An algorithmic proof of the Lovasz Local Lemma via resampling oracles

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Simultaneously avoiding many events

Lovasz Local Lemma: [Erdos-Lovasz '75]

Let $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n$ be events in a probability space. Suppose

- $\mathcal{E}_i$ is jointly independent of all but d events
- $\mathbb{P}[\mathcal{E}_i] \leq 1/ed$ for all i

Then $\mathbb{P}[\cap_i \overline{\mathcal{E}_i}] > 0$ (typically exponentially small).

Beyond what nature intended:
Finding a very rare object.

Simultaneously avoiding many events

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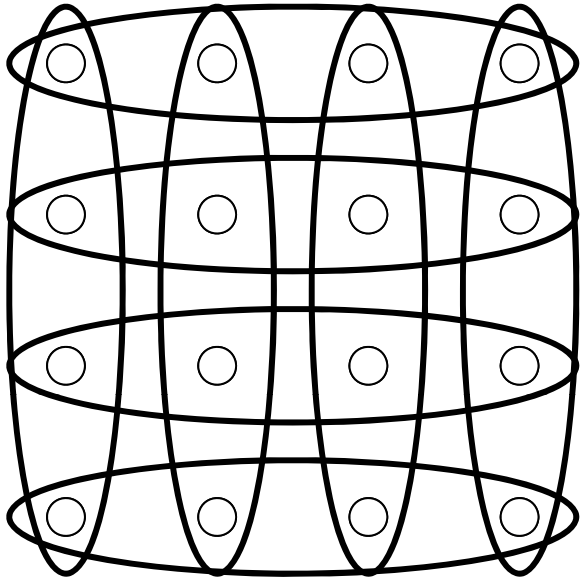
Some applications:

- Near-Ramanujan expanders via 2-lifts [Bilu-Linial]
- O(congestion + dilation) packet routing [Leighton-Maggs-Rao]
- O(1)-approx for Santa Claus problem [Feige], [Haeupler-Saha-Srinivasan]

Example: 2-coloring hypergraphs



- **Given:** System of sets of size k, each intersecting $\leq 2^{k-1}/e$ sets.
- **Goal:** Color vertices red/blue so that each set has both colors.
- **Random approach:** color each vertex independently red/blue.

- Let $\mathcal{E}_i$ be event i^{th} set is **all-red** or **all-blue**. Note $\mathbb{P}[\mathcal{E}_i]=2^{1-k}$.
- $\mathcal{E}_i$ is independent of all but $2^{k-1}/e =: d$ sets.
- Since $\mathbb{P}[\mathcal{E}_i]\leq 1/ed$, LLL implies $\mathbb{P}[\cap_i \overline{\mathcal{E}_i}]>0$.
- So, there is a coloring where no set is **all-red** or **all-blue**.

Algorithmic LLL

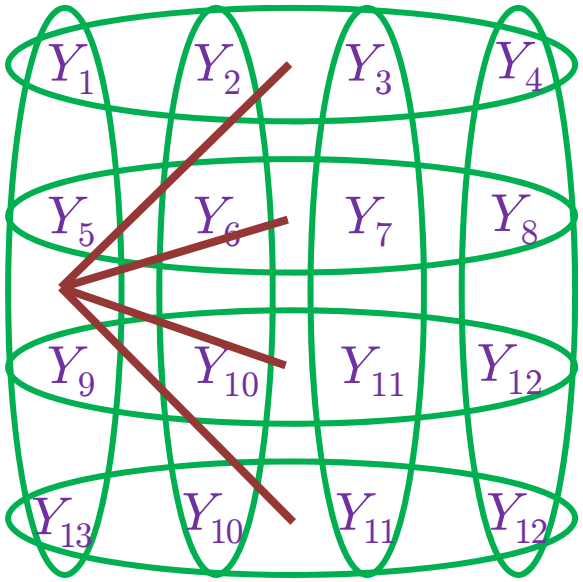
- Under the original distribution $\mathbb{P}$, avoinding $\mathcal{E}_1,\mathcal{E}_2,\ldots,\mathcal{E}_n$ is exponentially unlikely.
- Can we find another distribution (i.e. a randomized algorithm) in which it is likely to avoid $\mathcal{E}_1,\mathcal{E}_2,\ldots,\mathcal{E}_n$?
- Yes: Hypergraph case, weaker parameters: [Beck '91], [Alon '91], [Molloy-Reed '98], [Czumaj-Scheideler '00], [Srinivasan '08]

Beyond what nature intended:
Amplifying probability of rare event.

Moser-Tardos Variable Model

(dependency based on shared variables)

- Independent random variables $Y_1,\ldots,Y_m$
- "Bad" events $\mathcal{E}_1,\mathcal{E}_2,\ldots,\mathcal{E}_n$
- $\mathcal{E}_i$ depends on variables var($\mathcal{E}_i$)
- A dependency graph G:
  $i \sim j$ if var($\mathcal{E}_i$)$\cap$var($\mathcal{E}_j$)$\neq\emptyset$.



Sets sharing variables are neighbors

Moser-Tardos Algorithm [Moser-Tardos '08]

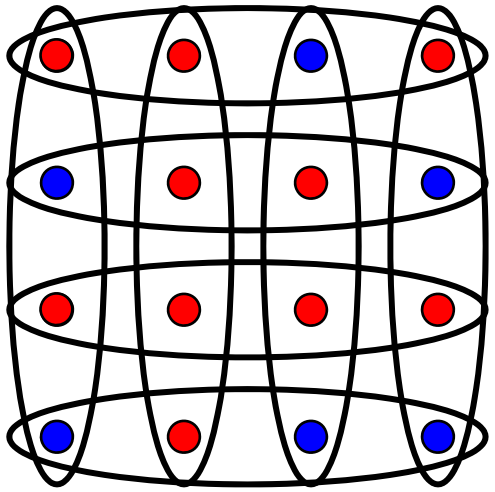
While some event $\mathcal{E}_i$ occurs

Resample all variables in var($\mathcal{E}_i$)

Theorem: Suppose max-degree < d and $\mathbb{P}[\mathcal{E}_i] \leq 1/ed$. The algorithm finds a point in $\cap_i \overline{\mathcal{E}_i}$ after $O(n)$ resampling operations, in expectation.

Algorithmic Improvements

- Many extensions of Moser-Tardos:
  - deterministic LLL algorithm [Chandrasekaran-Goyal-Haupler '10]
  - exponentially many events [Haupler-Saha-Srinivasan '10]
  - better conditions on probabilities [Kolipaka-Szegedy '11], [Harris '14]
- Require variable model (dependencies based on $Y_1,...,Y_m$)



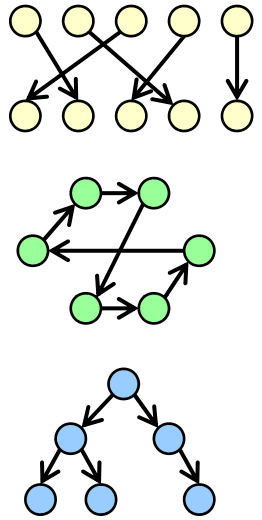
Hypergraph coloring

$$(\overline{x}_1 \vee x_2 \vee \overline{x}_3) \wedge (x_4 \vee \overline{x}_5 \vee x_6) \ \ldots$$

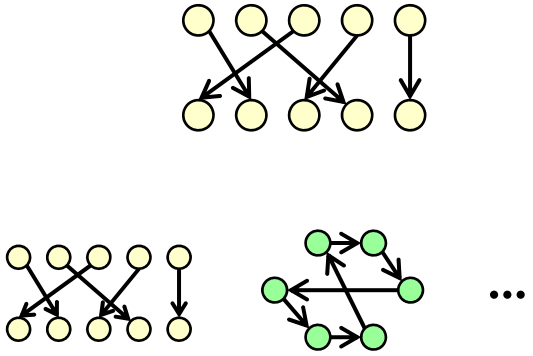
k-SAT

Algorithmic Improvements

- Original LLL works for arbitrary probability spaces
  - Permutations [Erdos-Spencer '91]
  - Hamilton cycles [Albert-Frieze-Reed '95]
  - Spanning trees [Lu-Mohr-Szekely '13]

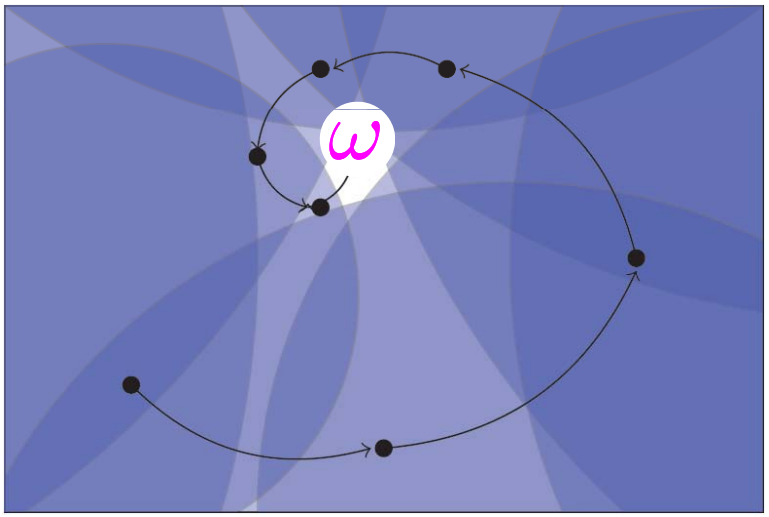


- Algorithms beyond variable model
  - Permutations [Harris-Srinivasan '14]
  - Abstract "flaw-correction" framework
    [Achlioptas-Iliopoulos '14], [Kolmogorov '15]



Algorithmic Local Lemma for general probability spaces?

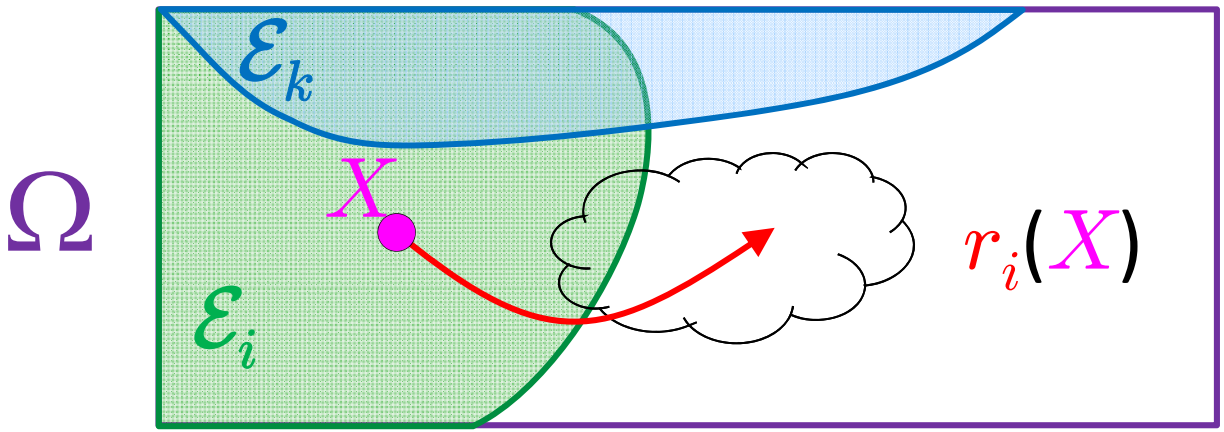
- For the LLL in **any** probability space, can we design a randomized algorithm to quickly find $\omega \in \bigcap_i \overline{\mathcal{E}_i}$?



- How can algorithm "move about" in general probability space?
  - Flaw-correcting actions [Achlioptas-Iliopoulos '14]
  - **Resampling oracles** [This paper]

Resampling Oracles

- Consider probability space Ω, measure $\mathbb{P}$, events $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n$ and dependency relation denoted $\sim$.
- A **resampling oracle** for $\mathcal{E}_i$ is a random function $r_i : \Omega \to \Omega$
  - Removes conditioning on $\mathcal{E}_i$:
    If X has measure $\mathbb{P}$ cond. on $\mathcal{E}_i$, then $r_i(X)$ has measure $\mathbb{P}$.
  - Does not cause non-neighbor events:
    If $\mathcal{E}_k \not\sim \mathcal{E}_i$ and $X \notin \mathcal{E}_k$, then $r_i(X) \notin \mathcal{E}_k$.



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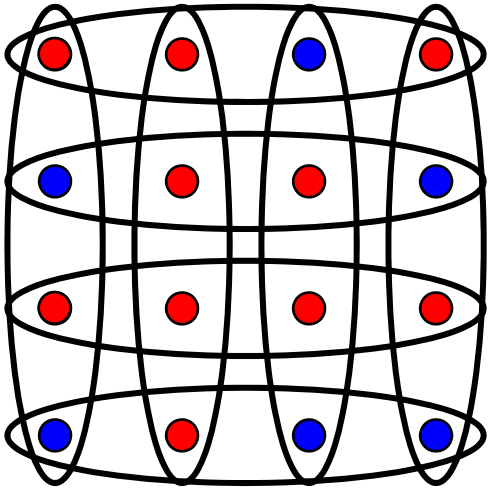
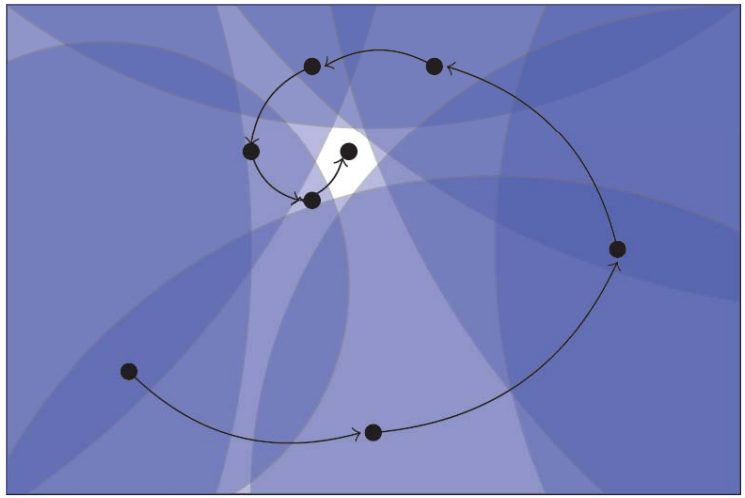
Example: Resampling Oracle for Hypergraph Coloring



Our Main Result:

An Algorithmic LLL in a General Setting

- Arbitrary probability space Ω
- Events $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_n$
- An arbitrary graph G
- A resampling oracle for each $\mathcal{E}_i$, with respect to G



- **Theorem:** [H.-Vondrak '15] Suppose max-degree $< d$ and $\mathbb{P}[\mathcal{E}_i] \leq 1/ed$. Our algorithm finds a point in $\cap_i \overline{\mathcal{E}_i}$ after $O(n^2)$ resampling operations, with high probability.
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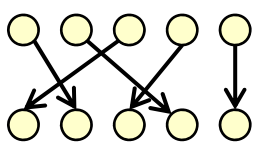
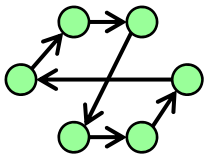
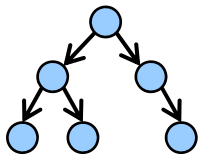
- Holds much more generally: Lovasz's conditions, Shearer's conditions...

- We design efficient resampling oracles for essentially every known application of the LLL (and generalizations)

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Algorithmic LLL via Resampling Oracles

MIS Resample

Draw ω from $\mathbb{P}$

Repeat

  $I \leftarrow \emptyset$

  While there is $i \notin \Gamma^+(I)$ s.t. $\mathcal{E}_i$ occurs in ω

    Pick smallest such i

    $\omega \leftarrow r_i(\omega)$

    $I \leftarrow I \cup \{i\}$

  End

Until $I = \emptyset$

Output ω

Like finding a maximal indep set

Similar to Moser & Tardos' parallel algorithm

Resampling Spanning Trees in K_n

- Let T be a uniformly random spanning tree in K_n
- For edge set A, let $\mathcal{E}_A = \{A \subseteq T\}$.
- **Dependency Graph:** Make $\mathcal{E}_A$ a neighbor of $\mathcal{E}_B$, unless A and B are vertex-disjoint.
- **Resampling oracle** r_A:
  - If T uniform conditioned on $\mathcal{E}_A$, want $r_A(T)$ uniform.
  - But, should not disturb edges that are vtx-disjoint from A.

- $\mathcal{E}_A = \{A \subseteq T\}$.
- **Resampling oracle** $r_A(T)$:
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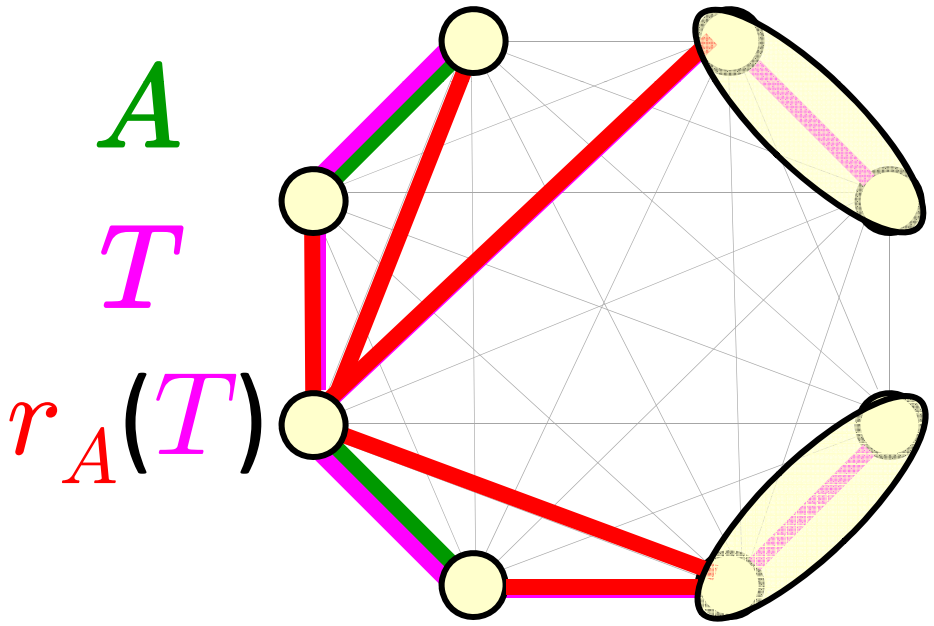
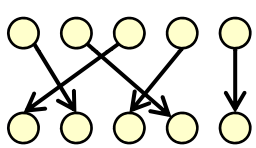
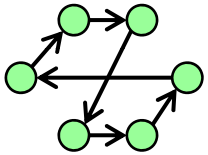
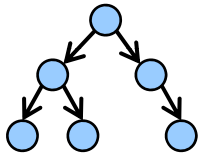


  - Contract edges of T vtx-disjoint from A
  - Delete edges adjacent to A
  - Let $r_A(T)$ be a uniformly random spanning tree in resulting (multi)-graph.

- **Lemma**: $r_A(T)$ is uniformly random.

Summary

- Our algorithmic proof of LLL works for **any** probability space and any events, under usual LLL conditions,
  so long as you can design **resampling oracles**.




...

- Efficiency is similar to Moser-Tardos, but quadratically worse
- Analysis is similar to Moser-Tardos, perhaps simpler
  – no "log" or branching processes
- Works under Lovasz's condition, cluster expansion condition, Shearer's condition (no slack needed!)