MOLLIFYING THE RIEMANN ZETA-FUNCTION

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Introduction

This talk is in two parts: the first part is about large values of ζ and serves to give some motivation about the behavior of ζ which necessitates 'the mollifying' or smoothing of ζ that the second part of the paper deals with.

Let

$$I_k = I_k(T) = \frac{1}{T}\int_0^T |\zeta(1/2+it)|^{2k}\, dt.$$

Hardy [H] showed that

$$I_1 \sim \sum_{n\le T}\frac{1}{n}$$

and Ingham [I] proved

$$I_2 \sim 2\sum_{n\le T}\frac{d(n)^2}{n}.$$

Conrey and Ghosh [CG] conjectured that

$$I_3(T) \sim 42\sum_{n\le T}\frac{d_3(n)^2}{n};$$

Conrey and Gonek [CGo] conjecture that

$$I_4(T) \sim 24024\sum_{n\le T}\frac{d_4(n)^2}{n}.$$

Is it true that

$$I_k(T) \sim c_k\sum_{n\le T}\frac{d_k(n)^2}{n}$$

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for some c_k? If so, how does c_k grow?

The work leading to the conjecture of Conrey and Gonek suggests that c_k grows at least as fast as an order 2 function; i.e.

$$c_k \geq 2^{k^2}(1+o(1))$$

(The conjecture of Keating and Snaith [KS] presented at this conference also supports this.)

It is known that

$$\sum_{n\leq T} \frac{d_k(n)^2}{n} \sim \frac{a_k}{\Gamma(k^2+1)} \log^{k^2} T$$

where (for integer k)

$$a_k = \prod_p \left(1-\frac{1}{p}\right)^{(k-1)^2} \sum_{j=0}^{k-1} \frac{\binom{k-1}{j}^2}{p^j}$$

It can be shown [CGo] that

$$\log a_k = -k^2 \log(2e^\gamma \log k) + o(k^2).$$

A lower bound for c_k in the shape $c_k > r^{k^2}$ together with this information about a_k suggests that

$$\max_{t\leq T} |\zeta(1/2+it)| \gg \exp\left((re^{-1-\gamma})^{1/2} \frac{\sqrt{L}}{\sqrt{\log L}}\right)$$

where $L = \log T$.

The best known result is due to Balasubramanian and Ramachandra [BR] who show

$$\max_{t\leq T} |\zeta(1/2+it)| \gg \exp\left(\frac{3}{4}\frac{\sqrt{L}}{\sqrt{\log L}}\right).$$

The conjecture of Keating and Snaith [KS] suggests that one can take $r = k - \epsilon$ above. (See [CF] for details.)

The best known upper bound, assuming RH is

$$\zeta(1/2+it) \ll \exp\left(c\frac{L}{\log L}\right)$$

for some c.

Generally, it has been conjectured that the omega result is closer to the true maximal order of ζ (see [M] for example). However, the above considerations suggest that it could be the upper bound which is actually closer to the truth.

Mollifier definition

The discussion of the last section shows that mean values of ζ are controlled by infrequent large values of ζ. In particular, the loss of 'logs' inherent in Cauchy's inequality

$$\int_0^T |\zeta(1/2+it)|\,dt \le \left(T\int_0^T |\zeta(1/2+it)|^2\,dt\right)^{1/2}$$

makes the use of a 'mollifier' essential for obtaining results about 'a positive proportion of zeros of ζ' on or near the critical line. In 1942 Selberg [S1] gave the first use of a mollifier in this context when he proved that a positive proportion of zeros of ζ are on the critical line. Levinson [L] invented a different approach in 1974 when he showed that at least 1/3 of the zeros of ζ are on the critical-line.

The mollifiers we are interested in are given by:

$$M(s,\theta) = \sum_{n\le y} \frac{\mu(n)P\left(\frac{\log \frac{y}{n}}{\log y}\right)}{n^s}$$

with $y = t^\theta$ and $P(0) = 0$.

Some pictures of mollifiers with $\theta = 1/2$ and $\theta = 1$ are presented at the end of the paper to illustrate the qualitative behavior of a mollifier.

Theorem 1. *(Conrey [C2]). If $\theta < 4/7$, then*

$$\frac{1}{T}\int_0^T \left|M\left(\frac{1}{2}+it,\theta\right)\zeta\left(\frac{1}{2}+it\right)\right|^2\,dt \sim |P(1)|^2 + \int_0^1 |P'(x)|^2\,dx.$$

The techniques used in proving the above lead to

Corollary. *More than 2/5 of the zeros of $\zeta(s)$ are simple and on the line $\sigma = 1/2$.*

Another application is to zeros of the derivatives of the Riemann ξ-function. This function is defined by

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

so that the functional equation for ζ is

$$\xi(s) = \xi(1-s).$$

It is not difficult to show that RH implies that all of the zeros of $\xi^{(m)}(s)$ are on the line $\sigma = 1/2$. Unconditionally we can show (see [C1],[F3])

Theorem 2. *At least 79.89% of the zeros of ξ' are on the critical line. At least 93.4% of the zeros of ξ'' are on the critical line. As $m \to \infty$ the percentage of zeros of $\xi^{(m)}$ on the critical line approaches 100%.*

Mean value of ζ mollified

A more general theorem (see [CGG1]) is given by

Theorem 3. *Suppose that $a, b, c, d \approx 1$. Let $M(s) = M(s, P)$ with $P(x) = x$ in the earlier notation. Then, for $\theta < 4/7$,*

$$\int_0^T \zeta\left(s + \frac{a}{L}\right) \zeta\left(1 - s + \frac{b}{L}\right) M\left(s + \frac{c}{L}\right) M\left(1 - s + \frac{d}{L}\right) dt$$

$$\sim T\left(1 + \frac{1}{\theta}\frac{\partial}{\partial u}\frac{\partial}{\partial v}\, I(u, v, \theta) J(u, v, \theta)\big|_{u=v=0}\right)$$

where

$$I(u, v, \theta) = \int_0^1 e^{-a(x+\theta u)-b(x+\theta v)}\, dx$$

and

$$J(u, v, \theta) = \int_0^1 (y + u)(y + v)e^{-c\theta(1-y-u)-d\theta(1-y-v)}\, dy.$$

Similar formulas exist for discrete means, i.e. for

$$\sum_{\gamma \le T} \zeta(\rho + a)\zeta(1 - \rho + b)M(\rho, P_1)M(1 - \rho, P_2).$$

These can be used to get conditional results such as (GLH stands for "Generalized Lindelöf Hypothesis)

Theorem 4. *(Conrey, Ghosh, and Gonek [CGG2]). Assuming RH + GLH, at least 19/27 of the zeros of $\zeta(s)$ are simple.*

Farmer's Conjecture

If we let $\theta \to \infty$ in the Theorem 3, we are led to

Conjecture. *(Farmer [F1]) If $a, b, c, d \approx 1/\log T$ with c and d positive, then*

$$\frac{1}{T}\int_0^T \frac{\zeta(s + a)\zeta(1 - s + b)}{\zeta(s + c)\zeta(1 - s + d)}\, dt \sim 1 + \frac{\left(1 - T^{-a-b}\right)(a - c)(b - d)}{(a + b)(c + d)}$$

as $T \to \infty$.

Consistency checks for Farmer conjecture

Limiting cases: $a = c$, $c \to \infty$, $d \to \infty$
These can be verified by other means.
Internal Consistency checks:

$$\partial_a\big|_{a=c} = \partial_c\big|_{c=a}$$

These obviously hold for the mean in question and for the conjectural formula.
External Consistency: (1) The formula for

$$\int_0^T \frac{\zeta'}{\zeta}(s+a)\frac{\zeta'}{\zeta}(1-s+b)\ dt$$

is equivalent to "almost everywhere Pair Correlation" (see [GGM]).
(2) The formula for

$$\int_0^T {\frac{\zeta'}{\zeta}}^2(s+a)\frac{\zeta'}{\zeta}(1-s+b)\ dt$$

is consistent with triple correlation predictions (see [F2]).

Sketch of proof of Theorem 1 using Estermann's function $D(s, h/k)$

We have

$$I = \int_0^T |\zeta M(1/2+it)|^2\ dt = \frac{1}{i}\int_{1/2}^{1/2+iT} \zeta(s)\zeta(1-s)M(s)M(1-s)\ ds$$
$$= \frac{1}{i}\int_{1/2}^{1/2+iT} \chi(1-s)\zeta(s)^2 M(s)M(1-s)\ ds$$

where

$$\chi(1-s) = \Gamma(s)\left((2\pi i)^{-s} + (-2\pi i)^{-s}\right).$$

We move the path to the region of absolute convergence and integrate term-by-term. This leads to

$$I \sim \sum_{h,k\le y} \frac{b(h)b(k)}{k} \sum_{n\le Tk/2\pi h} d(n)e(-nh/k).$$

Now

$$D(s, h/k) = \sum_{n=1}^{\infty} \frac{d(n)e(nh/k)}{n^s}$$

has meromorphic continuation to the whole plane with a double pole at $s = 1$ with principal part the same as that of

$$K^{1-2s}\zeta(s)^2$$

where $K = k/(h,k)$. We use Perron's formula on the sum over n. We move the path of integration past the poles at $s = 1$. The main terms then arise from the residues of these poles. The error terms can be handled by large sieve inequalities. To go beyond $\theta = 1/2$, we use the functional equation for D, followed by Vaughan's identity for μ, and then extimates for quintilinear exponential sums due to Deshouillers and Iwaniec [DI]. The latter depend on deep results using the theory of automorphic forms on Hecke congruence groups.

New approach to Theorem 1 via an explicit formula

We sketch a new proof of Theorem 1 with θ (currently) restricted to $\theta < 1/2$. It is hoped, however, that this method will suggest the possibility of obtaining theorems with mollifiers longer even than $\theta = 4/7$.

Let

$$M(s) = \sum_{n\le y} \frac{\mu(n)\log y/n}{n^s \log y} = \sum_{n \le y} \frac{b(m)}{m^s}.$$

Then, assuming (for notational convenience only) that all of the zeros of $\zeta(s)$ are simple, we have

$$\zeta(s)M(s) = 1 - \frac{1}{\log y}\frac{\zeta'}{\zeta}(s) + \frac{\zeta(s)}{\log y}\sum_{\rho}\frac{y^{\rho-s}}{\zeta'(\rho)(\rho-s)^2} + e(s)$$

where $e(s)$ is small. Hence,

$$\begin{aligned}\zeta(s)&\zeta(1-s)M(s)M(1-s)\\ &= \zeta(1-s)M(1-s)\left(\frac{1}{\log y}\frac{\chi'}{\chi}(s)\right) + \frac{M(1-s)\zeta'(1-s)}{\log y}\\ &\quad + \frac{\zeta(s)\zeta(1-s)M(1-s)}{\log y}\sum_{\rho}\frac{y^{\rho-s}}{\zeta'(\rho)(\rho-s)^2} + e_1(s).\end{aligned}$$

The integral of the first term is $\sim T(1+1/\theta)$. The integral of the second term is $o(T)$. The last term gives, after switching summation and integration and changing the ranges is

$$\frac{1}{i}\sum_{\gamma\le T}\frac{y^{\rho}}{\zeta'(\rho)\log y}\int_{(1/2)}\frac{\zeta(s)\zeta(1-s)M(1-s)}{(s-\rho)^2}y^{-s}\ ds.$$

Now let

$$T_\rho(x) = \frac{1}{2\pi i}\int_{(1/2)}\frac{\zeta(s)\zeta(1-s)}{(\rho-s)^2}x^{-s}\ ds.$$

Then

$$T_\rho(x) = \frac{-1}{x^\rho}\int_0^\infty\left(\sum_{n\le u}\frac{1}{n^\rho} - \frac{u^{1-\rho}}{1-\rho}\right)\left(\sum_{n\le ux}\frac{1}{n^{1-\rho}} - \frac{(ux)^\rho}{\rho}\right)\frac{du}{u}.$$

So we have

$$-2\pi \sum_{\gamma \le T} \frac{y^\rho}{\zeta'(\rho)\log y} \sum_{m \le y} \frac{b(m)}{m} \left(\frac{m}{y}\right)^\rho T_\rho(y/m).$$

Now we let

$$U_x(s) = \int_{1/x}^\infty \left(\zeta(s) - \sum_{n \le u} \frac{1}{n^s} + \frac{u^{1-s}}{1-s}\right)\left(\zeta(1-s) - \sum_{n \le xu} \frac{1}{n^{1-s}} + \frac{(xu)^s}{s}\right) \frac{du}{u}.$$

Then $U_x(s)$ is an entire function of s for each $x > 0$. Our sum in question is

$$\frac{-2\pi}{\log y} \sum_{\gamma \le T} \frac{\mathcal{B}(1-\rho)}{\zeta'(\rho)}$$

where

$$\mathcal{B}(s) = \sum_{m \le y} \frac{b(m) U_{y/m}(1-s)}{m^s}.$$

We write the sum over zeros as a contour integral

$$\frac{i}{\log y} \int_{\mathcal{C}_T} \frac{\mathcal{B}(1-s)}{\zeta(s)}\, ds$$

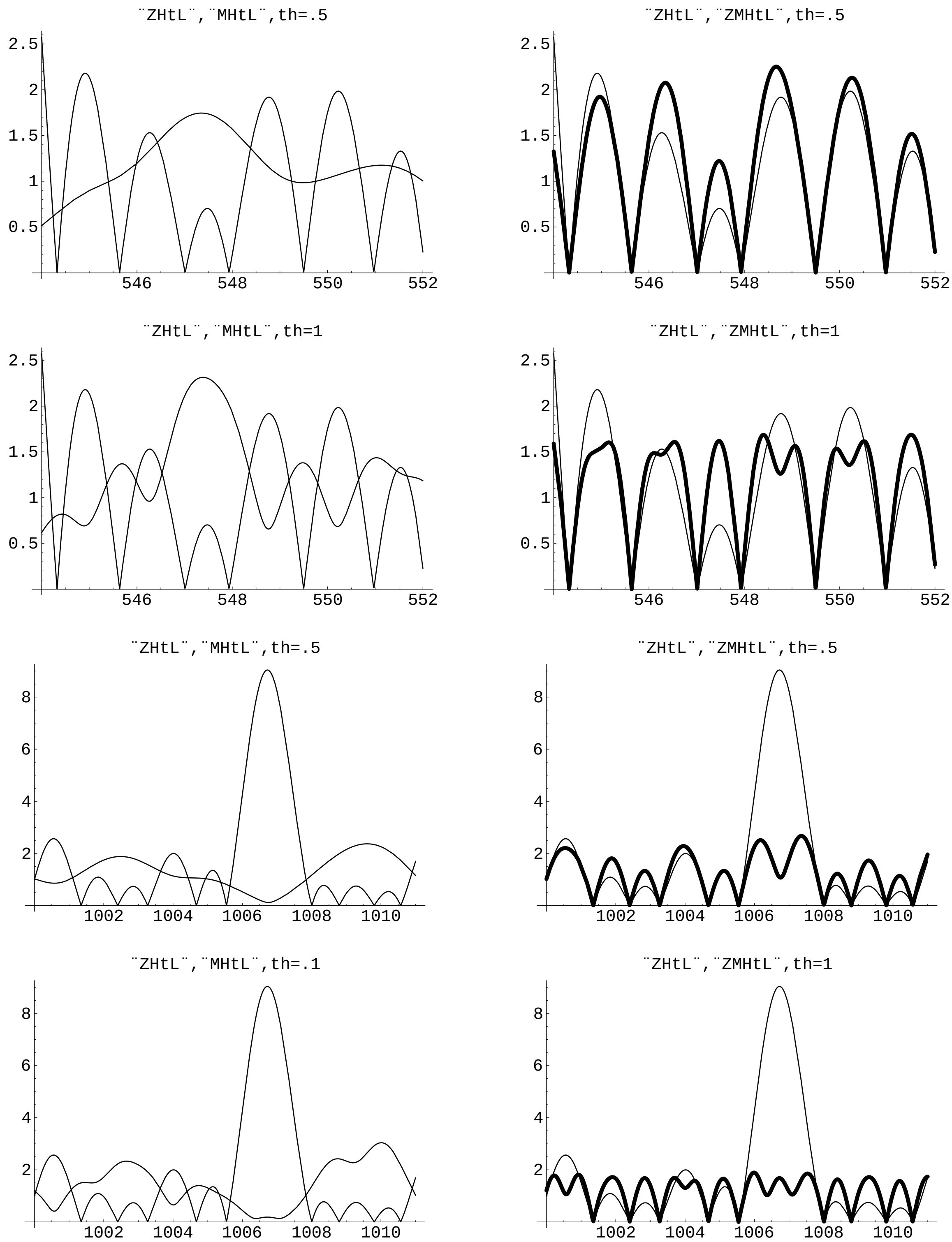
for an appropriate contour $\mathcal{C}_T$. The integral on the left side can be shown to be small. The integral on the right hand side can be evaluated by the mean value theorem for Dirichlet polynomials

$$\int_0^T |\sum_{n \le N} a_n n^{it}|^2 \, dt = \sum_{n \le N} (T + O(n))|a_n|^2.$$

After some arithmetical calculations we deduce that the whole sum is $O(T/\log y)$ for $\theta < 1/2$.

Conclusion

The goal of the new proof is to try to 'localize' the integral in question. By this I mean we want to try to model a proof as closely as possible to proofs of high moments of $S(t)$ or $\log|\zeta(1/2+it)|$ which have been treated by Selberg [S2] and others. These proofs use Dirichlet polynomials in a fundamentally different way than existing approaches to Theorem 1. In particular, the lengths of the Dirichlet polynomial approximations to $S(T)$ for example do not limit the size of the moment that can be treated in the way that such a limitation occurs in the (past) treatment of 'global' integrals. The techniques for the local integrals involve approximating the function in question through an explicit formula that features a sum of a Dirichlet polynomial and a sum over zeros. (We have tried to do the same for our mollified ζ integrals.) Then, the key step is that the sums over zeros can be eliminated by bounding them by Dirichlet polynomials! We are looking for a similar approach here, when after our many manipulations, we eliminate the sum over zeros and return to Dirichlet polynomials. However, we have not yet succeeded in doing this. Nevertheless, the above sketch may be of some interest.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