The Immersed Interface Method
Numerical Solutions of PDEs Involving Interfaces and Irregular Domains

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