



# A new signal subspace processing for DOA estimation

Ali Olfat\*, Said Nader-Esfahani

*Department of Electrical and Computer Engineering, University of Tehran, P.O. Box 14395-515, Tehran, Iran*

Received 8 February 2003; received in revised form 26 November 2003

## Abstract

We present a new subspace-based direction of arrival (DOA) estimation algorithm for narrow-band sources with high-resolution localization capabilities. The algorithm is based on the invariance property of noise subspace to power of emitters. We specifically compare the resolution of the proposed algorithm with the well-known high-resolution multiple signal classification (MUSIC) method. The proposed method performance is also superior to MUSIC algorithm in the case of correlated sources. Besides, the proposed algorithm is much less sensitive to the power level differences between adjacent sources compared to MUSIC, i.e., very weak sources in vicinity of strong ones can be resolved.

© 2003 Elsevier B.V. All rights reserved.

*Keywords:* Direction of arrival; Signal subspace; Noise subspace; Multiple signal classification

## 1. Introduction

In direction of arrival (DOA) estimation, a set of unknown parameters (propagation direction of the waves) should be estimated from noisy measurements collected by array of sensors. An important feature of any DOA estimation algorithm is its capability to resolve closely spaced sources. High-resolution DOA estimation is important in many applications such as radar, sonar and electronic surveillance. Recent applications include array processing for wireless mobile communications at the base station for increasing the capacity and quality of these systems [2,7,12].

Among the methods proposed for DOA estimation, the class of techniques known as eigen-

decomposition based algorithms is the most promising. These methods, by exploiting the underlying data model, try to separate the space spanned by the measured data into what are called noise and signal subspaces. Within this class of algorithms, the multiple signal classification (MUSIC) method [14] has received the most attention and has been widely studied [4,8,10,15]. Although the MUSIC is known as a high-resolution algorithm for DOA estimation, but in case of finite data samples it cannot resolve adjacent sources with large power level differences between them. In this paper a new method of DOA estimation, based on the invariance property of noise subspace to power of emitters, is presented. This method not only has a higher resolution than the MUSIC but also can resolve very weak sources in the vicinity of strong ones. The proposed algorithm can handle all array configurations and like the MUSIC method its DOA estimates are asymptotically exact, i.e., exact estimates are obtained

\* Corresponding author. Tel.: +98-21-271-1304; fax: +98-21-877-8690.

*E-mail addresses:* aolfat@ut.ac.ir (A. Olfat), nader@ut.ac.ir (S. Nader-Esfahani).

asymptotically as the number of measurements goes to infinity irrespective of the signal-to-noise ratios (SNR) and angular separations of the sources.

Another important feature of DOA estimation algorithms is their capability of handling of correlated sources. It is well known that for uncorrelated signals the asymptotic covariance of DOA estimates of MUSIC method coincides with the one of maximum likelihood (ML) method [15,10]. But the correlation between sources will cause the performance degradation of MUSIC compared to ML method. We will show in simulation results that the proposed method will perform much better than MUSIC in the case of highly correlated sources. The price paid for these advantages is higher computational complexity of proposed method over MUSIC. But with new emerging technologies in design and implementation of high-speed signal processors, this would not be a problem in many applications.

The paper is organized as follows. First, the signal and noise model is presented and DOA estimation problem is formulated. Next, the new method is presented and its asymptotic properties are investigated. Simulation results for comparison between the proposed algorithm and the MUSIC method are given in Section 4, followed by a summary of the main conclusions arising from this work.

## 2. The data model

Let an array of  $M$  sensors receive  $d$  narrow band plane waves from far-field emitters with the same known center frequency  $f_0$  and different directions as shown in Fig. 1.

With the narrow band assumption the  $i$ th signal complex envelop representation [16] can be shown as

$$s_i(t) = u_i(t)e^{j(2\pi f_0 t + \varphi_i(t))}, \quad i = 1, \dots, d \quad (1)$$

where  $u_i(t)$  and  $\varphi_i(t)$  are slowly varying functions of time that define the amplitude and phase of  $i$ th signal, respectively. Slowly varying means  $u_i(t) \approx u_i(t - \tau)$  and  $\varphi_i(t) \approx \varphi_i(t - \tau)$  for all possible propagation delays  $\tau$  between array sensors, and as a result of this the effect of a time delay on received waveforms is simply a phase shift, i.e.,

$$s_i(t - \tau) \approx s_i(t)e^{-j2\pi f_0 \tau}. \quad (2)$$

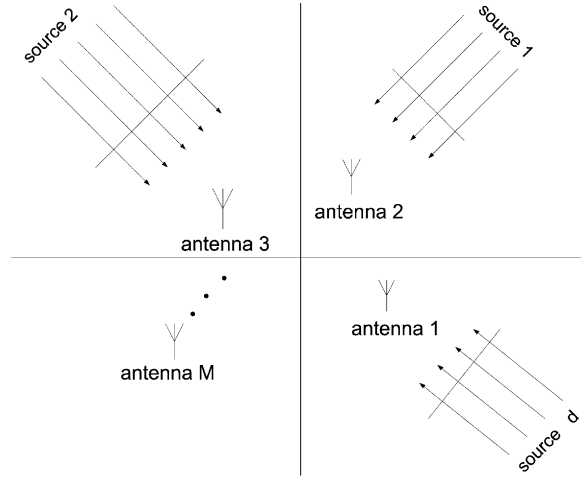


Fig. 1. An array with  $M$  sensors receiving signals from  $d$  narrow-band sources.

Then  $x_k(t)$ , the complex received signal of the  $k$ th sensor at time  $t$ , can be expressed as

$$\begin{aligned} x_k(t) &= \sum_{i=1}^d a_k(\theta_i) s_i(t - \tau_k(\theta_i)) + n_k(t) \\ &= \sum_{i=1}^d a_k(\theta_i) s_i(t) e^{-j2\pi f_0 \tau_k(\theta_i)} + n_k(t), \\ k &= 1, 2, \dots, M, \end{aligned} \quad (3)$$

where  $\tau_k(\theta_i)$  is the propagation delay between a reference point and the  $k$ th sensor for the  $i$ th wavefront impinging on the array from direction  $\theta_i$ ,  $a_k(\theta_i)$  is the corresponding sensor element complex response (gain and phase) at frequency  $f_0$ ,  $d$  is the number of sources present, and  $n_k(t)$  is additive noise at  $k$ th sensor element.

Using vector notation for the received signals of  $M$  sensors, the data model can be presented as

$$\underline{\mathbf{x}}(t) = \sum_{i=1}^d \underline{\mathbf{a}}(\theta_i) s_i(t) + \underline{\mathbf{n}}(t), \quad (4)$$

where

$$\underline{\mathbf{x}}(t) = [x_1(t), \dots, x_M(t)]^T,$$

$$\underline{\mathbf{n}}(t) = [n_1(t), \dots, n_M(t)]^T,$$

$$\underline{\mathbf{a}}(\theta_i) = [a_1(\theta_i)e^{-j2\pi f_0\tau_1(\theta_i)}, \dots, a_M(\theta_i)e^{-j2\pi f_0\tau_M(\theta_i)}]^T$$

$$i = 1, \dots, d \quad (5)$$

and superscript T denotes transpose of a matrix. The  $M \times 1$  vector  $\underline{\mathbf{a}}(\theta_i)$  is known as array response or array steering vector for direction  $\theta_i$ . With defining the  $M \times d$  matrix  $\mathbf{A} = [\underline{\mathbf{a}}(\theta_1), \dots, \underline{\mathbf{a}}(\theta_d)]$  and  $d \times 1$  vector  $\underline{\mathbf{s}}(t) = [s_1(t), \dots, s_d(t)]^T$ , relation (4) can be written as

$$\underline{\mathbf{x}}(t) = \mathbf{A}\underline{\mathbf{s}}(t) + \underline{\mathbf{n}}(t). \quad (6)$$

The DOA problem is finding proper estimations of  $\theta_i, i=1, \dots, d$  from a finite number ( $N$ ) of data samples or snapshots of  $\underline{\mathbf{x}}(t_j)$  taken at times  $t_j, j = 1, \dots, N$ .

For solving the DOA estimation the following assumptions are made:

- (1) The number of sources is known and is less than the number of sensors, i.e.,  $d < M$ .
- (2) The sources are uncorrelated zero mean stationary processes with the  $d \times d$  diagonal covariance matrix

$$\mathbf{R}_S = E\{\underline{\mathbf{s}}(t)\underline{\mathbf{s}}(t)^H\} = \text{diag}\{\eta_1^2, \eta_2^2, \dots, \eta_d^2\}, \quad (7)$$

where  $\eta_i^2 = E\{|s_i(t)|^2\}$  denotes the power (variance) of  $i$ th source,  $E$  is mathematical expectation operator and superscript H denotes conjugate transpose.

- (3) The additive noise at each sensor is a stationary zero mean complex white Gaussian process. The noise processes of different sensors are uncorrelated and with the covariance matrix

$$\mathbf{R}_n = E\{\underline{\mathbf{n}}(t)\underline{\mathbf{n}}(t)^H\} = \sigma_n^2 \mathbf{I}, \quad (8)$$

where  $\sigma_n^2$  is the noise power at each sensor and  $\mathbf{I}$  is an  $M \times M$  identity matrix.

- (4) The noise and signal waveforms are uncorrelated.
- (5) The array response vector  $\underline{\mathbf{a}}(\theta)$  is known for all  $\theta$  and the array is configured in such a way that the matrix  $\mathbf{A}$  in relation (6) has full column rank, i.e.,  $\text{rank}(\mathbf{A}) = d$ . This also implies the source directions to be different in space, i.e.,  $\theta_i \neq \theta_j$ .

From the above assumptions and (6) the  $M \times M$  covariance matrix of received data can be expressed as

$$\mathbf{R}_X = E\{\underline{\mathbf{x}}(t)\underline{\mathbf{x}}(t)^H\} = \mathbf{A}\mathbf{R}_S\mathbf{A}^H + \sigma_n^2 \mathbf{I}. \quad (9)$$

In practice, the data covariance matrix  $\mathbf{R}_X$  is not available but a maximum likelihood estimate  $\hat{\mathbf{R}}_X$  based on a finite number ( $N$ ) of data samples can be obtained as

$$\hat{\mathbf{R}}_X = \frac{1}{N} \sum_{j=1}^N \underline{\mathbf{x}}(t_j)\underline{\mathbf{x}}(t_j)^H \quad (10)$$

and estimation of direction of arrival of sources is based on this sample covariance matrix.

### 3. Proposed DOA estimation algorithm

The covariance matrix  $\mathbf{R}_X$  given in (9) is a positive definite matrix and we denote its eigenvalues (in decreasing order) and their corresponding eigenvectors by  $\lambda_i$  and  $\underline{\mathbf{e}}_i$ , i.e.,

$$\mathbf{R}_X \underline{\mathbf{e}}_i = \lambda_i \underline{\mathbf{e}}_i, \quad i = 1, \dots, M. \quad (11)$$

Using the assumptions made in previous section  $\mathbf{A}\mathbf{R}_S\mathbf{A}^H$  in (9) is a rank— $d$  matrix and it can be shown that [5,6,9]

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d \geq \lambda_{d+1} = \dots = \lambda_M = \sigma_n^2. \quad (12)$$

The range space of  $\mathbf{A}$  (space spanned by columns of  $\mathbf{A}$ ) is called the signal subspace and it can be verified that [5,6,9]

$$\text{Range}\{\mathbf{A}\} = \text{Span}\{\underline{\mathbf{e}}_1, \dots, \underline{\mathbf{e}}_d\}. \quad (13)$$

Relation (13) states that each column of  $\mathbf{A}$  that in fact is a steering vector corresponding to a source direction, is completely in the subspace spanned by the first  $d$  eigenvector of  $\mathbf{R}_X$  and consequently they are orthogonal to the last  $(M - d)$  eigenvectors  $\underline{\mathbf{e}}_j, j = d + 1, \dots, M$  and for this reason the subspace spanned by  $\{\underline{\mathbf{e}}_{d+1}, \dots, \underline{\mathbf{e}}_M\}$  is called the noise subspace. The MUSIC algorithm uses this property to estimate direction of sources [14].

If we keep the direction of sources fixed and change their powers (variances), then the last  $(M - d)$  eigenvalues and their corresponding eigenvectors of new covariance matrix are the same as before. In other words the noise subspace is invariant to power of sources and our proposed algorithm is based on this property.

We define a new matrix  $\mathbf{D}^\varphi$  as

$$\mathbf{D}^\varphi \triangleq \mathbf{R}_X + h\underline{\mathbf{a}}(\varphi)\underline{\mathbf{a}}(\varphi)^H, \quad (14)$$

where  $\mathbf{R}_X$  is given in (9),  $\mathbf{a}(\varphi)$  is  $M \times 1$  array response vector for direction  $\varphi$  given in (5) and  $h$  is a positive constant scalar. If  $\mu_k^\varphi$  and  $\mathbf{v}_k^\varphi$ ,  $k = 1, \dots, M$  denote eigenvalues (in decreasing order) and eigenvectors of  $\mathbf{D}^\varphi$ , respectively, i.e.,

$$\begin{aligned} \mathbf{D}^\varphi \mathbf{v}_k^\varphi &= \mu_k^\varphi \mathbf{v}_k^\varphi, & k = 1, \dots, M, \\ \mu_1^\varphi &\geq \mu_2^\varphi \cdots \geq \mu_M^\varphi, \end{aligned} \quad (15)$$

then it can be proved that while  $h$  in (14) is a positive scalar we have [5,17]

$$\begin{aligned} \mu_k^\varphi &\geq \lambda_k, \\ \forall \varphi, & \quad k = 1, \dots, M. \end{aligned} \quad (16)$$

Another important property of  $\mathbf{D}^\varphi$  is that when  $\varphi$  in (14) is set to one of the source directions, i.e.,  $\varphi = \theta_i$  for some  $1 \leq i \leq d$ , then the last  $(M-d)$  eigenvalues of  $\mathbf{D}^\varphi$  and  $\mathbf{R}_X$  are the same, i.e.,

$$\mu_k^\varphi = \lambda_k = \sigma_n^2, \quad k = d+1, \dots, M. \quad (17)$$

It is interesting to note that except the actual source directions no other value of  $\varphi$  has this property. The property stated in (17) does not depend on the value of scalar  $h$  in (14) explicitly and will hold while  $h$  is positive. The proof of (17) is given in the appendix. Our proposed algorithm finds the source directions by the following steps:

- (i) Compute  $\lambda_k$ ,  $k = 1, \dots, M$ , the eigenvalues of correlation matrix  $\mathbf{R}_X$  of (9) and they should satisfy (12).
- (ii) Choose a positive  $h$  and compute  $\mathbf{D}^\varphi$  given in (14) and its eigenvalues  $\mu_k^\varphi$ ,  $k = 1, \dots, M$  for all possible values of  $\varphi$ .
- (iii) The direction of sources are those values of  $\varphi$  that satisfy (17).

When exact covariance matrix  $\mathbf{R}_X$  given in (9) is available the above algorithm is applicable and exact direction of sources will be obtained. But in practice a sample covariance matrix  $\hat{\mathbf{R}}_X$  that is estimated from finite ( $N$ ) number of snapshots given in (10), is available. The above algorithm is not applicable to  $\hat{\mathbf{R}}_X$ , since relations (12), (13) and (17) do not hold and in fact it can be shown [1] that with probability one, the  $(M-d)$  smaller eigenvalues of  $\hat{\mathbf{R}}_X$  are different.

Our proposed DOA algorithm is a revised version of the above algorithm. Indeed, instead of unavailable

$\mathbf{R}_X$  in (14), we substitute  $\hat{\mathbf{R}}_X$  that is obtained from (10) and search for the directions that not exactly but as closely as possible satisfy (17). What we mean with as close as possible is stated in the following. The proposed algorithm includes the following steps:

- (1) From  $N$  data samples compute  $\hat{\mathbf{R}}_X$  from (10).
- (2) Compute  $\hat{\lambda}_k$ ,  $k = 1, \dots, M$ , the eigenvalues of  $\hat{\mathbf{R}}_X$  that are in decreasing order.
- (3) Substitute  $\hat{\mathbf{R}}_X$  in (14) for  $\mathbf{R}_X$ , choose a positive  $h$  and compute  $\hat{\mathbf{D}}^\varphi$  for all possible values of  $\varphi$  according to the following

$$\hat{\mathbf{D}}^\varphi \triangleq \hat{\mathbf{R}}_X + h \mathbf{a}(\varphi) \mathbf{a}(\varphi)^H. \quad (18)$$

- (4) For each value of  $\varphi$ , compute  $\hat{\mu}_k^\varphi$ ,  $k = 1, \dots, M$  the eigenvalues of  $\hat{\mathbf{D}}^\varphi$  in decreasing order and calculate  $F(\varphi)$  as

$$F(\varphi) = \frac{1}{\sum_{k=d+1}^M (\hat{\mu}_k^\varphi - \hat{\lambda}_k)}. \quad (19)$$

- (5) The direction of sources are those values of  $\varphi$  that correspond to  $d$  largest maximums of  $F(\varphi)$ .

Using (16), it can be verified that  $F(\varphi)$  is a positive function and by using (17) it can be shown that when  $\hat{\mathbf{R}}_X = \mathbf{R}_X$  then the denominator of (19) is zero when  $\varphi$  is set to actual direction of sources and as a result maximums of (19) are exact direction of sources. This proves the asymptotic consistency of proposed method, i.e., when the number of samples ( $N$ ) used for estimation of  $\mathbf{R}_X$  in (10) tends to infinity then  $\hat{\mathbf{R}}_X \stackrel{\text{w.p.1}}{=} \mathbf{R}_X$  [1,11] and therefore the maximums of  $F(\varphi)$  will tend to exact direction of sources. As will be shown in simulations the resolution of this algorithm is higher than MUSIC method and is applicable to all array configurations. Another important advantage of the proposed method is that unlike the MUSIC this method is insensitive to power level differences of closely spaced sources.

When there is correlation between sources, their covariance matrix  $\mathbf{R}_S$  defined in (7) is not diagonal. This may happen in practice for example when there is multipath propagation or there are smart jammers in communication applications. If some sources are fully correlated, i.e., they are linearly dependent then  $\text{rank}(\mathbf{R}_S) < d$  and conventional MUSIC method (and also proposed method) fails to estimate the DOA of sources. In this case usually symmetric-array

configurations are employed and after some kind of preprocessing such as forward-backward smoothing [9], conventional MUSIC (or proposed method) can be applied. When the sources are partially correlated the matrix  $\mathbf{R}_S$  is not diagonal but  $\text{rank}(\mathbf{R}_S) = d$ . In this case the MUSIC method can be applied but its performance highly deteriorates comparing to case of uncorrelated sources [10,15].

Another important advantage of proposed algorithm is that it can also be applied in the case of partially correlated sources and its performance is much better than MUSIC even the sources are highly correlated. In other words the performance deterioration of proposed method in case of partially correlated sources is much less than MUSIC compared to case of uncorrelated sources.

**4. Simulation results**

In all the simulations a uniform circular array of eight omni-directional sensors with half wavelength spacing is used as shown in Fig. 2. For applying the proposed algorithm, we should set a positive value for parameter  $h$  in (14). Theoretically all positive values are allowed but it has been verified by simulations that suitable values for  $h$  can increase the performance of algorithm. The suitable value that was experimentally obtained is

$$h = \frac{\text{tr}(\hat{\mathbf{R}}_X)}{M}, \tag{20}$$

where  $\text{tr}(\hat{\mathbf{R}}_X)$  denotes trace of  $\hat{\mathbf{R}}_X$  and  $M$  is the number of antenna elements in the array. The value of  $h$  in (20) has the property that  $\text{tr}(\hat{\mathbf{R}}_X) = \text{tr}(h\mathbf{a}(\varphi)\mathbf{a}(\varphi)^H)$ . In all the simulations we have set the value of  $h$  according to (20).

In the first experiment, we consider two uncorrelated sources ( $d = 2$ ) with direction of arrivals  $90^\circ$  and  $98^\circ$ . We have performed MUSIC and proposed method of this paper 1000 times for different SNR values using  $N = 100$  samples and the number of successful simulations for each algorithm versus SNR is plotted in Fig. 3. In all the simulations of this paper successful simulations are those that show two distinct peaks in their spectrum around source directions and unsuccessful simulations are those that show one peak and can't resolve two sources. It can be verified

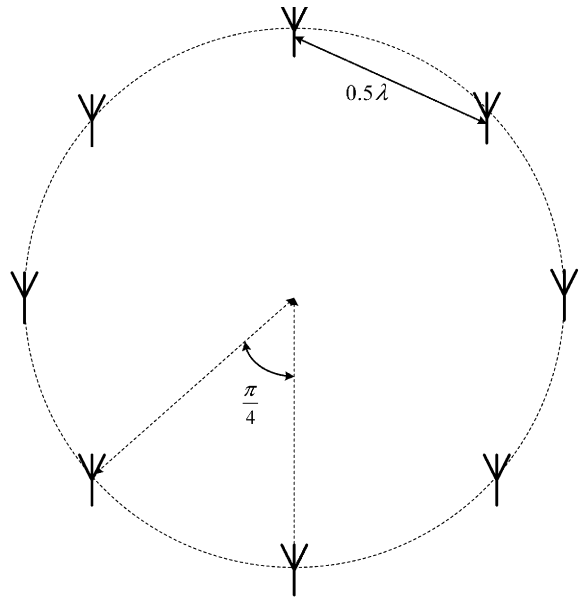


Fig. 2. An 8-element uniform circular array that is used for simulations of this paper.

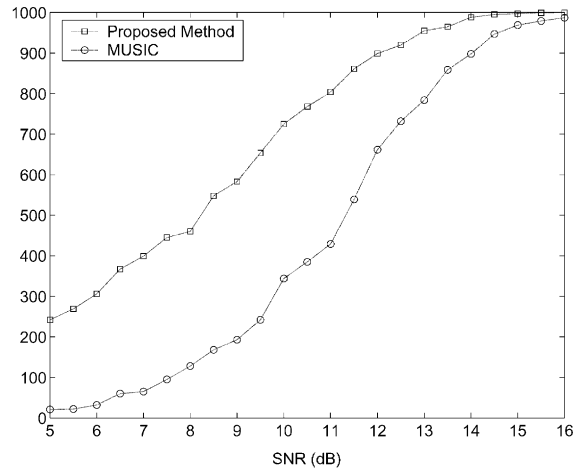


Fig. 3. Number of successful simulations (out of 1000 simulations) versus signal-to-noise ratio in dB for two uncorrelated sources with equal powers located at  $90^\circ$  and  $98^\circ$ .

from Fig. 3 that the resolution of the proposed method is higher than MUSIC method for sources with equal powers. It is important to note that in all successful simulations of MUSIC method the proposed algorithm has also been successful.

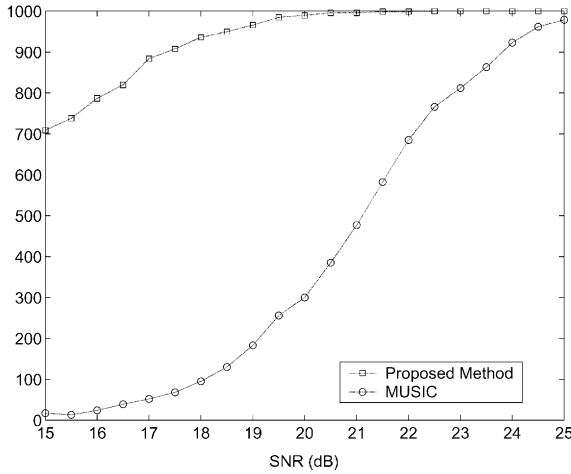


Fig. 4. Number of successful simulations (out of 1000 simulations) for two uncorrelated sources located at 90° and 98°. The power of source located at 90° is shown on horizontal axis and is 10 dB stronger than the other.

In the second experiment, we consider two uncorrelated sources ( $d = 2$ ) with direction of arrivals 90° and 98°. The SNR of second source (source located at 98°) is 10 dB less than first source (source located at 90°). We have performed MUSIC and proposed method of this paper 1000 times for different SNR values of first source using  $N = 100$  samples and the number of successful simulations for each algorithm versus SNR (SNR of first source) is plotted in Fig. 4. This figure shows higher capability of proposed method in resolving closely spaced sources with large power differences compared to MUSIC. The same as the first experiment in all successful simulations of MUSIC method the proposed algorithm has also been successful with a much distinct peaks. For comparison, the spectrum of proposed method and MUSIC when both of them are successful in second experiment is shown in Fig. 5.

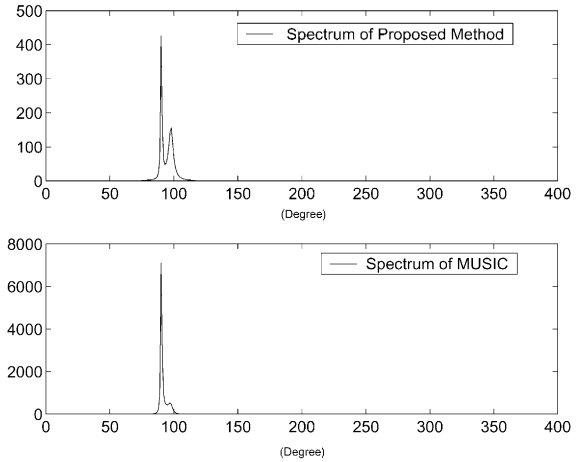


Fig. 5. The spectrum of proposed method and MUSIC for a successful simulation in the second experiment.

In the third experiment, we consider two uncorrelated sources ( $d = 2$ ) with direction of arrival of the first source fixed at 90° and vary the direction of arrival of second source. The SNR of both sources are 10 dB. At each location of second source we have performed MUSIC and proposed method of this paper 1000 times using  $N = 100$  samples and the number of successful simulations for each algorithm versus DOA of second source is plotted in Fig. 6.

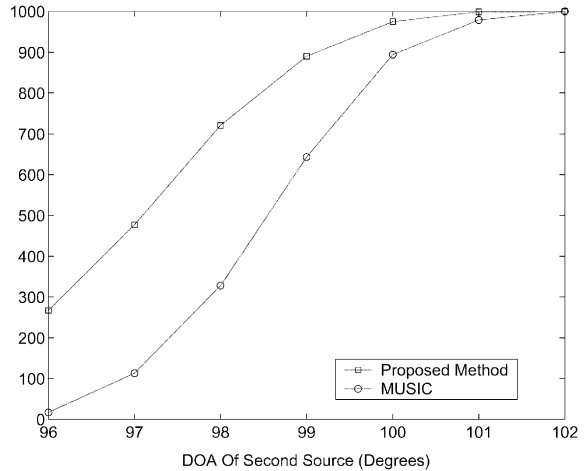


Fig. 6. Number of successful simulations (out of 1000 simulations) for two uncorrelated sources with first one fixed at 90° versus the location of the second one. The SNR of both sources is 10 (dB).

In the last (fourth) experiment, for investigating the power of proposed method in case of highly correlated sources we consider two correlated sources ( $d = 2$ ) with correlation coefficient  $r = 0.99$  and equal powers. In fact the covariance of the sources is given by

$$R_S = 10^{(0.1 \text{ SNR})} \begin{bmatrix} 1 & 0.99 \\ 0.99 & 1 \end{bmatrix}.$$

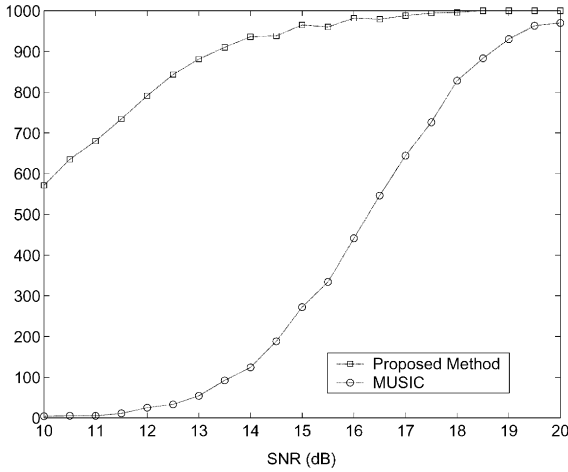


Fig. 7. Number of successful simulations (out of 1000 simulations) for two correlated sources with  $r = 0.99$  and equal powers versus SNR that are located at  $90^\circ$  and  $107^\circ$ .

Direction of arrival of the sources are  $90^\circ$  and  $107^\circ$ , respectively. We have performed MUSIC and proposed method of this paper 1000 times using  $N = 100$  samples for each SNR value and the number of successful simulations for each algorithm is plotted in Fig. 7.

More computational load is the price of performance gains of proposed method compared to MUSIC method in resolving uncorrelated and correlated sources. MUSIC method requires one time eigenvalue decomposition of correlation matrix of received data but for proposed method one eigenvalue decomposition for each direction of the spectrum is required. This computational load for new processors may still be low and tolerable but we can further reduce the computational load of the new method by first performing the MUSIC method and finding the peaks of MUSIC and then applying the proposed algorithm only around the peaks of MUSIC spectrum (not in all points of spectrum). This will highly reduce the computational load without any performance degradation. In this way those adjacent sources that result in only one peak in spectrum of MUSIC method, can be split in two separate peaks. In Table 1, the average processing times (in seconds) for MUSIC, proposed method and proposed method with reduced computations using MATLAB 6.5 software on a Pentium IV-2000 processor are listed. The resolution parameter shows the search resolution, i.e., resolution by

Table 1

The processing times (in seconds) for execution of MUSIC, proposed method and proposed method with reduced computations for a two source case on a Pentium IV-2000 processor using MATLAB 6.5

Algorithm	MUSIC	Proposed method	Proposed method with reduced computations
Resolution = $1^\circ$	0.115	0.328	0.174
Resolution = $0.1^\circ$	0.251	2.43	0.482

which the spectrum of different methods for finding peaks was searched. The averages are obtained from 1000 simulations of a two source situation located at  $90^\circ$  and  $98^\circ$ , using 8-element uniform circular array of Fig. 2.

### 5. Conclusion

We presented a new algorithm based on invariance property of noise subspace to source powers. The asymptotic consistency of the method was proved and its superiority over MUSIC method in resolving both equi-power sources and strong sources in vicinity of weak ones was shown through simulations. Simulations also proved the advantage of proposed method in resolving highly correlated sources. Simulation results have also shown similar superiority of the proposed method over two other well-known algorithms, that is ESPRIT [13] and root-MUSIC [3]. These algorithms are only applicable to arrays with certain geometries. But the MUSIC, like the proposed algorithm, is quite general and can be applied to all array configurations. For this reason only the results of comparison with MUSIC are presented in this paper.

### Appendix A.

If we consider the scenario shown in Fig. 1. that an arbitrary array of  $M$  sensors is receiving signals from  $d$  narrowband uncorrelated sources ( $d < M$ ) from directions  $\theta_i$ ,  $i = 1, \dots, d$ , then the received signals in vector form as stated in (4) will be

$$\underline{\mathbf{x}}(t) = \sum_{i=1}^d \underline{\mathbf{a}}(\theta_i) s_i(t) + \underline{\mathbf{n}}(t). \tag{A.1}$$

With the assumptions made in Section 2, the covariance matrix of received signal becomes

$$\begin{aligned} \mathbf{R}_X &= E\{\underline{\mathbf{x}}(t)\underline{\mathbf{x}}(t)^H\} \\ &= \sum_{i=1}^d \eta_i^2 \mathbf{a}(\theta_i)\mathbf{a}(\theta_i)^H + \sigma_n^2 \mathbf{I}, \end{aligned} \quad (\text{A.2})$$

where as stated in (7),  $\eta_k^2$  is the power of  $k$ th source and  $\sigma_n^2$  is the noise power at each sensor and  $\underline{\mathbf{a}}(\theta)$  is the array response vector for direction  $\theta$ . Each matrix  $\mathbf{R}_X$  with the form of (A.2) has the following property [5,6,9]

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d \geq \lambda_{d+1} = \dots = \lambda_M = \sigma_n^2, \quad (\text{A.3})$$

where  $\lambda_i$  denotes the eigenvalues of  $\mathbf{R}_X$  in decreasing order. If we define  $\mathbf{D}^\varphi$  as in (14) and set the value of  $\varphi$  to one of source directions, say  $\varphi = \theta_1$ , then we have

$$\begin{aligned} \mathbf{D}^{\theta_1} &\triangleq \mathbf{R}_X + h \underline{\mathbf{a}}(\theta_1) \underline{\mathbf{a}}(\theta_1)^H \\ &= (\eta_1^2 + h) \underline{\mathbf{a}}(\theta_1) \underline{\mathbf{a}}(\theta_1)^H \\ &\quad + \sum_{i=2}^d \eta_i^2 \underline{\mathbf{a}}(\theta_i) \underline{\mathbf{a}}(\theta_i)^H + \sigma_n^2 \mathbf{I}. \end{aligned} \quad (\text{A.4})$$

Comparing (A.4) and (A.2), we see that  $\mathbf{D}^{\theta_1}$  has the form of (A.2) and consequently its eigenvalues (denoted by  $\mu_i^{\theta_1}$ ) has the property in (A.3), i.e.,

$$\mu_k^{\theta_1} = \lambda_k = \sigma_n^2 \quad k = d+1, \dots, M. \quad (\text{A.5})$$

If we set  $\varphi$  to a value different from all source directions, i.e.,  $\varphi \neq \theta_j, j = 1, \dots, d$  then we have

$$\mathbf{D}^\varphi \triangleq \mathbf{R}_X + h \underline{\mathbf{a}}(\varphi) \underline{\mathbf{a}}(\varphi)^H = \sum_{i=1}^{d+1} \eta_i^2 \underline{\mathbf{a}}(\theta_i) \underline{\mathbf{a}}(\theta_i)^H + \sigma_n^2 \mathbf{I},$$

$$\eta_{d+1}^2 = h, \quad \theta_{d+1} = \varphi. \quad (\text{A.6})$$

Comparing (A.6) and (A.2), we find out that  $\mathbf{D}^\varphi$  has the form of (A.2) with the number of sources replaced by  $d+1$  and therefore from (A.3) its eigenvalues (denoted by  $\mu_i^\varphi$ ) has the property

$$\mu_k^{\theta_1} = \lambda_k = \sigma_n^2 \quad k = d+2, \dots, M. \quad (\text{A.7})$$

So we have proved that (17) holds while  $\varphi$  is set to a source direction and does not hold when  $\varphi$  is different from all source directions.

## References

- [1] T.W. Anderson, An Introduction to Multivariate Statistical Analysis, Wiley, New York, 1971.
- [2] J. Bakh Andersen, Antenna arrays in mobile communications: gain, diversity, and channel capacity, IEEE Antennas Propag. Mag. 42 (2) (April 2001) 12–16.
- [3] A. Barabell, Improving the resolution of eigenstructured based direction finding algorithms, in: Proc. ICASSP, Boston, MA, 1983, pp. 77–80.
- [4] B. Friedlander, A sensitivity analysis of the MUSIC algorithm, IEEE Trans. Acoust. Speech Signal Process. 38 (10) (October 1990) 1740–1751.
- [5] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, UK, 1989.
- [6] D.H. Johnson, D.E. Dudgeon, Array Signal Processing, Prentice-Hall, Englewood Cliffs, NJ, 1993.
- [7] A. Kavak, Adaptive antenna arrays for downlink capacity increase in third generation wireless CDMA, IEEE Radio and Wireless Conf., RAWCON 2001, 19–22 August 2001, Boston, MA, pp. 77–80.
- [8] M. Kaveh, A. Barabell, The statistical performance of the MUSIC and the minimum norm algorithms in resolving plane waves in noise, IEEE Trans. Acoust. Speech Signal Process. 34 (2) (April 1986) 331–341.
- [9] S.U. Pillai, Array Signal Processing, Springer, Berlin, 1989.
- [10] B. Porat, B. Friedlander, Analysis of the asymptotic relative efficiency of the MUSIC algorithm, IEEE Trans. Acoust. Speech Signal Process. 36 (4) (April 1988) 532–544.
- [11] C.R. Rao, Linear Statistical Inference and its Applications, Wiley, New York, 1973.
- [12] J. Razavilar, F. Rashid-Farrokhi, K.J.R. Liu, Traffic improvements in wireless communication networks using antenna arrays, IEEE J. Selected Areas Commun. 18 (3) (March 2000) 458–471.
- [13] R. Roy, T. Kailath, ESPRIT-Estimation of signal parameters via rotational invariance techniques, IEEE Trans. Acoust. Speech Signal Process. 37 (1989) 984–995.
- [14] R.O. Schmidt, Multiple emitter location and signal parameter estimation, in: Proc. RADCSpectrum Estimation Workshop, Rome, NY, 1979, pp. 243–258.
- [15] P. Stoica, A. Nehorai, MUSIC, maximum likelihood and Cramer-Rao bound, IEEE Trans. Acoust. Speech Signal Process. 37 (5) (May 1989) 720–741.
- [16] H.L. Van Trees, Detection, Estimation and Modulation Theory, Wiley, New York, 1971.
- [17] J.H. Wilkinson, The Algebraic Eigenvalue Problem, Oxford University Press, New York, 1965.