. In the original analysis of Brown and York [11] different boundary terms for the action functional are dictated by different boundary conditions and by different choices of the control modes of the boundary data. Before entering into the detais of the matter, we shall shortly summarize the geometric framework we shall need in the rest of the paper (we address the reader to [22], [36], [38] for a deeper and more rigorous mathematical exposition).

The *configuration bundle* for a Lagrangian field theory on a manifold M, with dimension $\dim(M)=m$, is a bundle $(B, M; \pi)$. A *field configuration* is a section σ of B; i.e. a map $\sigma : M \to B$. We can choose fibered coordinates (x^μ, y^i) on B and, in these coordinates, the field configuration can be locally represented as $\sigma^i : x^\mu \mapsto (x^\mu, y^i = \sigma^i(x^\mu))$. The jet prolongations of the configuration bundle are denoted by $(J^kB, M; j^k\pi)$. Local fibered coordinates $(x^\mu, y^i, y^i_\mu, .., y^i_{\mu_1..\mu_k})$ can be chosen on J^kB and they denote the spacetime derivatives of fields up to order k. The prolongation of a field configuration σ to a jet bundle is denoted by $j^k\sigma$.

A *vertical vector field* is a section X of the vertical bundle $V(B) = \mathrm{Ker}(T\pi)$, namely a vector-field which is everywhere tangent to the fibers; locally it can be written in fiber coordinates as $X = X^i \frac{\partial}{\partial y^i} \equiv \delta y^i \frac{\partial}{\partial y^i}$ and it describes the variation δy^i of the dynamical fields. Its prolongation to the k-order jet bundle of B is denoted by j^kX and describes the variation of the fields and of their derivatives up to order k. If we denote by Φ_t the 1-parameter flow generated by X on B, its prolongation to the jet bundle J^kB defines a flow on it, coherently denoted by $j^k\Phi_t$.

The variation of a generic morphism $R : J^kB \to \Lambda^n M$ (where $\Lambda^n M$ denotes the n-form bundle over M for $n \leq m$), along the flow of X at any given section σ can be defined as:

$$(\delta_X R)(\sigma) = \frac{d}{dt} R\Big(j^k\Phi_t \circ j^k\sigma\Big)\Big|_{t=0} = \frac{d}{dt} R\Big(j^k\sigma_t\Big)\Big|_{t=0} \tag{2}$$

where $\sigma_t = \Phi_t \circ \sigma$ denotes a 1-parameter family of field configurations on B obtained by dragging σ along the flow of X.

A k-order *Lagrangian* is a morphism from the k-order jet bundle J^kB to the bundle $\Lambda^m(M)$ of volume m-forms on M and locally it can be written as $L = \mathcal{L}(y^i, y^i_{\mu_1}, .., y^i_{\mu_1..\mu_k})ds$, where $\mathfrak{L}$ is scalar density called the *Lagrangian scalar density* and $ds = dx^1 \wedge dx^2 \wedge .. \wedge dx^m$ is the local volume m-form on M (we do not assume an explicit dependence of the Lagrangian on spacetime coordinates x^λ because this dependence is forbidden in natural theories, see [22]).

The *action functional*

$$A_D(\sigma) = \int_D (j^k\sigma)^* L \tag{3}$$

is defined by integrating the pull-back $(j^k\sigma)^*L$ of the Lagrangian L over a compact region $D \subset M$ with regular boundary ∂D. According to Hamilton's principle, field equations are obtained by imposing the action $A_D(\sigma)$ to be stationary along the flow generated by any compactly supported vertical field $X \in V(B)$. In this way one obtains Euler-Lagrange field equations $(\mathbb{E} \circ j^{2k}\sigma) = 0$ and sections σ of B which satisfy this variational equation are called *critical sections* (see [22], [36] and [45]).

The variation of the Lagrangian along the 1-parameter flow of the vertical vector field X can be defined as a global bundle morphism $\delta L : J^kB \to V^*(J^kB) \otimes \Lambda^m(M)$ by:

$$< \delta L \circ j^k\sigma \mid j^kX >= \frac{d}{dt}\Big(L \circ j^k\Phi_t \circ j^k\sigma\Big)\Big|_{t=0} \tag{4}$$

where $V^*(\cdot)$ denotes the dual bundle of the vertical bundle $V(\cdot)$ and $< \cdot \mid \cdot >$ denotes the natural duality between $V^*(\cdot)$ and $V(\cdot)$.
In this paper we consider only *natural theories*, namely those field theories for which the configuration bundle B is natural, i.e. each spacetime vector field $\xi = \xi^\mu \partial_\mu$ lifts naturally (i.e. preserving the commutators) to a unique vector field $\hat{\xi}$ over the configuration bundle as well as it uniquely lifts to any prolongation J^kB. This means that for each $\xi = \xi^\mu \partial_\mu \in \mathfrak{X}(M)$ we have a naturally lifted vector field $\hat{\xi} \in \mathfrak{X}(B)$ defined by:

$$\hat{\xi} = \xi^\mu(x^\mu)\partial_\mu + \xi^i(x^\mu, y^j)\partial_i \tag{5}$$

while $j^k\hat{\xi}$ is naturally defined as the prolongation of the vector field $\hat{\xi}$ to the jet bundle J^kB. We stress that the coefficients ξ^i in (5) depend on ξ^μ and their derivatives up to an algorithmically computable finite order r, which depends on the case considered, called the *total order of the natural bundle*. The sum $s = k + r$ of the order of the Lagrangian with the order of the natural bundle is called the *total order of the theory*.
A Lagrangian theory is *natural* or, "physically" speaking, *covariant* if each spacetime vector field ξ is an *infinitesimal Lagrangian symmetry*, which is equivalent to state that the following holds:

$$\pounds_\xi L =< \delta L \mid j^k \pounds_\xi \sigma > \tag{6}$$

where $\pounds_\xi \sigma$ is the *Lie derivative of a section* σ defined by

$$\pounds_\xi \sigma = T\sigma(\xi) - \hat{\xi} \circ \sigma \equiv (\pounds_\xi y^i)\partial_i \tag{7}$$

From this formula it is clear that $\pounds_\xi \sigma$ is a vertical vector field over B and it describes the evolution of the field configuration σ along the flow of ξ. In local coordinates formula (6) is equivalent to require that:

$$\pounds_\xi L = \Big(\frac{\partial L}{\partial y^i}\pounds_\xi y^i + \frac{\partial L}{\partial y^i_{\mu_1}}\pounds_\xi y^i_{\mu_1} + .. + \frac{\partial L}{\partial y^i_{\mu_1..\mu_k}}\pounds_\xi y^i_{\mu_1..\mu_k}\Big)ds \tag{8}$$

It is well known that General Relativity with the Hilbert Lagrangian is a covariant theory, i.e. it is natural; we will show in the sequel that also electromagnetism with the Einstein-Maxwell Lagrangian can be treated as a natural theory, provided the configuration bundle and the Lie derivatives of the electromagnetic vector potential A_μ are defined in an appropriate way [23], [25], [38].
The variation δL of the Lagrangian splits as follows by a covariant integration by parts:

$$< \delta L \circ j^k\sigma \mid j^kX >=< \mathbb{E}(L) \circ j^{2k}\sigma \mid X > +\mathrm{d} < \mathbb{F}(L) \circ j^{2k-1}\sigma \mid j^{k-1}X > \tag{9}$$

where $\mathbb{E}$ and $\mathbb{F}$ are called respectively the *Euler-Lagrange* and the *Poincarè -Cartan morphism* [22]. We stress that the Poincarè -Cartan morphism is not unique for theories of order higher than 2 (even though in natural theories uniqueness is achieved, for any order k, through the introduction of a dynamical spacetime connection; see [19] and [27] for details). When evaluated on a particular configuration and on a vertical vector field X, the morphisms $\mathbb{E}$ and $\mathbb{F}$ are respectively identified with a m-form and a $(m-1)$-form over M. In local coordinates δL can be expressed as:

$$\delta L(j^k\sigma, j^kX) = \mathbb{E}_i(j^{2k}\sigma)X^i ds + \mathbb{F}^\lambda(j^{2k-1}\sigma, j^{k-1}X)ds_\lambda \tag{10}$$

Substituting this expression into the variation $\delta_X A_D(\sigma)$ of the action functional (3), it is easy to see that $\mathbb{E}_i$ represent the Euler-Lagrange equations of motion, while $\mathbb{F}^\lambda$ is a boundary term which vanishes if suitable conditions are imposed on X (usually one states that X is vanishing on the boundary ∂D together with all its derivatives, which is equivalent to state that it is "strongly" compactly supported).

If we deal with natural field theories, Noether's theorem allows us to construct a covariantly conserved current and a conserved quantity for each vector field ξ on M. This object can be globally constructed and well defined from a geometric point of view [21], [22]. The Noether current, for a solution σ is locally defined as a $(m-1)$-form over M:

$$\mathcal{E}^\lambda(L,\xi,\sigma) = \mathbb{F}^\lambda(j^{2k-1}\sigma, j^{k-1}(\pounds_\xi\sigma)) - \xi^\lambda \mathcal{L}(j^k\sigma) \tag{11}$$

It is well known that there exists a unique and global decomposition of the Noether current as the sum of a $(m-1)$-form $\tilde{\mathcal{E}}$ of M (called the *reduced current*, which vanishes on shell) and the divergence of a $(m-2)$-form $\mathcal{U}$, named the *superpotential* of the theory (see [22]):

$$\mathcal{E}(L,\xi,\sigma) = \mathcal{E}^\lambda(L,\xi,\sigma)ds_\lambda = \tilde{\mathcal{E}}^\lambda(L,\xi,\sigma)ds_\lambda + [d_\mu \mathcal{U}^{\lambda\mu}(L,\xi,\sigma)]ds_\lambda \tag{12}$$

Both the superpotential and the reduced current are linear in the components of the infinitesimal simmetry generator ξ and their derivatives up to order $(k+r-2)$ and $(k+r-1)$ respectively, where $k+r$ is the total order of the theory (see our discussion about formula (5)).

Covariantly conserved quantities are naturally defined as the integral of the current $\mathcal{E}$ on a $(m-1)$-dimensional submanifold $\Sigma \subset M$ with a compact boundary $\partial\Sigma \subset \Sigma \subset M$:

$$Q_\Sigma(L,\xi,\sigma) = \int_\Sigma \mathcal{E}^\lambda(L,\xi,\sigma)ds_\lambda = \int_\Sigma \tilde{\mathcal{E}}^\lambda(L,\xi,\sigma)ds_\lambda + \frac{1}{2}\int_{\partial\Sigma} \mathcal{U}^{\mu\lambda}(L,\xi,\sigma)ds_{\mu\lambda} \tag{13}$$

When evaluated on shell, since $\tilde{\mathcal{E}}$ is proportional to the equations of motion, the conserved quantities are pure boundary terms integrated on $\partial\Sigma$:

$$Q_\Sigma(L,\xi,\sigma) = \frac{1}{2}\int_{\partial\Sigma} \mathcal{U}^{\lambda\mu}(L,\xi,\sigma)ds_{\lambda\mu} \tag{14}$$

which means (see (12)) that the conserved Noether currents are exact on-shell. When evaluated for a particular theory and for a fixed ξ these quantities do not reproduce the expected physical values because of the "anomalous factor problem" [43]. The solution to this problem relies on the so called *ADM covariant formalism* [23], [27]. According to this, we compute the variation of the conserved current (11) along a vertical vector field X as:

$$\begin{aligned} \delta_X\mathcal{E}(L,\xi,\sigma) &= \delta_X\mathbb{F}(L,\pounds_\xi\sigma) - i_\xi(\delta_X L) = \\ &= \delta_X\mathbb{F}(L,\pounds_\xi\sigma) - i_\xi\mathbb{E}(L,X) - i_\xi[d\mathbb{F}(L,\pounds_\xi\sigma)] = \\ &= \delta_X\mathbb{F}(L,\pounds_\xi\sigma) - \pounds_\xi\mathbb{F}(L,X) - i_\xi\mathbb{E}(L,X) + d[i_\xi\mathbb{F}(L,\pounds_\xi\sigma)] = \\ &= \omega(L,X,\pounds_\xi\sigma) - i_\xi\mathbb{E}(L,X) + \mathrm{d}(i_\xi\mathbb{F}(L,X)) \end{aligned} \tag{15}$$

where δ_X is defined by formula (2) and we shortly set $\mathbb{F}(L,\pounds_\xi\sigma) = \mathbb{F}^\lambda(j^{2k-1}\sigma, j^{k-1}(\pounds_\xi\sigma))ds_\lambda$. The *symplectic current* ω (see [13]) is a $(m-1)$-form over M

$$\omega(L,X,\pounds_\xi\sigma) = \delta_X\mathbb{F}(L,\pounds_\xi\sigma) - \pounds_\xi\mathbb{F}(L,X) \tag{16}$$

which suitably defines a symplectic structure for the field theory. The covariant ADM method consists in defining the variation of the conserved quantity, pushing the boundary terms appearing in the right hand side of (15) into the definition itself, namely:

$$
\begin{aligned}
\delta_X \hat{Q}_\Sigma(L,\xi,\sigma) &= \int_\Sigma \delta_X \mathcal{E}(L,\xi,\sigma) - \int_{\partial\Sigma} i_\xi \mathbb{F}(L,X) = \\
&= \int_\Sigma \delta_X \tilde{\mathcal{E}}(L,\xi,\sigma) + \int_{\partial\Sigma} [\delta_X \mathcal{U}(L,\xi) - i_\xi \mathbb{F}(L,X)] = \\
&= \int_\Sigma \omega(L,X,(\pounds_\xi\sigma)) - \int_\Sigma i_\xi \mathbb{E}(L,X)
\end{aligned}
\tag{17}
$$

This definition generalises to all natural theories the analysis originally given by Regge and Teitelboim [43] for the ADM Hamiltonian with asymptotically flat solutions and gives the physically expected results for all the conserved quantities of the theory.

The Hamiltonian structure of the theory naturally arises from this definition [7], [23], [28], [37]. We define the *variation of the Hamiltonian* as the variation of the conserved quantity, relative to a Cauchy surface Σ. The fundamental requirement is that the infinitesimal generator ξ of the symmetry, which defines δH, is transverse to the surface Σ. From equation (17) we have that:

$$
\begin{aligned}
\delta_X \hat{H}(L,\xi,\sigma) &\equiv \int_\Sigma \delta_X \mathcal{E}(L,\xi,\sigma) - \int_{\partial\Sigma} i_\xi \mathbb{F}(L,X) = \\
&= \int_\Sigma \omega(L,X,\pounds_\xi\sigma) - \int_\Sigma i_\xi \mathbb{E}(L,X)
\end{aligned}
\tag{18}
$$

The variation of the energy is defined as the on-shell variation of the Hamiltonian [7], [37]; in this case $\mathbb{E}(L,X)$ vanishes and the variation of energy is defined as the integral of the symplectic current ω on Σ. We remark that if both X and $\pounds_\xi\sigma$ are solutions of the linearized field equations, from (16) we have that ω is closed on-shell [13], [19].
The Poincarè -Cartan form is linear in X and its derivatives up to order $(k-1)$ and is linear in ξ and its derivatives up to order $(k+r-1)$, where r is the order of the natural bundle. When we integrate (16) on Σ it is then possible to split the symplectic current into two terms (through an integration by parts):

$$
\int_\Sigma \omega(L,X,\pounds_\xi\sigma) = \int_\Sigma \tilde{\omega}(L,X,\pounds_\xi\sigma) + \int_\Sigma \mathrm{d}[\tau(L,X,\pounds_\xi\sigma)]
\tag{19}
$$

where $\tilde{\omega}$ is a $(m-1)$-form, while τ is a $(m-2)$-form which can be integrated on the boundary $\partial\Sigma$ using Stokes' theorem[1]. Our aim is thence to suitably correct the definition of the Hamiltonian and consequently of energy in a way that the symplectic structure of the theory is completely defined by $\int_\Sigma \tilde{\omega}$, i.e. by an integral over the $(m-1)$-surface Σ, while energy is a pure boundary term, i.e. an integral over $\partial\Sigma$.

[1] It is the analogy with Classical Mechanics which suggests how to perform the splitting: roughly speaking, our purpose is to generalize to natural field theories the well-known formula $\delta H = \dot{q}\,\delta p - \dot{p}\,\delta q$ for Hamilton equations in Classical Mechanics. Hence, in the integral $\int_\Sigma \omega(L,X,\pounds_\xi\sigma)$ all the terms which are not of the form $\int_\Sigma [(\pounds_\xi\sigma)\,\delta_X p - (\pounds_\xi p)\,\delta_X\sigma]\,d^3x$ have to be pushed, through integration by parts, into the boundary term $\int_{\partial\Sigma} \tau(L,X,\pounds_\xi\sigma)$.

According with the prescription of the covariant ADM method we can redefine a corrected variation of the Hamiltonian $\delta_X H$ by pushing the new boundary term τ into the definition of $\delta_X H$ itself, i.e.:

$$\begin{aligned} \delta_X H(L,\xi,\Sigma) &= \delta_X \hat{H}(L,\xi,\Sigma) - \int_{\partial\Sigma} \tau(L,X,\pounds_\xi \sigma) = \qquad (20) \\ &= \int_\Sigma \delta_X \tilde{\mathcal{E}}(L,\xi,\sigma) + \int_{\partial\Sigma} [\delta_X \mathcal{U}(L,\xi) - i_\xi \mathbb{F}(L,X) - \tau(L,X,\pounds_\xi \sigma)] \qquad (21) \end{aligned}$$

The new variation of the Hamiltonian does no longer contain any boundary term. In fact boundary terms arising from the variation of the reduced current $\delta_X \tilde{\mathcal{E}}$ are completely cancelled by the variation of the superpotential together with the correction terms $i_\xi \mathbb{F}$ and τ, so that, from equations (18) and (19), $\delta_X H$ finally results to be:

$$\delta_X H(L,\xi,\Sigma) = \int_\Sigma \tilde{\omega}(L,X,\pounds_\xi \sigma) - \int_\Sigma i_\xi \mathbb{E}(L,X) \qquad (22)$$

We can now define the *variation of energy* as the on-shell value of the variation of the Hamiltonian [28]. It turns out to be:

$$\delta_X E(L,\xi) = \int_{\partial\Sigma} [\delta_X \mathcal{U}(L,\xi) - i_\xi \mathbb{F}(L,X) - \tau(L,X,\pounds_\xi \sigma)] \qquad (23)$$

because $\delta_X \tilde{\mathcal{E}}$ in (21) is identically zero since we have assumed that X be a solution of the linearized equations of motion. This *master formula* (23) gives us a recipe to define the energy $E(L,\xi)$ once the variational equation $\delta_X E$ is solved. We stress that the definition (21) of the variation of the Hamiltonian and consequently the definition (23) of energy variation δE do not depend on any divergence term possibly added to the Lagrangian.
We indeed remark that the final formula of δH is invariant under a change of the Lagrangian by pure divergence terms. This means that the variation of the Hamiltonian and the variation of energy are defined for an equivalence class $[L]$ of Lagrangians, rather than for a single one, defined by the equivalence relation:

$$L \sim L' \Leftrightarrow \exists T \in \Lambda^{m-1}(M) \Rightarrow L = L' + \mathrm{d}T$$

which implies that L and L' generate the same field equations. Alternative definitions of energy that can be found in literature define δH by adding some boundary terms to the Lagrangian or to the canonical Hamiltonian [7], [11], [31]. These boundary terms which do not affect the equations of motion modify the definition of energy and consequently the control mode of the fields on the boundary [37]. Suitable terms have then to be added a posteriori to the action functional to obtain the predefined boundary control mode.
The symplectic structure of the theory and consequently its phase space are instead defined in literature starting from the symplectic current $\tilde{\omega}$ defined on the surface Σ and by a symplectic structure on the boundary $\partial\Sigma$, which is dictated by the choice of the boundary term we add to the Lagrangian; see [3].
The definition of the variation of the Hamiltonian we propose gets rid of all these problems, since the symplectic structure is defined only by the surface part $\int_\Sigma \tilde{\omega}$ of the symplectic current. In analogy with Classical Mechanics, as shown in [3], [28], [37], the variation of the Hamiltonian contains a term which determines the symplectic structure of the theory as well as a term which contains the

Lagrangian equations of motion. This is just the case of (22), where $\tilde{\omega}$ is related to the symplectic structure while the second term is linear in the Lagrangian equations of motion so that, eventually, formula (21) can be considered as a well-defined expression for the variation of the Hamiltonian.
Moreover, boundary conditions in this framework do not play any role either in the definition of the variation of energy nor in the definition of the symplectic structure of the theory. They will instead play a key role in the integration of the variational equation $\delta_X E$ where imposing boundary conditions means to fix the values of the vertical vector field X (i.e. of the variations of fields) at the boundary $\partial\Sigma$. The choice of boundary conditions, namely the choice of the vertical vector field X, together with a reference "zero level" for energy, provides us with the definition of energy [10], [28].
We will see that this formalism applies to Einstein gravitational field and also to the electromagnetic field in the framework of a geometric natural formulation of Einstein-Maxwell theory. This is possible as the total order of the electromagnetic theory results to be $s = 3$.
The theory has been formulated to calculate energy for a bounded, compact region of spacetime. However it is possible to consider the case of a system of infinite spatial extent, i.e. $\partial\Sigma$ in (23) is now the spatial infinity. Results obtained using the quasilocal definition of conserved quantities reproduce the results obtained through the standard ADM definitions. Moreover the definition of the variation of energy (23) is homologically invariant, if ξ is a global Killing vector field [28]. In the case that spatial infinity is homologic to a finite bounded surface B, conserved quantities evaluated on B correspond to the total conserved quantities at spatial infinity.

3 General Relativity

The main application of the formalism exposed in the previous chapter is General Relativity in vacuum with dim$M$=4. The Lagrangian of the theory is the Hilbert Lagrangian:

$$L_H = \frac{1}{2k}\sqrt{g}g^{\mu\nu}R_{\mu\nu}ds \tag{24}$$

where $k = 8\pi$ in geometric units, with $G = c = 1$. This argument has been analysed in a previous paper [28] (for details on calculations and notations see also [10]); we give here just a brief overview to recall the main results. We consider a four dimensional manifold $D \subset M$ of spacetime and we assume D to be diffeomorphic to the product $\Sigma \times I\!R$, here Σ is a 3-dimensional closed manifold with boundary $\partial\Sigma = B$. For any $t \in I\!R$ there exists an immersion:

$$\varphi_t : \Sigma \to \Sigma_t \tag{25}$$

and we require Σ_t to be a portion of a spacelike Cauchy hypersurface. The set of all leaves Σ_t defines a *foliation* of D in spacelike hypersurfaces. Each Σ_t intersects ∂D in a two dimensional surface B_t which is diffeomorphic to B. We denote by $\mathcal{B}$ the timelike hypersurface $\mathcal{B} = \partial D = \bigcup_t B_t$.
The time evolution is generated by a vector field ξ transverse to Σ_t; we can require that $\xi^\mu \nabla_\mu t = 1$ and ξ is tangent to the boundary $\mathcal{B}$. We denote by u^μ the timelike future directed normal to Σ_t and by n^μ the outward pointing spacelike normal to B_t in Σ_t. The vector field ξ can be decomposed as

$$\xi^\mu = Nu^\mu + N^\mu \tag{26}$$

where the *shift* N^μ lies in $T\Sigma_t$, i.e. $N^\mu u_\mu = 0$ (this matter will be discussed later in detail).
We define $\bar{n}^\mu$ the outward pointing spacelike unit normal to $\mathcal{B}$ while by $\bar{u}^\mu$ we define the timelike

future directed normal to B_t in $\mathcal{B}$. Barred and unbarred vectors are related by boost relations (see [10]):

$$\bar{u}^\mu = \gamma u^\mu + \gamma v n^\mu$$
$$\bar{n}^\mu = \gamma n^\mu + \gamma v u^\mu$$

where we have set:

$$\begin{aligned} \gamma v &= \bar{u}_\mu n^\mu = -\bar{n}_\mu u^\mu = sinh(\theta) \\ v &= \frac{N^\mu n_\mu}{N} = -\frac{\xi^\mu n_\mu}{\xi^\mu u_\mu} \end{aligned} \tag{27}$$

The parameter θ is usually called the *velocity parameter*, while v is the boost velocity and $\gamma = (1-v^2)^{-1/2}$. In fact, if we consider a point on B_t at a time t we can observe its evolution according to the unboosted (unbarred) observers, which means observers at rest on Σ_t and evolving with the normal vector u^μ. Otherwise we can describe its evolution with respect to a boosted (barred) observer evolving with the four velocity ξ^μ and seeing the vectors $\bar{u}^\mu, \bar{n}^\mu$ as normals to B_t. The scalar v is in relation to the boost radial velocity between the two classes of observers, while θ describes the non-orthogonality of the foliation; see [10].
The metrics induced on the surface Σ_t and $\mathcal{B}$ by the projector operators are defined respectively as:

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$$
$$\bar{\gamma}_{\mu\nu} = g_{\mu\nu} - \bar{n}_\mu \bar{n}_\nu$$

while the metric on B_t can be defined with respect to boosted or unboosted observers as:

$$\sigma_{\mu\nu} = g_{\mu\nu} - \bar{n}_\mu \bar{n}_\nu + \bar{u}_\mu \bar{u}_\nu = g_{\mu\nu} - n_\mu n_\nu + u_\mu u_\nu \tag{28}$$

In the sequel we denote by $g, h, \bar{\gamma}, \sigma$ the absolute values of the corresponding metric determinants. The *extrinsic curvatures* $K_{\mu\nu}$ of Σ_t in M, $\bar{\Theta}_{\mu\nu}$ of $\mathcal{B}$ in M and $\mathcal{K}_{\mu\nu}$ of B_t in Σ_t are defined respectivly as:

$$K_{\mu\nu} = -h^\alpha_\mu \nabla_\alpha u_\nu$$
$$\bar{\Theta}_{\mu\nu} = -\bar{\gamma}^\alpha_\mu \nabla_\alpha \bar{n}_\nu$$
$$\mathcal{K}_{\mu\nu} = -\sigma^\alpha_\mu D_\alpha n_\nu$$

where D is the metric covariant derivative with respect to the metric h of Σ_t . We denote by K, $\bar{\Theta}$, $\mathcal{K}$ the above quantities contracted with the corresponding metrics. We shall instead denote by $\bar{\mathcal{K}}_{\mu\nu}$ the extrinsic curvature of B_t with respect to the normal $\bar{n}^\mu$, i.e. $\bar{\mathcal{K}}_{\mu\nu} = \gamma \mathcal{K}_{\mu\nu} - \gamma v \sigma^\alpha_\mu \sigma^\beta_\nu \nabla_\alpha u_\beta$; see [28].
The *three momenta* $P^{\mu\nu}$ of the surfaces Σ_t and $\Pi^{\mu\nu}$ of $\mathcal{B}$ are defined in terms of the extrinsic curvatures of the 3-hypersurfaces in M:

$$\begin{cases} P^{\mu\nu} = \frac{\sqrt{h}}{2k}(K h^{\mu\nu} - K^{\mu\nu}) \\ \bar{\Pi}^{\mu\nu} = -\frac{\sqrt{\bar{\gamma}}}{2k}(\bar{\Theta}\bar{\gamma}^{\mu\nu} - \bar{\Theta}^{\mu\nu}) \end{cases}$$

The *time evolution vector field* can be decomposed, into a normal part N to Σ_t called *lapse* and a tangent part N^μ called *shift*, by:

$$\xi^\mu = N u^\mu + N^\mu$$

Equivalently it can be decomposed with respect to the boosted observers:

$$\xi^\mu = \bar{N}\bar{u}^\mu + \bar{N}^\mu$$

where $\bar{N} = \frac{N}{\gamma}$ and $\bar{N}^\mu$ is the projection of N^μ on B_t, i.e. $\bar{N}^\mu = \sigma^\mu_\alpha N^\alpha$.

3.1 Variation of the Hamiltonian in General Relativity

It is now possible to calculate explicitly the symplectic current and the variation of the Hamiltonian for General Relativity in vacuum, which is governed by the Hilbert Lagrangian (details and calculations can be found in [28]).
On a leaf Σ_t of the foliation, the reduced symplectic current $\tilde{\omega}$ and the boundary part τ in (19), turns out to be:

$$\begin{aligned} \tilde{\omega}(L_H, X, \pounds_\xi g) &= [(\pounds_\xi h_{\mu\nu})\delta_X P^{\mu\nu} - (\pounds_\xi P^{\mu\nu})\delta_X h_{\mu\nu}]d^3x & (29) \\ \tau(L_H, X, \pounds_\xi g) &= \frac{1}{2k}[\pounds_\xi(\sqrt{\sigma}n_\mu \delta_X u^\mu) - \delta_X(\sqrt{\sigma}n_\mu \pounds_\xi u^\mu)]d^2x & (30) \end{aligned}$$

From these expressions for the reduced symplectic current $\tilde{\omega}$ and τ it is clear that the symplectic structure and the phase space of the theory are both completely determined on the Cauchy surface Σ_t in terms of the 3-metric $h_{\mu\nu}$ and its conjugated momentum $P^{\mu\nu}$.
The variation of the Hamiltonian (21) has been calculated in [28] and turns out to be:

$$\delta_X H(L_H, \xi, \Sigma_t) = \int_{\Sigma_t} \{\mathcal{H}\delta_X N + \mathcal{H}_\alpha \delta_X N^\alpha + [h_{\mu\nu}]_\xi \delta_X P^{\mu\nu} - [P^{\mu\nu}]_\xi \delta_X h_{\mu\nu}\}d^3x \qquad (31)$$

where $[h_{\mu\nu}]_\xi$ and $[P^{\mu\nu}]_\xi$ follow directly from the 3-dimensional Einstein equations. Notice that no boundary term appears in (31). Comparing (29) with (31) the Hamiltonian equations (22) read as follows:

$$\begin{cases} -\frac{\sqrt{h}}{k}G^{\mu\nu}u_\mu u_\nu = \mathcal{H} = 0 \\ -\frac{\sqrt{h}}{k}G^{\mu\nu}h_{\alpha\mu}u_\nu = \mathcal{H}_\alpha = 0 \\ \pounds_\xi h_{\mu\nu} = [h_{\mu\nu}]_\xi \\ \frac{\sqrt{h}}{2k}G^{\alpha\beta}h^\mu_\alpha h^\nu_\beta = \pounds_\xi P^{\mu\nu} - [P^{\mu\nu}]_\xi = 0 \end{cases}$$

For our later purposes it is not necessary to formally integrate (31) and to give the explicit expression of H (see [10] and [28] for details). Using the formalism developed, in fact, one can directly turn to calculate the time rate of change of the Hamiltonian. It is easy to show that the Hamiltonian is conserved along the flow of a vector field ξ iff ξ is a Killing vector for the boundary metric on $\mathcal{B}$, i.e. $\pounds_\xi \bar{\gamma}_{\mu\nu} = 0$.
The variation of the energy turns out to be equal to (see again [28]):

$$\begin{aligned} \delta_X E(L_H, \xi) &= \int_{B_t} \Big\{\bar{N}\delta_X(\sqrt{\sigma}\bar{\epsilon}) - \bar{N}^\alpha \delta_X(\sqrt{\sigma}\bar{j}_\alpha) + \frac{\bar{N}\sqrt{\sigma}}{2}\bar{s}^{\mu\nu}\delta_X \sigma_{\mu\nu}\Big\}d^2x + & (32) \\ &+ \frac{1}{k}\int_{B_t}[\pounds_\xi(\sqrt{\sigma})\delta_X(\theta) - \delta_X(\sqrt{\sigma})\pounds_\xi(\theta)]d^2x \end{aligned}$$

where we have set:

$$\begin{cases} \bar{\epsilon} = (\frac{1}{k})\bar{\mathcal{K}} \\ \bar{j}_\alpha = -\frac{2}{\sqrt{\bar{\gamma}}}\sigma_{\alpha\mu}\bar{\Pi}^{\mu\nu}\bar{u}_\nu \\ \bar{s}^{\mu\nu} = \frac{1}{k}[(\bar{n}^\alpha\bar{a}_\alpha)\sigma^{\mu\nu} - \bar{\mathcal{K}}\sigma^{\mu\nu} + \bar{\mathcal{K}}^{\mu\nu}] \\ \bar{a}^\nu = \bar{u}^\mu(\nabla_\mu\bar{u}^\nu) \end{cases}$$

To integrate the expression (32) we need to impose suitable boundary conditions. C.-M. Chen and J. M. Nester have shown that there exist only two different Hamiltonian boundary terms which correspond to covariant boundary conditions and they are respectively the Dirichlet and the Neumann control, which define two different energies [14]. The energy corresponding to Dirichlet boundary conditions is usually accepted as the internal energy; Dirichlet boundary conditions fix the 3-metric $\bar{\gamma}$ on the boundary $\mathcal{B}$:

$$\delta_X\bar{N}\mid_{\mathcal{B}} = \delta_X\bar{N}^\mu\mid_{\mathcal{B}} = \delta_X\sigma_{\mu\nu}\mid_{\mathcal{B}} = 0 \tag{33}$$

which can be considered as a restriction on the vector field X and its flow. This last conditions allow to integrate (32) and we obtain:

$$E(L_H, g, B_t) - E_0(L_H, g_0, B_t) = \int_{B_t}\left\{\sqrt{\sigma}(\bar{N}\bar{\epsilon} - \bar{N}^\alpha\bar{j}_\alpha) + \frac{1}{k}\pounds_\xi(\sqrt{\sigma})(\theta)\right\}d^2x \tag{34}$$

We stress that we are integrating along a 1-parameter family of solutions all satisfying the same fixed boundary conditions (33). The term $E_0(g_0, B_t)$ corresponds to the quasilocal energy of a background solution g_0 inside the 1-parameter family and it becomes the "zero level" for energy. Owing to the properties $\bar{N} = \bar{N}_0\mid_{\mathcal{B}}$, $\bar{N}^\alpha = \bar{N}_0^\alpha\mid_{\mathcal{B}}$ and $\sigma_{\mu\nu} = \sigma_{0\mu\nu}\mid_{\mathcal{B}}$, we obtain the equivalent expression:

$$E(L_H, g, B_t) = \int_{B_t}\left\{\sqrt{\sigma}[\bar{N}(\bar{\epsilon} - \bar{\epsilon}_0) - \bar{N}^\alpha(\bar{j}_\alpha - \bar{j}_{0\alpha})] + \frac{1}{k}\pounds_\xi(\sqrt{\sigma})(\theta - \theta_0)\right\}d^2x \tag{35}$$

If we assume the vector field ξ to be a Killing vector on the boundary, this guarantees the conservation of energy in time and the term $\pounds_\xi(\sqrt{\sigma})$ identically vanishes, so that formula (35) agrees with the definition of energy given in [10].

4 Electromagnetic field

In this Section we shall describe the Einstein-Maxwell theory and we shall calculate its energy in relation with different boundary conditions. The energy of the system is related to the control-mode we choose. It is possible to imagine a "gedanken experiment" which reproduces in a laboratory the same conditions described; controlled quantities and energy have in this case an interpretation as physical osservables [7], [37].

The approach we use here is different from the the ones usually found in literature. We treat electromagnetism, described by the Maxwell Lagrangian, as a natural theory ([23] and [25]), which implies that we consider it as a $U(1)$-gauge theory based on a natural principal bundle. This choice implies that is possible to construct a fiber bundle of geometric objects on spacetime (describing the configurations of the electromagnetic field) which is a principal bundle over M. An immediate consequence of this fact is that the magnetic charge of the configurations considered is identically zero. This is the same case usually treated in literature to describe the Hamiltonian formalism for Einstein-Maxwell theory using a trace-K action [7], [8], but in this latter case the vanishing of magnetic charge is assumed (somewhat equivalently) as a restriction on the class of solutions.

4.1 Einstein-Maxwell theory

We consider a Maxwell theory described as a gauge theory with gauge group $U(1)$, based on a fibred principal *natural* bundle $(P, M, U(1); \pi)$, whose existence has been proven in [24] and [25]. A configuration is a section of the connection bundle $\frac{J^1(P)}{U(1)}$, which we denote by $A : M \to \frac{J^1(P)}{U(1)}$ where $A = A_\mu(x)dx^\mu$; it is called the *quadripotential* of the theory. The coefficients A_μ are $u(1)$-valued, where $u(1) = i\mathbb{R}$ is the Lie algebra of the Lie group. This means that A_μ can be interpreted as a $U(1)$-connection and its curvature $F_{\mu\nu}$ is the field strength. We are particulary interested in the interaction between electromagnetic fields and gravity, with configuration bundle $(\frac{J^1(P)}{U(1)} \times \mathrm{Lor}(M), M; \pi)$ and described by the Einstein-Maxwell Lagrangian density:

$$\mathcal{L}_{EM}(j^2 g, j^1 A) = \mathcal{L}_H(j^2 g) + \mathcal{L}_M(g, j^1 A) \tag{36}$$

with:

$$\mathcal{L}_M(g, j^1 A) = -\frac{1}{2k}\sqrt{g} g^{\alpha\mu} g^{\beta\nu} F_{\alpha\beta} F_{\mu\nu} \tag{37}$$

We have defined $F_{\alpha\beta}$ as the *curvature* of the quadripotential

$$F = dA = \frac{1}{2} F_{\alpha\beta} dx^\alpha \wedge dx^\beta = \frac{1}{2}(\partial_\alpha A_\beta - \partial_\beta A_\alpha) dx^\alpha \wedge dx^\beta \tag{38}$$

while the *naive momentum* with respect to the curvature F is denoted by:

$$f^{\alpha\beta} = -k\frac{\partial L_M}{\partial F_{\alpha\beta}} = \sqrt{g} g^{\alpha\mu} g^{\beta\nu} F_{\mu\nu} \tag{39}$$

The components of the curvature F can be decomposed on the Cauchy surface Σ_t, hence defining the electric flux density and the magnetic induction of the electromagnetic field: they represent the physically observable quantities. The Lie derivative of the electromagnetic potential A_μ, according to [23] and [25], is defined as:

$$\pounds_\xi A_\mu = \xi^\nu(\partial_\nu A_\mu) + A_\nu(\partial_\mu \xi^\nu) + q d_\mu(d_\nu \xi^\nu) = \dot{A}_\mu + q d_\mu(d_\nu \xi^\nu) \tag{40}$$

where $\dot{A}_\mu$ is a shortcut for $\dot{A}_\mu = \xi^\nu(\partial_\nu A_\mu) + A_\nu(\partial_\mu \xi^\nu)$ and q is a dimensional constant which represents the geometric charge of the field A_μ carried by the natural representation

$$R_q : X \in GL(m, \mathbb{R}) \to \exp\Big(i\frac{q}{e} \mid det(X) \mid\Big) \in U(1) \tag{41}$$

(here $e < 0$ is the unit charge of the electron). This representation (41) defines the principal bundle $(P, M, U(1); \pi)$ as a bundle associated to the frame bundle (or in other words it is the representation which defines $(P, M, U(1))$ as a natural bundle; see [24], [25]). Notice from (40) that the bundle of connections $\frac{J^1(P)}{U(1)}$ turns out to be a natural bundle of order $r = 2$. Indeed, the Lie derivatives $\pounds_\xi A$ of its sections depend on the coefficients ξ^μ and their derivatives up to order two. Being the Maxwell Lagrangian (37) of order $k = 1$ the total order of the electromagnetic theory turns out to

be $s = 3$.
The Maxwell Lagrangian will be considered separately for the sake of convenience; the generalization to the Einstein-Maxwell theory can be easily obtained recalling the results obtained in Section 3. Euler-Lagrange equations of motion follow for $\mathcal{L}_M$ immediately from the variational principle (10): we obtain that the electromagnetic field must satisfy the equation $J^\mu = 0$, where

$$J^\mu = -\frac{2}{k}(\partial_\alpha f^{\mu\alpha}) = -\frac{2}{k}(\nabla_\alpha f^{\mu\alpha}) \tag{42}$$

is called the *current density.* We remark that the Poincarè -Cartan form is obtained by the prescription (9) as:

$$\mathbb{F}^\lambda(L_M, X) = \frac{2}{k} f^{\mu\lambda}\delta_X A_\mu \tag{43}$$

Using the definition of the Noether current, from formula (11) it follows that:

$$\mathcal{E}^\lambda(L_M, \xi) = \frac{2}{k} f^{\lambda\beta}(\pounds_\xi A_\beta) - \xi^\lambda \mathcal{L} \tag{44}$$

and from (40) we obtain the natural splitting of the current into a reduced current term and the divergence of the superpotential

$$\mathcal{E}^\lambda(L_M, \xi) = \tilde{\mathcal{E}}^\lambda(L_M, \xi) + d_\mu \mathcal{U}^{\lambda\mu}(L_M, \xi) \tag{45}$$

where in local coordinates we have

$$\begin{cases} \tilde{\mathcal{E}}^\lambda(L_M, \xi) = \mathcal{T}^\lambda_\nu \xi^\nu - J^\lambda A_\alpha \xi^\alpha - qJ^{(\lambda}\delta^{\nu)}_\alpha \partial_\nu \xi^\alpha + q\partial_\nu\{J^{[\lambda}\delta^{\nu]}_\alpha\}\xi^\alpha \\ \mathcal{U}^{\lambda\mu}(L_M, \xi) = \frac{2}{k}[f^{\mu\lambda}(qd_\alpha\xi^\alpha + A_\alpha\xi^\alpha)] - qJ^{[\lambda}\delta^{\mu]}_\alpha \xi^\alpha \end{cases}$$

having defined the *stress-energy tensor* of the electromagnetic field by setting:

$$\mathcal{T}^\lambda_\nu = -\frac{2}{k} g^{\lambda\mu}(F^\alpha_\mu F_{\alpha\nu} - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}) \tag{46}$$

The Lagrangian of the Einstein-Maxwell theory is the sum of L_H and $L_M = \mathcal{L}_M ds$. Accordingly Noether currents are simply obtained for this theory as the sum of (45) with the analogous expression for General Relativity (see [21]).

4.2 Energy for the Electromagnetic Field in a $(3 + 1)$ perspective

We are now going to apply the definition (23) for the variation of energy to obtain an expression for the Einstein-Maxwell Lagrangian, writing for convenience only the terms regarding the electromagnetic field, since the ones for the gravitational field are well known from the previous Section. To obtain the final result we need to integrate the superpotential and the correction terms τ on the boundary B_t of the spacetime slice Σ_t, which in turn is assumed to be a Cauchy surface. We shall consider a $(3+1)$ splitting of the dynamical fields evolving with respect to boosted observers, which means that we put ourself in a reference frame which corresponds to observers evolving with respect to ξ.
We define the electric vector field E and magnetic vector field B in the usual way:

$$\begin{cases} E_\mu = F_{\nu\mu} u^\nu \\ B_\mu = u^\nu \epsilon_{\nu\mu\alpha\beta} F^{\alpha\beta} \end{cases}$$

where $\epsilon_{\nu\mu\alpha\beta}$ is the totally skew-symmetric Levi-Civita tensor on M. Conversely the curvature can be defined in terms of the electric and the magnetic vector fields as:

$$F_{\alpha\beta} = (E_\alpha u_\beta - E_\beta u_\alpha) + u^\nu \epsilon_{\nu\alpha\beta\mu} B^\mu \tag{47}$$

We define the *electrostatic potential* and the *normal component of the electric field* as:

$$\begin{cases} \Phi = -A_\alpha u^\alpha \\ E^\perp = F_{\nu\mu} u^\nu n^\mu = E_\mu n^\mu \end{cases}$$

Analogously, by $(\hat{A}_\mu, \hat{E}_\mu, \hat{B}_\mu)$ we denote the quantities projected on the 2-dimensional surface B_t, so that, e.g.:

$$\hat{A}_\mu = \sigma^\nu_\mu A_\nu \Rightarrow A_\mu = \hat{A}_\mu + (A_\nu n^\nu) n_\mu + \Phi u_\mu \tag{48}$$

(and analogous expressions for the other quantities). The symplectic current defined in (19) owing to (43) turns out to be:

$$\omega^\lambda(L_M, X, \pounds_\xi \sigma) = \frac{2}{k}[(\delta_X f^{\mu\lambda})(\pounds_\xi A_\mu) - (\pounds_\xi f^{\mu\lambda})(\delta_X A_\mu)] \tag{49}$$

This expression justifies the definition of $f^{\mu\lambda}$ as the conjugate momentum with respect to the electromagnetic field. The pullback of the symplectic current (49) on the surface Σ_t can be written as:

$$\begin{aligned} \omega(L_M, X, \pounds_\xi \sigma) &= [\delta_X \mathcal{E}^\mu (\pounds_\xi A_\mu) - \pounds_\xi \mathcal{E}^\mu (\delta_X A_\mu)] d^3x = \\ &= [\delta_X \mathcal{E}^\mu (\pounds_\xi A_\mu) - \dot{\mathcal{E}}^\mu (\delta_X A_\mu)] d^3x \end{aligned} \tag{50}$$

where we have defined the 3-*dimensional electric density* $\mathcal{E}^\mu$ and its time rate of change by:

$$\begin{cases} \mathcal{E}^\mu = \frac{2}{k}\sqrt{h} E^\mu \\ \dot{\mathcal{E}}^\mu = \pounds_\xi \mathcal{E}^\mu \end{cases}$$

Using the formula (40) and recalling that the total order of Maxwell-Einstein theory is $s = 3$, we can split $\int_\Sigma \omega$, in accordance with (19), into two bulk terms and a boundary term:

$$\begin{aligned} \int_{\Sigma_t} \omega &= \int_{\Sigma_t} \Big\{ \delta_X \mathcal{E}^\mu (\dot{A}_\mu) - \dot{\mathcal{E}}^\mu (\delta_X A_\mu) \Big\} d^3x + \\ &- q \Big\{ \int_{\Sigma_t} \delta_X (D_\mu \mathcal{E}^\mu) d_\rho \xi^\rho d^3x + \delta_X \int_{B_t} \Big[\frac{\mathcal{E}^\mu n_\mu}{\sqrt{h}} d_\rho \xi^\rho \sqrt{\sigma} \Big] d^2x \Big\} \end{aligned} \tag{51}$$

Finally it is possible to identify the reduced symplectic current and the term τ (see (19)):

$$\tilde{\omega}(L_M, X, \pounds_\xi \sigma) = [\delta_X \mathcal{E}^\mu (\dot{A}_\mu) - \dot{\mathcal{E}}^\mu (\delta_X A_\mu)] d^3x \tag{52}$$

$$\tau(L_M, X, \pounds_\xi \sigma) = \frac{2q}{k} \delta_X \{E^\mu n_\mu \sqrt{\sigma}\} (d_\rho \xi^\rho) d^2x \tag{53}$$

while the remaining term in (51) is identically vanishing on shell. This last expression encodes the symplectic structure of the theory and defines both the phase space and the conjugate momenta [13]. We remark that the boundary part $\tau(L_M, X, \pounds_\xi \sigma)$ of the symplectic current is non vanishing since this holds whenever the total order of the theory is higher than 2: for electromagnetic theory,

treated as a natural theory, the total order is in fact $s = 3$. This term τ_M will influence the definition of the variation of energy in accordance with (23).
The reduced current (45) for this theory can be rewritten as:

$$\begin{aligned}\tilde{\mathcal{E}}(L_M,\xi) = & - \{(N\mathcal{T}^{\alpha}_{\nu}u_\alpha u^\nu) + (\mathcal{T}^{\lambda}_{\alpha}u_\lambda N^\nu)\}\sqrt{h}d^3x + [\mathcal{G}A_\rho\xi^\rho]\sqrt{h}d^3x + \qquad (54)\\ & + q[u_\alpha J^{(\alpha}\delta^{\nu)}_{\sigma}\partial_\nu\xi^\sigma - \partial_\nu(J^{[\alpha}\delta^{\nu]}_{\sigma})\xi^\sigma u_\alpha]\sqrt{h}d^3x \qquad (55)\end{aligned}$$

where we have set:

$$\mathcal{G} = -D_\beta\mathcal{E}^\beta \qquad (56)$$

which is the spatial part of Gauss' constraint and vanishes on-shell. If we define:

$$\begin{cases}\mathcal{H}^{\mathcal{M}} = -\sqrt{h}(\mathcal{T}^{\lambda}_{\nu}u_\lambda u^\nu) = \frac{\sqrt{h}}{k}(E^\alpha E_\alpha + B^\alpha B_\alpha)\\ \mathcal{H}^{\mathcal{M}}{}_\alpha = \sqrt{h}h^\beta_\alpha(\mathcal{T}^{\lambda}_{\beta}u_\lambda) = \frac{2\sqrt{h}}{k}(\epsilon_{\alpha\lambda\beta}E^\lambda B^\beta)\end{cases}$$

we finally have that:

$$\begin{aligned}\int_{\Sigma_t}\tilde{\mathcal{E}}(L_M,\xi) = & \int_{\Sigma_t}[N(\mathcal{H}^{\mathcal{M}} - \Phi\mathcal{G}) + N^\alpha(\mathcal{H}^{\mathcal{M}}{}_\alpha + A_\alpha\mathcal{G})]d^3x + \qquad (57)\\ & +q\int_{\Sigma_t}[u_\alpha J^{(\alpha}\delta^{\nu)}_{\sigma}\partial_\nu\xi^\sigma - \partial_\nu(J^{[\alpha}\delta^{\nu]}_{\sigma})\xi^\sigma u_\alpha]\sqrt{h}d^3x\end{aligned}$$

To calculate the variation of the energy by means of (23) we calculate now the expression for the superpotential, evaluated on B_t:

$$\begin{aligned}\int_{B_t}\mathcal{U}(L_M,\xi)d^2x = & \frac{2}{k}\int_{B_t}E_\perp[-N\Phi + \hat{A}_\rho N^\rho + vN(A_\nu n^\nu)]\sqrt{\sigma}d^2x + \qquad (58)\\ & - q\delta_X\int_{B_t}J^{[\alpha}\delta^{\nu]}_{\sigma}u_\nu n_\alpha\xi^\sigma\frac{\sqrt{\sigma}}{\sqrt{g}}d^2x + \frac{2q}{k}\delta_X\int_{B_t}\{E^\mu n_\mu d_\rho\xi^\rho\sqrt{\sigma}\}d^2x\end{aligned}$$

together with the correction term, which turns out to be:

$$\int_{B_t}i_\xi\mathbb{F}(L_M,X) = \frac{2}{k}\int_{B_t}N(E_\perp u^\mu + vE^\mu + u_\delta n_\nu\epsilon^{\delta\nu\mu\beta}B_\beta)\delta_X A_\mu\sqrt{\sigma}d^2x \qquad (59)$$

Neglecting terms which vanish on shell, from (53), (58) and (59) we obtain that the variation of energy for the electromagnetic part of the Einstein-Maxwell theory is:

$$\begin{aligned}\delta_X E(L_M,\xi) = & \int_{B_t}[\delta_X\mathcal{U} - i_\xi\mathbb{F} - \tau]d^2x = \qquad (60)\\ = & -\frac{2}{k}\int_{B_t}N[\Phi - v(n^\alpha A_\alpha)]\delta_X(E^\perp\sqrt{\sigma})d^2x + \frac{2}{k}\int_{B_t}N^\alpha\delta_X(E^\perp\hat{A}_\alpha\sqrt{\sigma})d^2x + \\ - & \frac{2}{k}\int_{B_t}N[v\hat{E}^\alpha + u_\nu n_\beta\epsilon^{\nu\beta\alpha\mu}\hat{B}_\mu]\delta_X\hat{A}_\alpha\sqrt{\sigma}d^2x\end{aligned}$$

We will see that the first term in the last expression is directly related with the charge of the field, the second is related with the rotational degrees of freedom and the third one is determined by the

values of the electric and the magnatic field projected onto the surface B_t.
We remark that this expression of energy exactly cancels the boundary terms in the variation of the reduced current (57) so that, in accordance with the general theory, we obtain that the variation of the Hamiltonian (21)

$$\delta_X H(L_M,\xi,\Sigma)=\int_\Sigma \delta_X\tilde{\mathcal{E}}(L_M,\xi,\sigma)+\int_{\partial\Sigma}[\delta_X\mathcal{U}(L_M,\xi)-i_\xi\mathbb{F}(L_M,X)-\tau(L_M,X,\pounds_\xi\sigma)] \qquad (61)$$

is a pure bulk term, which, according to (22) and (52), generates the Hamiltonian equations of motion.

4.3 Energy and boundary conditions

In the Einstein-Maxwell theory we have that Hilbert Lagrangian is minimally coupled with the Maxwell Lagrangian, so that the conserved quantities are additive:

$$\delta E(L_{EM},\xi)=\delta E(L_H,\xi)+\delta E(L_M,\xi) \qquad (62)$$

We have previously defined in (35) the quasilocal internal energy for the gravitational field contained in a region of spacetime bounded by a surface $\mathcal{B}$, imposing the Dirichlet metric boundary conditions (33). In analogy, we want to derive now the energy contribution due to the pure electromagnetic part of the Einstein-Maxwell theory. To perform calculations we rewrite equation (60) in terms of boosted observers:

$$\begin{aligned}\delta_X E(L_M,\xi) \;&=\; -\frac{2}{k}\int_{B_t}\bar{N}\bar{\Phi}\delta_X(\bar{E}^\perp\sqrt{\sigma})d^2x+\frac{2}{k}\int_{B_t}\bar{N}^\alpha\delta_X(\bar{E}^\perp\hat{A}_\alpha\sqrt{\sigma})d^2x \\ &-\; \frac{2}{k}\int_{B_t}\bar{N}[\bar{u}_\nu\bar{n}_\beta\epsilon^{\nu\beta\alpha\mu}\hat{\bar{B}}_\mu\delta_X\hat{A}_\alpha]\sqrt{\sigma}d^2x\end{aligned} \qquad (63)$$

where we recall from from the definitions given above that $E^\perp=\bar{E}^\perp$, $\bar{\Phi}=-A_\alpha\bar{u}^\alpha$, $\bar{B}=\bar{u}^\nu\epsilon_{\nu\mu\alpha\beta}F^{\alpha\beta}$ and $\bar{u}_\nu\bar{n}_\beta\epsilon^{\nu\beta\alpha\mu}=\epsilon^{\nu\beta\alpha\mu}u_\nu n_\beta$.
To integrate this variational equation and consequently to obtain the energy contribution in the region bounded by B_t we have to consider a 1-parameter family of solutions of field equations which admit the same physical data fixed on the boundary. We can choose different control modes which correspond to two different physical situations, namely a system which is electrically isolated from outside or an adiabatic system (see [37]).

- We can choose a control mode for the boundary components of the electromagnetic potential $\hat{A}_\alpha$ (i.e. we control the magnetic flux through B_t) and for the electrostatic potential $\bar{\Phi}$, which is equivalent to state that the laboratory is electrically isolated from outside. In other words we assume $\delta_X\hat{A}_\alpha\mid_{B_t}=0$ and $\delta_X\bar{\Phi}\mid_{B_t}=0$ in (63). Recalling that Dirichlet conditions (33) are chosen for the gravitational field $\delta_X\bar{\gamma}\mid_{\mathcal{B}}=0$, we obtain from (63) that:

$$\delta_X E(L_M,\xi)=\delta_X\Big\{-\frac{2}{k}\int_{B_t}\bar{N}\bar{\Phi}\bar{E}^\perp\sqrt{\sigma}d^2x+\frac{2}{k}\int_{B_t}(\bar{N}^\alpha\bar{E}^\perp\hat{A}_\alpha\sqrt{\sigma})d^2x\Big\} \qquad (64)$$

This expression is integrable so that:

$$E(L_M,\xi) - E_0(L_M,\xi) = -\frac{2}{k}\int_{B_t}(\bar{N}\bar{\Phi} - \bar{N}^\alpha \hat{A}_\alpha)\bar{E}^\perp \sqrt{\sigma} d^2x = \tag{65}$$
$$= \frac{2}{k}\int_{B_t}(\xi^\mu A_\mu)\bar{E}^\perp \sqrt{\sigma} d^2x$$

We recall that the variation is performed along a one parameter family of solutions, all admitting the same boundary conditions. Hence E_0 is the energy corresponding to a reference solution inside this family.
Expression (65) for the energy of the electromagnetic field, owing the boundary conditions for the gravitational and the electromagnetic field ($\bar{\Phi}\mid_{B_t} = \bar{\Phi}_0 \mid_{B_t}$, $\hat{A}_{0\alpha}\mid_{B_t} = \hat{A}_\alpha \mid_{B_t}$, $\bar{\gamma}_{\mu\nu} = \bar{\gamma}_{0\mu\nu}$), can be rewritten in an analogous way as:

$$E(L_M,\xi) = -\frac{2}{k}\int_{B_t}(\bar{N}\bar{\Phi} - \bar{N}^\alpha \hat{A}_\alpha)(\bar{E}^\perp - \bar{E}_0^\perp)\sqrt{\sigma} d^2x \tag{66}$$

where the subscript 0 clearly refers to the reference solution. This formula reproduces the energy content of the electromagnetic field and it stresses that the response variable, with this boundary condition, is the normal component of the electric field. The one parameter curve of solutions is parametrized by $\bar{E}^\perp$, i.e. by its electric charge.

- If we choose instead a control mode for $\hat{A}_\alpha$ and $\bar{E}^\perp$, (i.e. $\delta_X \hat{A}_\alpha \mid_{B_t} = 0$ and $\delta_X \bar{E}^\perp \mid_{B_t} = 0$) we state that there is no flux of magnetic and electric field through B_t. In analogy with classical thermodynamics this system can be defined to be *adiabatic*. In this case calculations are even more trivial. From equation (63) and always assuming Dirichlet boundary conditions (33) we find:

$$\delta E(L_M,\xi) = 0 \Rightarrow E(L_M,\xi) = \text{const} \tag{67}$$

  This result, which appears to be quite shocking, can be interpreted by saying that the contribution to energy coming from the pure electromagnetic Lagrangian is neglectable. This is in accordance with the fact that the system is adiabatic. Nevertheless the pure gravitational contribution (35) to the total energy keeps track of the electromagnetic field through the metric solution g of Einstein equations $G_{\mu\nu} = k\mathcal{T}_{\mu\nu}$, where $\mathcal{T}_{\mu\nu}$ is the electromagnetic stress tensor defined in (46). In other words, the quasilocal energy (35) of the gravitational field takes account of the presence of the electromagnetic field.

These results are in accordance with the analysis on control modes proposed by Kijowski in [37], using a symplectic type analysis in the framework of Legendre transformation and Hamiltonian formalism. We remark that formula (65) generalizes this analysis to the case of Einstein-Maxwell theory in a spacetime region with non-orthogonal boundaries.

5 Rigidly rotating horizon

In this Section we are going to recall the definitions of *isolated horizon*, *weakly isolated horizon* and *rigidly rotating horizon*. The concept of isolated horizon was first introduced by A. Ashtekar and coworkers in [2] and later developed in [5] to obtain a viable and more general definition. This new concept allows to generalize the laws of black holes thermodynamics to the case of black holes

which do not admit an event horizon and a global Killing vector field. This situation is much more realistic with respect to the quasi-static laws of black holes mechanics based on the existence of an event horizon and referred to equilibrium situations and small perturbations out of them. The definition of isolated horizons is given intrinsecally and this implies that we do not need to know the whole hystory of spacetime to define this geometric surface (as we usually need for the event horizon in non stationary spacetimes). We can admit isolated horizons with radiation infinitesimally near the horizon surface, meaning non-stationary spacetimes. Conditions are imposed only on the internal geometry of the horizon, not on the whole spacetime, to ensure that the black hole itself is "isolated". This is reasonable also in analogy with the classical approach to thermodynamics [15] where equilibrium is just defined in function of the internal degrees of freedom of the system.
Physical examples of isolated horizons can be found in collapsing stars or in cosmological horizons in de-Sitter spacetime (see [3] for a review on the matter); in the latter case no singularities are present in spacetime, but it is still possible to define thermodynamics for such surfaces.
The matter covered in this subsequent Section is (apart from minor changes needed to adapt to our notations) a brief account of Ashtekar's definitions of [5], which we consider worth of being recalled for the sake of completeness and clarity.

5.1 Geometry of horizons

The fundamental requirement is to ask the isolated horizon Δ (a full definition will be given later) to be a null non expanding 3-surface, which means that the 3-metric on it admits a Killing vector; this encloses the fundamental concept that Δ is isolated in a suitable physical sense. If any matter or electromagnetic field is in interaction with the gravitational field we have to state that the flux of radiation and matter through the horizon is zero, which for the electromagnetic field is equivalent to say that the relevant component of the Poynting vector is zero on Δ. With more precision we can start with the following definition:

Definition 1 *A null hypersurface Δ in a 4-dimensional spacetime M is a 3-dimensional submanifold of the spacetime admitting a null normal vector $l = l^\mu \partial_\mu$ (which for definition is also tangent to the same surface Δ).*

In the sequel we will consider only null surfaces which are diffeomorphic to the product $\Delta \simeq I\!R \times S^2$ such that for each $t \in I\!R$ there exists a diffeomorphism $\psi_t : S^2 \to \Delta_t$, where Δ_t is a two dimensional spacelike surface.
The metric defined on Δ is clearly degenerate for definition (see [46]). We now consider a region D of spacetime such that $\Delta \subset D$; like in the previous Section, we assume D to be foliated by 3-dimensional spacelike hypersurfaces Σ_t, which induce a preferred foliation $\hat{\psi}$ of Δ and we denote again by u^μ the future directed causal normal to Σ_t and by n^μ the outward pointing normal to Δ_t in Σ_t.
To define the metric on Δ we have to introduce a null vector field $\hat{l}$ transverse to Δ. We ask this vector field to be normalized relatively to l so that $\hat{l}^\alpha l_\alpha = -1$. Notice that $l^\alpha \in T\Delta$, $\hat{l}_\alpha \in T^*\Delta$, while transversality requires $l_\alpha \notin T^*\Delta$ and $\hat{l}^\alpha \notin T\Delta$. Due to this normalization condition it is possible to express l and $\hat{l}$ in terms of u^μ and n^μ as:

$$
\begin{aligned}
l_\alpha &= f(u_\alpha + n_\alpha) \\
\hat{l}_\alpha &= \frac{1}{2f}(u_\alpha - n_\alpha)
\end{aligned}
$$

where f is a non-vanishing function on M. Now the "metric" $q_{\alpha\beta}$ and the projector operator q^{ν}_{β} onto Δ can be defined as:

$$\begin{cases} q_{\alpha\beta} = g_{\alpha\beta} + 2l_{(\alpha}\hat{l}_{\beta)} \\ q^{\nu}_{\beta} = g^{\nu\alpha}q_{\alpha\beta} \end{cases}$$

Since $q_{\alpha\beta}$ is degenerate the "inverse" $q^{\alpha\beta}$ of this "metric" is not unique but any reasonable definition of it remains unchanged under the addition of terms of the form $l^{(\alpha}V^{\beta)}$, with V tangent to Δ.
The non-degenerate metric σ over Δ_t can be obtained by considering either Δ_t as a submanifold of Δ or a submanifold of Σ_t. We have:

$$\begin{cases} \sigma_{\alpha\beta} = g_{\alpha\beta} + 2l_{(\alpha}\hat{l}_{\beta)} = g_{\alpha\beta} - n_\alpha n_\beta + u_\alpha u_\beta \\ \sigma^{\nu}_{\beta} = \delta^{\nu}_{\beta} + l^{\nu}\hat{l}_{\beta} + \hat{l}^{\nu}l_{\beta} \end{cases}$$

where the analytical expression of $\sigma_{\alpha\beta}$ is the same of $q_{\alpha\beta}$, but has not to be confused with $q_{\alpha\beta}$ which is defined in a different space. The time evolution, defined by a vector field ξ^α transverse to Σ_t, see (26), can be expressed as an element of $T\Delta$ as:

$$\xi^\alpha = \tilde{N}l^\alpha + \bar{N}^\alpha \tag{68}$$

where $\tilde{N} = \frac{N}{f}$ and $\bar{N}^\alpha = \sigma^{\alpha}_{\beta}N^\beta$.
For each null hypersurface Δ it is possible to select (non uniquely) a vector l^μ (which is for definition a null normal vector) and define the *tensor expansion* of Δ as:

$$\theta_{(l)}{}_{\alpha\beta} = q^{\nu}_{\alpha}q^{\mu}_{\beta}\nabla_\nu l_\mu \tag{69}$$

The expansion plays for null surfaces the same role that extrinsic curvature plays for spacelike or timelike hypersurfaces. As l^α is normal to Δ we have immediately, from Frobenius' theorem, that the twist of the expansion is zero $\theta_{(l)}{}_{[\alpha\beta]} = 0$, so that $\theta_{\alpha\beta}$ can be decomposed as:

$$\theta_{(l)}{}_{\alpha\beta} = \frac{1}{2}\theta_{(l)}\sigma_{\alpha\beta} + \zeta_{\alpha\beta} \tag{70}$$

where $\zeta_{\alpha\beta} = \zeta_{(\alpha\beta)}$ is called the *shear* of the expansion tensor. The scalar:

$$\theta_{(l)} = \theta_{(l)}{}_{\alpha\beta}\sigma^{\alpha\beta} = \frac{1}{\sqrt{\sigma}}\pounds_l(\sqrt{\sigma}) \tag{71}$$

is properly called *expansion* of the null surface (see [46]). We are now ready to give the definition [2]:

Definition 2 *A non-expanding horizon $\Delta \subset M$ is a 3-dimensional submanifold of spacetime (M, g) with the following properties:*

- *Δ is a null hypersurface and it is topologically the product of a two-dimensional sphere and the real line: $\Delta \cong S^2 \times I\!R$*
- *The expansion $\theta_{(l)}$ of any null normal to the surface vanishes on Δ*
- *Equations of motion hold at Δ and the stress energy tensor for matter interacting with gravity is such that $(-T_{\alpha\beta}l^\alpha)\mid_\Delta$ is future directed and causal for each future directed null normal l to Δ*

We remark that the fundamental property to identify an isolated horizon is to impose $\theta_{(l)} = 0$. The topological condition is equivalent to the requirement that there exists a foliation of the hypersurface Δ in "time" constant surfaces. The matter condition is very weak and it is implied for example from the dominant energy condition; it is equivalent to state that there is no flux of matter through the horizon; see [2] for a detailed presentation.
Since the twist of $\theta_{(l)}$ vanishes, from the Raychaudhuri equation [42], we obtain that every null normal l is free of expansion, twist and shear on the non-expanding horizon, i.e. $\theta_{(l)_{[\alpha\beta]}} \mid_\Delta = 0$, $\theta_{(l)} \mid_\Delta = 0$ and $\zeta_{\alpha\beta} \mid_\Delta = 0$.
The same Raychaudhuri equation ensures that there are some restrictions on the Ricci tensor and in particular that the following should hold

$$R_{\alpha\beta} l^\alpha l^\beta \mid_\Delta = 0$$

The *natural connection* 1*-form* ω is defined on Δ (since l is expansion and twist free) as:

$$q^\mu_\alpha \nabla_\mu l^\beta = (\omega_\alpha l^\beta) \tag{72}$$

which implies the property for l to be a Killing vector of the 3-metric q:

$$\pounds_l q_{\alpha\beta} = 0$$

The *acceleration* $k_{(l)}$ of l, defined as:

$$l^\mu \nabla_\mu l^\beta = k_{(l)} l^\beta \tag{73}$$

is expressed in terms of the natural connection as $k_{(l)} = \omega_\alpha l^\alpha$ [2]. Notice that $k_{(l)}$ is the counterpart for isolated horizons of the surface gravity for Killing horizons [46].
Finally we stress that from the third condition in Definition 2 it follows that the stress energy tensor contracted twice with l^α is identically zero, i.e. $T_{\alpha\beta} l^\alpha l^\beta = 0$. This implies that in the case of Einstein-Maxwell theory we are dealing with, $f_\mu = F_{\alpha\beta} l^\alpha \sigma^\beta_\mu = 0$ and $\hat{f}_\mu = F_{\alpha\beta} \hat{l}^\alpha \sigma^\beta_\mu = 0$. This is equivalent to state that the electric and the magnetic components tangential to Δ_t are identically zero.

Definition 3 *A non-expanding horizon becomes a weakly isolated horizon* $(\Delta, [l])$ *if it is equipped with an equivalence class* $[l]$ *of null normals satisfying the equation:*

$$\pounds_l \omega = 0, \forall l \in [l] \tag{74}$$

where the equivalence relation is defined as: $l \sim l' \iff l = cl'$*;* $c \in I\!R$ *is a non zero constant.*

We remark that a Killing horizon is automatically a weakly isolated horizon. As it happens for Killing horizons it is possible to define (apart from a constant factor, which depends on the representative inside $[l]$) the surface gravity for weakly isolated horizons if we equip them (as it has been done in the definition) with a preferred family of null normals. The value of $k_{(l)}$ depends on the element of the class we choose. One can otherwise uniquely define $k_{(l)}$ by demanding it to be a suitably defined function on the horizon; see [3].
The properties of weakly isolated horizons ensure that the *zero law* holds true. This is equivalent

to say that the surface gravity is constant on a weakly isolated horizon. In fact from the property $\pounds_l \omega = 0$, we obtain that:

$$d(k_{(l)}) \mid_{\Delta} = 0 \tag{75}$$

We will consider for applications only the case of rigidly rotating horizons interacting with electromagnetic fields, i.e. the Einstein-Maxwell described above. The theory developed hereafter is also true in the case of isolated horizons, which can be considered as particular cases of rigidly rotating horizons (roughly speaking with vanishing angular momentum). [2]

Definition 5 *A weakly isolated horizon* $(\Delta, [l])$ *is said to be a rigidly rotating horizon* $(\Delta, [l], \varphi)$ *if it admits a rotational simmetry* φ *(with* φ *tangent to the surfaces* Δ*) with closed and circular orbits, such that* $\pounds_\varphi l^\alpha = 0$*,* $\pounds_\varphi \omega_\alpha = 0$ *and* $\pounds_\varphi q_{\alpha\beta} = 0$*.*

This last definition is a restriction on the class of weakly isolated horizons we are considering and it is analogous to state that the 3-dimensional geometry of the hypersurface is axisymmetric, with φ as its infinitesimal rotational symmetry generator [5].
In this case, it is possible to choose a foliation of Δ in a preferred way so that the surfaces Σ_t intersect Δ at the preferred 2-surfaces Δ_t topologically diffeomorphic to S^2 and we state that there exists a tangent vector $\varphi^\alpha \in T\Delta_t$, which generates the desired symmetry. Correspondingly, the time flow transverse to the surface Σ_t is adapted to Δ if $\xi^\alpha = T^\alpha = l^\alpha - \Omega_{(l)} \varphi^\alpha$ on Δ.
In the case of interaction with matter, and particulary in the case of Einstein-Maxwell theory we are going to analyse, it is natural to require φ to be a symmetry also for the electromagnetic field projected on Δ, so that $\pounds_\varphi (q^\alpha_\beta q^\mu_\nu F_{\alpha\mu}) = 0$.
In stationary and asymptotically flat spacetimes one usually sets the electrostatic potential to be $\Phi = -\xi^\mu A_\mu$ and requires a gauge fixing on the electromagnetic potential such that A_μ tends to zero at infinity and $\pounds_\xi A_\mu = 0$ everywhere in spacetime. In analogy with this fact we define a *gauge fixing adapted to the weakly isolated horizon* by setting:

$$\pounds_l A_\mu \mid_{\Delta} = 0 \tag{77}$$

This choice of the gauge is only formally analogous to the condition (74) imposed on ω (i.e. $\pounds_l \omega = 0$) because this latter is a restriction on the form of the gravitational field, while the gauge fixing can be imposed without constraining the electromagnetic field itself. We can now define the horizon *electrostatic potential* as:

$$\Phi_{(l)} = -l^\mu A_\mu \tag{78}$$

and the gauge fixing condition ensures that $\Phi_{(l)}$ is constant on the horizon [2]

$$\pounds_l A_\mu \mid_{\Delta} = 0 \Rightarrow d(\Phi_{(l)}) \mid_{\Delta} = 0 \tag{79}$$

[2] We will not use the above definition in the sequel, but we report it for completeness:

Definition 4 *A weakly isolated horizon becomes an isolated horizon* $(\Delta, [l])$ *if it is equipped with an equivalence class of null normals to it satisfing the equation:*

$$[\pounds_l, D]V = 0 \tag{76}$$

for any vector field V *tangential to* Δ *and* $l \in [l]$*, where* D *denotes the covariant derivative with respect to the metric on* Δ*.*

We stress however that properties holding for weakly isolated horizons are also true for isolated horizons. In fact it is easy to show that every isolated horizon is a weakly isolated horizon (see for details [3]).

This property, together with the zero principle and the boundary conditions which define the geometry of a rigidly rotating horizon, will ensure that the parameters $\Phi_{(l)}, k_{(l)}, \Omega_{(l)}$, which appear in the first principle of thermodynamics for rigidly rotating horizons, are constant on Δ.

5.2 Energy for rigidly rotating horizon

The formalism developed for calculating the variation of energy of the gravitational and the electromagnetic field naturally applies to define the variation of energy of a region of spacetime bounded by an isolated horizon.

We do not mind in the sequel what happens on the outer boundary $\mathcal{B}$; this case has been deeply examined in [28] to which we refer the reader for details. We rewrite the expression for energy variation in the case of Einstein-Maxwell theory, using a 2-surface with no a priori defined geometry. Gluing together formula (32) and formula (63), we find:

$$\begin{aligned}
\delta E(L_{EM},\xi) &= \int_{B_t}\Big\{\bar{N}\delta_X(\sqrt{\sigma}\bar{\epsilon}) - \bar{N}^\alpha\delta_X(\sqrt{\sigma}\bar{j}_\alpha) + \frac{\bar{N}\sqrt{\sigma}}{2}\bar{s}^{\mu\nu}\delta_X\sigma_{\mu\nu}\Big\}d^2x + \\
&+ \frac{1}{k}\int_{B_t}[\pounds_\xi(\sqrt{\sigma})\delta_X(\theta) - \delta_X(\sqrt{\sigma})\pounds_\xi(\theta)]d^2x \\
&- \frac{2}{k}\int_{B_t}\bar{N}\bar{\Phi}\delta_X(\bar{E}^\perp\sqrt{\sigma})d^2x + \frac{2}{k}\int_{B_t}\bar{N}^\alpha\delta_X(\bar{E}^\perp\hat{A}_\alpha\sqrt{\sigma})d^2x + \\
&- \frac{2}{k}\int_{B_t}\bar{N}[\bar{u}_\nu\bar{n}_\beta\epsilon^{\nu\beta\alpha\mu}\hat{\bar{B}}_\mu\delta_X\hat{A}_\alpha]\sqrt{\sigma}d^2x
\end{aligned} \tag{80}$$

This formula expresses the variation of energy for the Einstein-Maxwell theory evaluted on a generic surface B_t. From now we shall specialize this formula to the case of rigidly rotating horizons, i.e. to the case of $B_t = \Delta_t$. We stress that in that case we chose a null evolution of the boundary and this implies that the boost velocity, defined in (27), is $v = 1$.

Using the boost relations (see [10]), we obtain:

$$\begin{cases}
\bar{\epsilon} = \frac{1}{k}[\gamma\mathcal{K} + \gamma v l] \\
\bar{j}_\alpha = \frac{1}{k}\sigma^\beta_\alpha n^\gamma K_{\beta\gamma} - \frac{1}{k}\nabla_\alpha\theta = j_\alpha - \frac{1}{k}\nabla_\alpha\theta \\
\bar{n}^\alpha\bar{a}_\alpha = \gamma n^\alpha a_\alpha - \gamma v u^\alpha b_\alpha + \bar{u}^\alpha\nabla_\alpha\theta
\end{cases}$$

where we have defined $l_{\mu\nu} = -\sigma^\alpha_\mu\sigma^\beta_\nu\nabla_\alpha u_\beta$, so that $l = l_{\mu\nu}\sigma^{\mu\nu}$ and $b^\nu = n^\alpha\nabla_\alpha n^\nu$.

After a long calculation (details can be found in [10], [28] for the gravitational part, while for the electromagnetic part we refer to formula (60)), the equation which defines the variation of energy on a generic null surface can be rewritten, starting from (80) , as:

$$\begin{aligned}
\delta E_\Delta(L_{EM},\xi) &= -\frac{1}{k}\int_{\Delta_t}\Big\{\tilde{N}\delta_X(\sqrt{\sigma}\theta_{(l)}) - \bar{N}^\alpha\delta_X(\sqrt{\sigma}\hat{\omega}_\alpha) + \frac{\tilde{N}\sqrt{\sigma}}{2}s_\Delta{}^{\mu\nu}\delta_X\sigma_{\mu\nu}\Big\}d^2x + \\
&- \frac{2}{k}\int_{\Delta_t}\tilde{N}\Phi_{(l)}\delta_X(E^\perp\sqrt{\sigma})d^2x + \frac{2}{k}\int_{\Delta_t}\bar{N}^\alpha\delta_X(E^\perp\hat{A}_\alpha\sqrt{\sigma})d^2x \\
&- \frac{2}{k}\int_{\Delta_t}\tilde{N}f^\alpha\delta_X\hat{A}_\alpha\sqrt{\sigma}d^2x + \int_{\Delta_t}\{(\pounds_\xi P^{\sqrt{\sigma}}_\Delta)\delta_X\sqrt{\sigma} - (\pounds_\xi\sqrt{\sigma})\delta_X P^{\sqrt{\sigma}}_\Delta\}d^2x
\end{aligned} \tag{81}$$

where we have set:

$$\begin{cases}
s^{\alpha\beta}_\Delta = [\theta_{(l)}{}^{\alpha\beta} - (k_{(l)} + \theta_{(l)})\sigma^{\alpha\beta}] \\
P^{\sqrt{\sigma}}_\Delta = k\ln(f)
\end{cases}$$

The acceleration $k_{(l)}$, in accordance with definition (73), is expressed by:

$$k_{(l)} = f[n^\alpha a_\alpha - u^\alpha b_\alpha] + k \pounds_l (P_\Delta^{\sqrt{\sigma}}) \tag{82}$$

and the natural connection 1-form $\hat{\omega}_\alpha = \sigma_\alpha^\beta \omega_\beta$, in accordance with definition (72), turns out to be:

$$\hat{\omega}_\alpha = \sigma_\alpha^\beta \omega_\beta = \sigma_\alpha^\beta n^\gamma \nabla_\beta u_\gamma + k d_\alpha P_\Delta^{\sqrt{\sigma}} = -k j_\alpha + k d_\alpha P_\Delta^{\sqrt{\sigma}} \tag{83}$$

To perform the calculations from (80) to (81) we also made use of the following relations, which can be easily obtained from (68) and (69):

$$\begin{cases} \theta_{(l)\alpha\beta} = -\xi(\mathcal{K}_{\alpha\beta} + l_{\alpha\beta}) \\ \pounds_\xi \sqrt{\sigma} = \tilde{N}\sqrt{\sigma}\theta_{(l)} + \sqrt{\sigma} d_\beta \bar{N}^\beta \end{cases}$$

Up to now formula (81) holds true for any cross section Δ_t of a generic null hypersurface Δ. In the case of a rigidly rotating horizon it is possible to obtain explicitly a first principle of thermodynamics if we recall the more relevant geometric properties we explained in section (5.1), which hold true on that particular null surface:

- $\theta_{(l)} \mid_\Delta = 0$
- $k_{(l)}$ and $\Phi_{(l)}$ are constant on Δ
- $f_\mu = F_{\alpha\beta} l^\alpha \sigma_\mu^\beta = 0$ and $\hat{f}_\mu = F_{\alpha\beta} \tilde{l}^\alpha \sigma_\mu^\beta = 0$
- $\xi^\alpha = T^\alpha = l^\alpha - \Omega_{(l)} \varphi^\alpha$ on Δ
- φ is a symmetry for both the 3-metric and the electromagnetic field

With these assumptions, which follow directly from the definition of rigidly rotating horizon, it is straightforward to rewrite (81) as:

$$\delta E_\Delta(L_{EM}, \xi) = \frac{k_{(l)}}{k} \delta_X A_\Delta + \Phi_{(l)} \delta_X Q_\Delta + \Omega_{(l)} \delta_X J_\Delta \tag{84}$$

where we define the area, the charge and the angular momentum of the horizon as follows:

$$\begin{cases} A_\Delta = \int_{\Delta_t} \sqrt{\sigma} d^2x \\ Q_\Delta = -\frac{2}{k} \int_{\Delta_t} E_\perp \sqrt{\sigma} d^2x \\ J_\Delta = -\frac{1}{k} \int_{\Delta_t} \varphi^\alpha (\hat{\omega}_\alpha + 2E_\perp \hat{A}_\alpha) \sqrt{\sigma} d^2x \end{cases}$$

and owing to the definition of rigidly rotating horizon the quantities A_Δ and Q_Δ are independent on time, i.e. on the slice Δ_t we are integrating on. The definition of rigidly rotating horizon ensures that also J_Δ is independent on the slice Δ_t we are integrating on; see [5] and [8] for a detailed discussion.
The integrability of this expression depends on the calibration of the parameters (namely the "temperature" $= \frac{k_{(l)}}{k}$, the angular velocity $\Omega_{(l)}$ and the electrostatic potential $\Phi_{(l)}$). This argument are extensively treated in other papers and we refer for details to [8], [5]. We just want to point out that the Noether covariant approach together with a proper analysis of boundary terms in the definition of energy allows us to reproduce the first law of thermodynamics also in the case of rigidly rotating horizons (84) in exactly the same way as it was formulated in [5].

6 Conclusions

Our analysis allowed us to formulate a first principle for rigidly rotating horizons analogous to the ones given in [5] and [8], but the approach and the formalism used here are completely different. The approach used in [5] is metric-affine, based on tetrad gravity, and uses a symplectic type analysis to define the conserved quantities. On the other hand, Booth uses in [8] a purely metric approach based on a canonical decomposition of a "modified" trace-K action functional. Indeed he needs to add suitable boundary terms in the action functional to correct the conserved quantities.
We worked instead in a purely metric framework and we considered a definition of the conserved quantities which naturally arises from the Noether theorem. Closed in spirit with the methods employed in the covariant ADM approach ([23], [27]) and the Regge-Teitelboim analysis of boundary terms in the Hamiltonian variation, we are able to correctly define the variation of the Hamiltonian and the variation of energy by pushing all the boundary terms arising in the variation of the Noether current into the definition itself.
This allowed us to treat succesfully Einstein-Maxwell theory, where Electromagnetism is considered as a natural theory. The framework allows us to define the theory from a geometric point of view, without a priori assumptions on the magnetic field and on the existence of a global potential. Energy definitions arising from the imposed boundary conditions have a nice physical interpretation. Finally this theory find applications in defining a first law for rigidly rotating horizons, which have recently assumed importance in trying to generalize the quasi static treatments of thermodynamics of black holes. These surfaces allow us to define the first principle also for non-static solutions, that admit radiation near the horizon; quasilocal formalism is indispensable in this context.
We have treated here the case of minimal interaction between the gravitational field and the electromagnetic radiation using Einstein-Maxwell Lagrangian. It is straightforward to include into the theory a dilatonic coupling with the Maxwell Lagrangian and generalize the framework to include also more general Yang-Mills fields. This arguments will be treated in a forthcoming paper [1].

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