Cramér–Rao Lower Bounds for QAM Phase and Frequency Estimation

Feng Rice, Bill Cowley, Bill Moran, and Mark Rice

Abstract—In this paper, we present the true Cramér–Rao lower bounds (CRLBs) for the estimation of phase offset for common quadrature amplitude modulation (QAM), PSK, and PAM signals in AWGN channels. It is shown that the same analysis also applies to the QAM, PSK, and PAM CRLBs for frequency offset estimation. The ratio of the modulated to the unmodulated CRLBs is derived for all QAM, PSK, and PAM signals and calculated for specific cases of interest. This is useful to determine the limiting performance of synchronization circuits for coherent receivers without the need to simulate particular algorithms. The bounds are compared to the existing true CRLBs for an unmodulated carrier wave (CW), BPSK, and QPSK. We investigated new and existing QAM phase estimation algorithms in order to verify the new phase CRLB. This showed that new minimum distance estimator performs close to the QAM bound and provides a large improvement over the power law estimator at moderate to high signal-to-noise ratios.

Index Terms—Cramér–Rao lower bounds, frequency estimation, phase estimation, synchronization, quadrature amplitude modulation.

I. INTRODUCTION

QUADRATURE amplitude modulation (QAM) is a highly bandwidth efficient transmission technique for digital communications. It makes use of multiple signal phase and amplitude levels to carry multiple bits per symbol. This requires accurate and robust carrier phase and frequency estimation in the receiver. The parameters often have to be estimated with no knowledge of the transmitted sequence (blind operation) and wide tolerances in the local oscillator reference. In order to analyze system performance and design appropriate equipment, it is important to understand the theoretical estimator performance limits and methods to approach them. The Cramér–Rao lower bound (CRLB) is useful in this analysis.

The CRLBs for unmodulated signals have been known for some time [1] and are still frequently employed as a benchmark for modulated signals. The modified CRLB (MCRB) [2]–[4] and the asymptotic CRLB (ACRB) [5] are good approximations for the true bound for MPSK/QAM modulated signals at higher signal-to-noise ratios (SNRs). However, they depart significantly from the true CRLB at low SNR. The true CRLBs for BPSK and QPSK phase and frequency estimation were presented in [6].

This paper is concerned with evaluating the CRLB for QAM, with special attention given to square 16QAM and 64QAM as popular modulation schemes. We assume a static flat channel with no knowledge of the transmitted sequence, i.e., preamble or pilot symbols. Several authors have suggested phase estimators for this scenario [7]–[9], [10]. Without knowledge of the true CRLB, it is impossible to know how far these techniques are from the fundamental performance limits.

In Section II, we derive a general CRLB expression for symmetric signal constellations and evaluate it using two methods. A comparison of CRLBs for CW, MPSK, 16QAM, and 64QAM is given, together with the ACRB and MCRB for the modulated signals. It is shown that the ratio $F_{\text{QAM}}(N_o/2E_s)$ of CRLB for a modulated signal to the CRLB for CW is the same for phase estimation and frequency estimation. $F_{\text{QAM}}(N_o/2E_s)$ is particularly useful to designers in determining the limiting performance of synchronization circuits for coherent receivers for an arbitrary observation interval. The CRLB is evaluated for frequency estimation first with known phase and then with joint frequency and phase estimation. These are shown to be equivalent when evaluating at the middle of the signal vector, which is consistent with the signal tone case presented by Rife [1].

Section III starts with a brief review of QAM phase estimators. These include Georgiades’ Histogram Algorithm [9], the 2-stage-conjugate algorithm of [10], and a method based on Euclidean distance evaluation which we refer to as the minimum distance estimator (MDE). It is shown that little improvement is possible relative to the best known algorithms. Simulation results of phase estimation algorithms are compared to the CRLBs evaluated in Section III for 16QAM and 64QAM square constellations. We include comparisons with the power law estimator (PLE) variance expressions developed in [8]. Brief conclusions are presented in Section IV.

II. TRUE CRLBS FOR QAM PHASE AND FREQUENCY ESTIMATION

A. A Phase CRLB for QAM

We initially assume that a received QAM square constellation signal has an unknown fixed phase offset $\phi$ and that the signal...
has been ideally filtered and sampled at the optimum sampling instant. In this case, the received samples are

\[ x_k = a_k e^{j\phi_k} + w_k, \quad k = 0, 1, \ldots, N - 1 \] (1)

where \( a_k \) are the transmitted symbols of a QAM constellation \( \mathcal{C} \) of a unit average energy. For example, with BPSK, \( \mathcal{C}_{\text{bpsk}} = \{-1, 1\} \) and for Square 16QAM \( \mathcal{C}_{\text{16qam}} = (1/\sqrt{10}) \{\pm1 \pm j, \pm3 \pm j, \pm j, \pm3 \pm j\} \). Denote this range of \( k \) by \( \mathcal{K}_N \). The \( w_k \)'s are independent zero-mean complex Gaussian random variables whose real and imaginary parts have variance \( \sigma^2 \) representing noise. The symbol SNR is \( (E_s/N_0) = (1/2\sigma^2) \) [11].

We will use the convention that \( p_x(x) \) is the probability density function (pdf) for a random variable \( X \) and use boldface symbols to denote vectors; where the meaning is clear we will drop the subscript.

The CRLB on the variances of an unbiased estimator of \( \phi \), namely \( \hat{\phi} \), for a sequence of \( N \) symbols is given by [12]

\[ \text{CRLB}(\phi) = \frac{1}{-E \left[ \frac{\partial \ln p(x|\phi)}{\partial \phi^2} \right]} \] (2)

where \( X = (X_0, X_1, \ldots, X_{N-1}) \) and \( E[\cdot] \) denotes statistical expectation with respect to the pdf \( p_x(x|\phi) \). For a single received sample, the pdf of the corresponding complex random variable \( x_0 \) is given by

\[ p(x_0 | \phi) = p_x(x_0 | \phi) = \sum_{a_i \in \mathcal{C}} p_X(x_0 | \phi, a_i) p_A(a_i) 
= \sum_{a_i \in \mathcal{C}} \frac{p_A(a_i)}{2\pi\sigma^2} e^{-\frac{|a_i|^2}{2\sigma^2}} e^{-\frac{|x_0 - a_i e^{j\phi}|^2}{2\sigma^2}} \] (3)

where \( A \) is a random symbol and \( a_i \) is an element of the set \( \mathcal{C} \).

Since the received samples \( X \) are independent random variables, for \( N \) samples, we have the pdf below, where we have assumed that the transmit symbols are equally likely, i.e., \( p_A(a_i) \) is independent of \( i \)

\[ p(x | \phi) = p_X(x | \phi) = p(x_0 | \phi) p(x_1 | \phi) \cdots p(x_{N-1} | \phi) 
= \prod_k p(x_k | \phi) 
= \prod_k \left[ \frac{p_A(a)}{2\pi\sigma^2} e^{-\frac{|a|^2}{2\sigma^2}} \sum_{a_i \in \mathcal{C}} e^{-\frac{|x_0 - a_i e^{j\phi}|^2}{2\sigma^2}} \right] \] (4)

Note that products over the index \( k \) range from 0 to \( N - 1 \) and the summation over all points in the constellation.

Expanding (4), we obtain

\[ p(x | \phi) = \prod_k \left[ \frac{p_A(a)}{2\pi\sigma^2} e^{-\frac{|a|^2}{2\sigma^2}} \sum_{a_i \in \mathcal{C}} e^{-\frac{|x_0 - a_i e^{j\phi}|^2}{2\sigma^2}} \right] \] (5)

where \( a_i^* \) indicates complex conjugate.

From the Appendix, we obtain the pdf

\[ p(x | \phi) = \prod_k \left\{ \frac{p_A(a)}{\pi\sigma^2} e^{-\frac{|a|^2}{2\sigma^2}} \sum_{a_i \in \mathcal{Q}_1} e^{-\frac{|x_k - a_i e^{j\phi}|^2}{2\sigma^2}} \right\} \times \sum_{r=1}^{N-1} \cosh \left( \frac{|a_i|^2}{\sigma^2} x_k e^{-j(\phi + r \arctan \left( \frac{a_i}{x_k} \right))} \right) \] (6)

where \( \mathcal{Q}_1 \) consists of the constellation points in the first quadrant and \( \mathfrak{R} \) and \( \mathfrak{I} \) stand for real and imaginary parts. Taking the logarithm and retaining the terms dependent on \( \phi \), we obtain the corresponding log-likelihood function

\[ \sum_k \ln \left\{ \sum_{a_i \in \mathcal{Q}_1} e^{-\frac{|a_i|^2}{2\sigma^2}} \sum_{r=1}^{N-1} \cosh \left( \frac{|a_i|^2}{\sigma^2} x_k e^{-j(\phi + r \arctan \left( \frac{a_i}{x_k} \right))} \right) \right\} \] (7)

To simplify the notation, we define

\[ \xi(\phi, x_k, a_i, r) = |a_i| x_k e^{-j(\phi + r \arctan \left( \frac{a_i}{x_k} \right))} \] (8)

Considering (7) and (8), we obtain the second partial derivative (9), shown at the bottom of the next page, where \( \Lambda(x_k) \) is defined by the three terms within the large curly brackets.

The expectation of the second derivative is

\[ E \left[ \frac{\partial^2 \ln p(x | \phi)}{\partial \phi^2} \right] = \frac{1}{\sigma^2} E \left[ \sum_k \Lambda(x_k) \right] \]

where

\[ E \left[ \sum_k \Lambda(x_k) \right] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_k \Lambda(x_k) p(x | \phi) d|x| \]

\[ = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_k \Lambda(x_k) p(x_1 | \phi) \cdots p(x_{N-1} | \phi) d|x_1| \cdots d|x_{N-1}| \]

where \( d|x| = dR(x) d\mathfrak{I}(x) \).

Combining the fact that \( \Lambda(x_0) \) depends only \( x_0 \) and not on the remaining \( x_k \)'s, and

\[ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(x_1 | \phi) d|x_1| \cdots d|x_{N-1}| = 1 \]

we see that the expectation of \( \Lambda(x_0) \) is just an integral over the variable \( x_0 \). The same applies for all \( x_k \)'s and so we can drop the \( x_k \)'s subscript in (9), and

\[ E \left[ \frac{\partial^2 \ln p(x | \phi)}{\partial \phi^2} \right] = \frac{N}{\sigma^2} E[\Lambda(x)] \] (10)

Now define

\[ F(\sigma^2) = -E[\Lambda(x)] \] (11)
Thus, from (2) and (10), we obtain
\[ \text{CRLB}(\phi) = \sigma^2 \frac{1}{N} F(\sigma^2) \]
\[ = \frac{1}{2N} \frac{\mathbb{E}_F}{\mathbb{E}_N} F \left( \frac{N}{2\mathbb{E}_N} \right). \]  
(12)

We may note that \(1/(2N(E_N/E_\text{N}))\) in (12) corresponds to the CW CRLB for estimation of the carrier phase over \(N\) symbol intervals. \(F^{-1}(N_0/E_\text{N})\) is therefore the ratio of the CRLB for random QAM signals to the CRLB for an unmodulated carrier of the same power.

In order to evaluate \(F^{-1}(N_0/E_\text{N})\), substituting (1) into (8) we have
\[ \xi(\phi, N_0, N_0) = |\alpha_i| (A_k + we^{-j\phi}) e^{-j\pi \text{arctan} \left( \frac{\mathbb{R}(\alpha_i)}{\mathbb{I}(\alpha_i)} \right)} \]  
(13)

where \(A_k\) are the transmitted symbols in all four quadrants, and \(\alpha_i\) are constellation points in the first quadrant only. Since multiplication of a complex Gaussian process by a random phase simply produces another complex Gaussian process with the same statistics, then let
\[ \xi(n, n_0) = n I + j n Q \]  
(14)

where the superscripts \(I\) and \(Q\) denote the real and imaginary parts, respectively. Since the rotated noise samples \(n I\) and \(n Q\) are independent variables with the same distribution as \(u I\) and \(u Q\), (13) becomes
\[ \xi(\phi, n I, n Q, A_k, a_i, r) = |a_i| (A_k + n I + j n Q) \]
\[ \times e^{-j\pi \text{arctan} \left( \frac{\mathbb{R}(a_i)}{\mathbb{I}(a_i)} \right)} \]  
(15)

Hence,
\[ F(\sigma^2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \lambda(\xi(\phi, n I, n Q, A_k, a_i, r)) \]
\[ \times p(n I, n Q) dn I dn Q. \]  
(16)

Note that (12) and (16) are general CRLB expressions for symmetric signal constellations, where \(A_k = \pm \mathbb{R}(\alpha_i) \pm j \mathbb{I}(\alpha_i)\) and \(i\) extends over the number of elements in \(Q_1\).

**B. Evaluation of Phase CRLB**

It is clear from (16) that an analytical solution for \(F(\sigma^2)\) is not feasible. We have evaluated \(F(\sigma^2)\) for square 16QAM, 64QAM, and 8PSK modulations. In general, numerical integration works well at lower SNRs. Fig. 1 shows the three-dimensional (3-D) plots of the integrand \(A(\phi, n I, n Q)\) for 16QAM at SNR = 15 dB. The plot indicates that when SNR > 15 dB, the integrand has distinct spikes, so numerical integration is more difficult. However, \(A(\phi, n I, n Q)\) alone can be evaluated using Monte Carlo evaluation (MCE) as it is a relatively smooth function. A Gaussian random noise generator is employed to produce random noise at different values of SNR. Based on (9), MCE calculates the average value of \(A(\phi, n I, n Q)\) at each SNR. At intermediate SNRs, we have found that MCE results readily match the integral results. However, when SNR < -9 dB, MCE requires a much larger number of trials to obtain accurate results, and numerical integration obtained consistent results.

We used both approaches to evaluate the CRLB. The execution time at low SNR favored the use of the numerical method (by a factor of 3 at -8 dB), whilst at moderate to high SNR the MCE was more efficient (by a factor of 1000 at 10 dB).

Similarly, we have calculated the ratios of \(F^{-1}(\phi)\) for 8PSK and 64QAM using both numerical integration and MCE. All the ratios \(F^{-1}(\phi)\) for 8PSK and 64QAM are shown in Fig. 2 together with \(F^{-1}(\phi)\) for BPSK and QPSK [6]. From Fig. 2, the ratios \(F^{-1}(\phi)\) \((\phi)\) for BPSK, QPSK, etc.) eventually converge to 1 as SNR increases, i.e., the CRLBs converge to the CRLB of an unmodulated carrier. As expected, the more spectrally efficient modulations require a higher SNR to approach the CW CRLB. At low SNR, the simpler signal set has a lower CRLB.

It may be observed that the 16QAM and 64QAM bounds converge at low SNRs. Once the symbols become indistinguishable due to noise, the shape of the constellation determines the CRLB rather than the number of elements in the signal set. This also
provides some explanation of the much worse performance of 8PSK. As might be expected, at $E_s/N_0$ below 8 dB, the circular shape of the constellation makes phase estimation difficult. In contrast, at higher SNRs, for all signal sets considered it is the minimum distances between members of the set which determines when each bound departs from the CW bound. Therefore, the QAM characteristics have two distinct regions. Fig. 2 shows the transition between these two regions at approximately 9 dB for 16QAM and 14 dB for 64QAM. At high SNR, all estimator variances approach CW performance.

C. Frequency CRLB for QAM With Known Phase

We may apply much of the above derivation to a CRLB for QAM frequency estimation. The received signal is

$$x_k = a_k e^{j(k\omega + \phi_0)} + w_k, \quad k = -\frac{N-1}{2}, \ldots, \frac{N-1}{2}$$

where $\omega$ is the frequency offset in radians per symbol period. $k$ is selected so that the sampling times are symmetrically located about zero which maximizes the value of the bound [1]. Denote this range of $k$ by $K$. We assume a fixed, arbitrary, and known phase offset at $k = 0$ of $\phi_0$, and that the signal vector contains an odd number of symbols $N$. We note that frequency estimation assuming known carrier phase is not usually a realistic situation, since practical carrier frequency estimation often occurs in the presence of an unknown carrier phase. However, the known carrier phase situation can be approximated when one or more phase estimates are made directly from reference symbols. We also present this result as a useful intermediate step in the derivation for the CRLB of joint frequency and phase estimation given in the next subsection. We have assumed that the frequency offset is relatively small so that intersymbol interference may be neglected. The probability density is similar to that given in (6), with $\phi$ replaced by $k\omega + \phi_0$, resulting in (18), shown at the bottom of the next page.

Then the second partial derivative is

$$\frac{\partial^2 \ln p(x|\omega)}{\partial \omega^2} = \frac{1}{\sigma^2} \sum_k k^2 \Lambda(x),$$  

(19)

Recalling that $E\{\Lambda(x)\}$ is independent of $k$, then

$$k^2 = \frac{N(N^2 - 1)}{12}$$

and for $N$ odd we obtain the reciprocal frequency CRLB with known phase for QAM as

$$\text{CRLB}_{\phi_0}(\omega)^{-1} = 2\frac{E_s}{N_0} \frac{N(N^2 - 1)}{12} \left(\frac{N_0}{2E_s}\right).$$

(20)

This is consistent with the single tone result given in [1, eq. (16)], and the BPSK/QPSK result given in [6, eq. (9)]. In a practical receiver, the phase is usually unknown and uniformly distributed. In order to treat the phase as a random variable, we should average (18) across the phase distribution as discussed.
in [13, p. 87]. This leads to a highly complex expression, the evaluation of which is left for future work. We expect that the random phase bound is no less than that for known phase, and as discussed in the next section, the two bounds become asymptotically equal for large \( N \). Fig. 3 presents the known phase frequency offset CRLB.

The figure shows 16QAM, 64QAM, MPSK, and CW CRLBs for frequency estimation with an \( N = 200 \) symbol block. The vertical axis is the standard deviation of frequency error normalized by symbol rate.

D. CRLBs for Frequency and Phase Joint Estimation

In this section, we derive the CRLBs of the joint phase and frequency estimation for modulated signals. We wish to estimate both parameters \( \omega \) and \( \phi \). Again we assume that the received signal has perfect symbol timing and the frequency offset is relatively small so that the received signal is free of intersymbol interference. The received signals, \( x_k \), are then given by

\[
x_k = a_k e^{j(k\omega + \phi)} + w_k
\]

where the range of \( k \) is defined later.

For the modulated signal, the pdf of the random vector \( x \) with deterministic parameter vector \( \beta = (\omega, \phi) \) and known equiprobable symbols \( \alpha = (a_0, a_1, \ldots, a_{N-1}) \) is given by

\[
p(x | \beta) = \prod_k p(x_k | \beta)
\]

\[
= \prod_k p_{\mathcal{A}}(\alpha) \sum_{a \in \mathcal{C}} \frac{1}{2 \pi \sigma^2} e^{-\frac{1}{2 \sigma^2} |x_k - a e^{j(k\omega + \phi)}|^2}
\]  

(22)

where \( \mathcal{C} \) is a unit average energy constellation.

Lower bounds on the variances of estimators for the components of \( \omega \) and \( \phi \) are given in terms of the diagonal elements of the inverse of the Fisher information matrix (FIM) \( \mathbf{I}^{-1} \) [12]. The elements of \( \mathbf{I} \) are

\[
\mathbf{I} = \begin{pmatrix}
-\mathbb{E} \left[ \frac{\partial^2 \ln p(x | \beta)}{\partial \omega^2} \right] & -\mathbb{E} \left[ \frac{\partial^2 \ln p(x | \beta)}{\partial \omega \partial \phi} \right] \\
-\mathbb{E} \left[ \frac{\partial^2 \ln p(x | \beta)}{\partial \phi^2} \right] & -\mathbb{E} \left[ \frac{\partial^2 \ln p(x | \beta)}{\partial \phi \partial \omega} \right]
\end{pmatrix}
\]

Since we have discussed the QAM phase CRLB in detail in the previous section, we omit some of the mathematical details. We can modify (6) which corresponds to unknown phase only to give joint estimation of frequency and phase by substituting \( \phi \) with \( k\omega + \phi \). The joint pdf is given in (23), shown at the bottom of the page where \( \mathcal{Q}_1 = \{ \text{all constellation points in the first quadrant} \} \).

Using similar expressions to (9) and (19) and deriving the second partial derivative for phase and frequency

\[
\frac{\partial^2 \ln p(x | \beta)}{\partial \omega \partial \phi} = \frac{1}{\sigma^2} \sum_k k \Lambda(x)
\]  

(24)

we generate the FIM for the modulated signal

\[
\mathbf{I} = \frac{1}{\sigma^2} \left( \sum_k k^2 \sum_{a \in \mathcal{Q}_1} \frac{1}{\sigma^2} \right) \mathbf{F}(\sigma^2).
\]

After simplification, for the joint estimation case when \( k \) is in the range \( \mathcal{K}_N \), the CRLBs for joint frequency and phase estimation are

\[
\text{Joint CRLB}(\hat{\omega}) = \frac{12 \sigma^2}{N(N^2 - 1)} \mathbf{F}(\sigma^2) \frac{1}{\mathbf{F}(\sigma^2)}
\]

(25)

\[
= \text{CRLB}_{\text{CW}}(\hat{\omega}) \frac{1}{\mathbf{F}(\sigma^2)} \left( \frac{N}{2\pi} \right)
\]

\[
\text{Joint CRLB}(\hat{\phi}) = \frac{\sigma^2 (2N - 1)}{N(N + 1)} \mathbf{F}(\sigma^2)
\]

(26)

\[
= \text{CRLB}_{\text{CW}}(\hat{\phi}) \frac{1}{\mathbf{F}(\sigma^2)} \left( \frac{N}{2\pi} \right).
\]
These results are the same as the unmodulated carrier (CW) case for equal average power, except for the $F_m(N_0/2E_n)$ factor. Thus, $F_m^{-1}(N_0/2E_n)$ is again the ratio of the CRLB for modulated random signals to the CW CRLB.

Note that when the joint estimator is analyzed relative to the middle of the signal vector (i.e., symmetrically, $k$ in the range $K_S$), then the FIM becomes diagonal and the frequency and phase estimators are decoupled. In this case, the phase CRLB is equal to (12), and, for large $N$, approaches a quarter of (26). The frequency CRLB in this case is equivalent to the known phase result derived in (20).

In statistical terms, when the FIM is diagonal, $\omega$ and $\phi$ are orthogonal parameters (see [14]). In this case, it is evident that we should obtain the same CRLB as in (12) and we do. Orthogonality has a number of consequences [14, p. 2]. In particular, the maximum likelihood estimators $\hat{\omega}$ and $\hat{\phi}$ are asymptotically (for large $N$) independent. It is of more relevance to us that the asymptotic CRLB for estimating $\omega$ is the same whether $\phi$ is regarded as known or unknown, so that integrating out the $\phi$ parameter over a prior (such as the uniform distribution) does not change the asymptotic CRLB.

A further consequence of orthogonality, which while not of direct relevance to this work is of some interest to us, is that the maximum likelihood estimate (MLE) $\hat{\omega}_{\phi_0}$ of $\omega$ for given $\phi_0$ varies only slowly with $\phi$. Specifically, for large $N$, if $(\hat{\omega}, \hat{\phi})$ is the MLE for the pair $(\omega, \phi)$, then $|\hat{\omega}_{\phi_0} - \hat{\omega}|$ goes to zero at least as fast as $1/N$.

In summary, for finite $N$, we deduce the following relationship between the frequency CRLB’s for known phases, joint estimation, and unknown phase cases, respectively:

$$\text{CRLB}_{K_S}(\hat{\omega}; \phi_0) < \text{CRLB}_{K_S}(\hat{\omega}_{\phi_0})$$

$$\leq \text{CRLB with unknown phase},$$

(27)

Asymptotically, the right-hand side becomes an equality, and $\text{CRLB}_{K_S}(\hat{\omega}; \phi_0)$ becomes a quarter of $\text{CRLB}_{K_S}(\hat{\omega}_{\phi_0})$.

For the phase case

$$\text{Joint CRLB}(\phi) = \frac{1}{2N} \frac{1}{K_N} F \left( \frac{N}{2E_n} \right) \gamma$$

(28)

where

$$\gamma = \begin{cases} \frac{1}{N} & \text{K_S (decoupled)} \\ \frac{2(N-1)}{N+1} & \text{K_N} \\ 4, & \text{K_{NS}, N \to \infty} \end{cases}$$

(29)

These results apply to any QAM, PSK, and PAM modulation, including the previously known CW case.

## III. COMPARISON OF NEW CRLB AND PHASE ESTIMATORS

In this section we review some existing phase estimators for QAM and also present some new results for variations of the algorithms. The performance is compared with the 16QAM and 64QAM CRLBs derived above.

### A. Power Law Estimator

Moenecleay and de Jonghe have extended the power law estimator performance analysis in moderate to high SNR regions for general rotationally symmetric signal constellations, such as MPSK and QAM. The phase estimate may be written as [8]

$$\hat{\theta} = \frac{1}{M} \arg \left( E \left[ a_k^M \sum_{k=0}^{N-1} x_k^M \right] \right)$$

(30)

where $a_k$ is the transmitted symbol and $M = 2$ for BPSK and $M = 4$ for QPSK and QAM.

The variance of the phase estimation error is approximated by [8, eq. (14)] and is plotted at high SNR for 16QAM and 64QAM in Figs. 7 and 8, respectively (referred to as the PLE Approximation). In addition, simulation results for the PLE have been included.

Moenecleay and de Jonghe pointed out the estimator has a constellation-dependent noise penalty as compared to the CRLB as well as suffering from self-noise which is introduced by the constellation.

### B. Histogram Algorithm

The Histogram Algorithm (HA) [9] is a simple algorithm for blind carrier phase acquisition that can be used for square, star [11], and cross constellations. The algorithm is based on the likelihood function

$$L(\theta) = \prod_{k=0}^{N-1} \sum_a \exp \left( -\frac{1}{2\sigma^2} || x_k - a \exp(\theta - \theta_k) ||^2 \right)$$

(31)

where $\psi$ is the angle of a QAM symbol $a$, $\theta_k$ is the angle of $x_k$.

The terms of the inner sum of (31) where the amplitude of $x_k$ is close to that of $a$ have the most significant contribution and other terms are not considered. The selected values of $a$ with corresponding phase angle $\theta_k$ are compared with $\theta_k$ and mapped into the first quadrant by

$$\hat{\phi}_k = (\theta_k - \theta_0) \bmod \left( \frac{\pi}{2} \right).$$

(32)

A histogram of all quantized $\hat{\phi}_k$ is generated and the largest bin entry is selected. This bin corresponds to the estimated phase.

The implementation of the algorithm is straightforward. We ran the simulations for 128QAM, 16QAM, and 64QAM with 1° per quantization interval. The performance is plotted in Figs. 7 and 8 for 16QAM and 64QAM (the simulation details are given in Section III-E). The 128QAM results are very similar to those of [9, in Fig. 2]. Note that the HA standard deviation at low SNRs, for $N = 200$, begins to asymptote toward the constant value $\pi/(2\sqrt{2})$ as expected from a uniformly distributed phase estimate over the range $[-\pi/4, \pi/4]$.

Georghiades also derived the modified histogram algorithm (MHA) [9] which improved the performance significantly at high SNR. However, he notes that neither the HA, MHA, or the power-law algorithm approaches the performance of the ML algorithm.

### C. Two-Stage-Conjugate Algorithm

The authors have independently developed the 2-stage-conjugate (2SC) algorithm which was subsequently found to be very similar to the Two-Pass algorithm of [10, p. 33]. This method is applicable to any square constellation and results are given here for 16QAM and 64QAM.
As illustrated in Fig. 4, the 16QAM constellation points can be partitioned into two groups. The signals close to the inner and outer rings are called Class I ($C_1$) and the others close to the middle ring are called Class II ($C_2$). The algorithm works by separating the received signals into two groups of signals based on their amplitudes

$$x_k \in \begin{cases} C_1, & \text{if } |x_k| \leq T_1, \\ C_2, & \text{if } T_1 < |x_k| < T_2 \end{cases}, \quad \text{or } |x_k| \geq T_2, \quad (33)$$

where $k = 0, 1, \ldots, N - 1$, and $T_1$ and $T_2$ are the thresholds for the Class I and II. For the results given in this paper, the thresholds are $T_1 = \sqrt{2} + (\sqrt{2}/2)$ and $T_2 = (\sqrt{10} + \sqrt{18})/2$, i.e., the mean radii between two sets of constellation points. We have found that slightly increasing or decreasing the thresholds may give very small performance improvements for selected values of $E_b/N_0$ and $N$.

Viterbi and Viterbi phase estimation (VVPE) [15] is applied to the Class I signal treated as QPSK symbols to obtain the estimated coarse phase offset $\hat{\phi}_2$ (first stage)

$$\hat{\phi}_2 = \frac{1}{4} \tan^{-1} \sum_{x_k \in C_1} \frac{\Im \left( x_k^* \right)}{\Re \left( x_k^* \right)}, \quad (34)$$

We correct the phase for all received signals $x_k$ using $\hat{\phi}_2$

$$x'_k = x_k e^{-j(\hat{\phi}_1 - \hat{\phi}_2)}, \quad k = 0, 1, \ldots, N - 1. \quad (35)$$

The next step is to estimate the transmitted symbols by demodulating the phase-corrected signals $x'_k$ and choosing the corresponding closest symbols $\hat{a}_k$ based on equidistant decision regions. The estimated transmitted symbols $\hat{a}_k$ may be different from the transmitted symbols $a_k$ due to the effect of Gaussian noise and the residual phase offset. The phase-corrected signals $x'_k$ are then multiplied with the conjugate of the estimated transmitted signals to give

$$y_k = x'_k \hat{a}_k^*, \quad k = 0, 1, \ldots, N - 1$$

where $*$ indicates complex conjugate.

Then the residual phase (second stage) can be calculated as

$$\hat{\phi}_2 = \tan^{-1} \sum_{k=0}^{N-1} \frac{\Im (y_k)}{\Re (y_k)}. \quad (36)$$

Hence the estimated phase offset of the received signal is

$$\hat{\phi} = \hat{\phi}_1 + \hat{\phi}_2. \quad (37)$$

We demonstrated that the 2SC algorithm performance is very close to the new CRLB through Monte Carlo simulation. These new results for 16QAM and 64QAM are shown in Figs. 7 and 8. For the 64QAM case, the first stage used the outer corner constellation points.

### D. Minimum Distance Estimator

Comparison of the methods described above with the new bound raises the question whether any other scheme can achieve better performance? Fig. 6 shows a straightforward method designed to give good performance for any QAM constellation at the cost of increased computational complexity. We call this the minimum distance estimator (MDE). The received signals are rotated by $\phi = (\phi_0, \phi_1, \ldots, \phi_{I-1})$ in the range $[-45^\circ, 45^\circ]$ to obtain a series of $I$ hypothesis sets. Then, making hard decisions for each of the hypothesis sets, we have $\hat{a}_{ik}, i = 0, 1, \ldots, I - 1$. The Euclidean distance ($D_k$) is calculated between the received signal $x_k, k = 0, 1, \ldots, N - 1$ and the $i$th hypothesis set signals $\hat{a}_{ik}, i.e.,$

$$D_k = \sum_{k=0}^{N-1} |x_k e^{j\hat{\phi}_i} - \hat{a}_{ik}|^2 \quad (38)$$

whose minimum is $D_i = \min(D_0, D_1, \ldots, D_{I-1}).$ Then we use the $i$th hypothesis set to calculate the residual phase offset by

$$\hat{\psi} = \tan^{-1} \sum_{k=0}^{N-1} \frac{\Im (x_k e^{j\hat{\phi}_i} \hat{a}_{ik}^*)}{\Re (x_k e^{j\hat{\phi}_i} \hat{a}_{ik})}. \quad (39)$$

Hence the estimated phase offset of the received signal is

$$\hat{\phi} = \hat{\phi}_e + \hat{\psi}.$$

### E. Simulation Results

In order to determine the performance of the HA, 2SC and MDE phase estimators were simulated using a Monte Carlo technique. The 16QAM and 64QAM signals are randomly generated. Then the signal is multiplied by $e^{j\phi}$, where $\phi$ is a static phase offset taken from a random uniform distribution. Noise is generated by using a complex Gaussian noise generator, scaled
Finally, the calibrated Gaussian noise is added to the signals. The Monte Carlo simulation results are shown in Fig. 7 for 16QAM. The number of symbols $N$ in the simulation is 200, with a minimum of 1000 trials to ensure accuracy. The performance of the HA estimator, the 2SC estimator, and the MDE estimator with every 2$^n$ hypothesis set are compared to the phase CRLB of square 16QAM. The approximate analytic expression [8] for the power low estimator (PLE) is also plotted for comparison. The vertical scale indicates the standard deviation of phase error in radians and the horizontal scale shows the SNR. The solid line in Fig. 7 is the true square 16QAM CRLB which was derived above. The figure shows that the MDE and 2SC estimators follow the CRLB curve. At $(E_s/N_o) = 12$ dB the SNR losses are 0.4 dB for MDE and 0.9 dB for 2SC. At $(E_s/N_o) = 14$ dB the losses are 0.2 dB for MDE and 0.7 dB for 2SC. However, between 5 and 9 dB, the 2SC algorithm performs slightly better than MDE. There is only a fraction of a decibel difference between the CRLB and the two estimators in the range of 10 dB $\leq (E_s/N_o) \leq 16$ dB and both estimators converge to the CRLB for $(E_s/N_o) \geq 16$ dB.

Fig. 8 shows the true 64QAM phase CRLB and the Monte Carlo simulation results of HA, 2SC, and MDE. Both 2SC and MDE estimators follow the 64QAM CRLB. At $(E_s/N_o) = 18$ dB, the SNR losses are 0.8 dB for MDE and 2.2 dB for 2SC. At $(E_s/N_o) = 14$ dB, the losses are 0.75 dB for MDE and 1.8 dB for 2SC. However, between 8 dB and 14 dB, the 2SC algorithm performance is better than MDE. Figs. 7 and 8 show that the HA method is significantly worse than the CRLB in all regions (MHA will have better performance than HA).

The PLE approximation agrees well with the PLE simulation at high SNR and both deviate significantly from the CRLB due to the self-noise of the PLE. At low SNR, the PLE approximation is not valid, in fact it predicts a lower variance than the CRLB. The PLE simulation gives good results at low SNRs, being slightly better than both the MDE and the 2SC below 10 dB for 16 QAM and 15 dB for 64 QAM. We assume this is due to the lack of any classification errors in the simpler PLE approach.

**IV. Conclusion**

We have derived the CRLBs for the estimation of phase and frequency offset of general rotationally symmetric signal constellations. At lower SNR, the CRLBs are much tighter than the ACRB and the MCRB. At higher SNR, the CRLBs converge to the CRLBs for an unmodulated carrier. The CRLBs for modulated signals have been expressed as a ratio of the well-known CW CRLBs with the same power. This ratio $F_{1/n}(N_r/2E_s)$ is a function of the geometry of the signal set and $E_s/N_o$. It applies to both phase and frequency estimation. With some insight into the effect of the constellation shape on the bound, this technique could assist in the design of specific signal sets for given requirements of spectral efficiency, packet length, and phase and frequency variance.

For frequency offset estimation, the case of known phase at the middle of the signal vector is equivalent to the joint phase and frequency estimation case. The unknown phase case remains an open research problem, however, asymptotically it equals the known phase case.

For phase estimation, the joint frequency case is in general increased by a factor of $\gamma$ compared to the phase only case. In
the special case where the joint estimator is analyzed relative to the middle of the signal vector, the joint phase and frequency estimation is decoupled. This is usually the case in practical algorithm designs.

We have evaluated the CRLBs for 16QAM and 64QAM and compared them with existing and new results for several phase estimators. Only small improvements in estimator performance are possible and this work allows us to understand the achievable limits. The results given and CRLB calculation techniques presented here provide a faster and simpler method of determining the performance limit of coherent receivers’ phase/frequency synchronization compared with estimator simulation. The new MDE estimator performs close to the QAM bound and provides a large improvement over the power law estimator at moderate to high SNR. At low SNR, the power law estimator works well.

APPENDIX

PDF DERIVATION FOR 16QAM CONSTELLATION

We select 16QAM as an example to illustrate mathematical details, however, a similar derivation applies to other symmetric constellations.

Equation (5) gives the pdf by

\[ p(x|\phi) = \prod_{k} \frac{p_k(a_k)}{2\pi\sigma^2} e^{-\frac{|x_k|^2}{2\sigma^2}} \sum_{a_i \in \mathbb{C}} e^{-\frac{|a_i|^2}{2\sigma^2}} e^{\frac{\Re(x_k a_i^* e^{-j\phi})}{\sigma^2}} \]

(40)

where \( C = (1/\sqrt{10})\{\pm1 \pm j, \pm3 \pm 3j, \pm3 \pm j, \pm1 \pm 3j\} \).

We split the 16QAM constellation points into four groups corresponding to fixed signal power levels and expand (40) as

\[ p(x|\phi) = \prod_{k} \left\{ \frac{1}{32\pi\sigma^2} e^{-\frac{|x_k|^2}{2\sigma^2}} e^{\frac{\Re(x_k a_i^* e^{-j\phi})}{\sigma^2}} \right\} \times \sum_{a_i \in Q_1} \sum_{r=1}^{4} \cos \left( \frac{|x_k|^2}{\sigma^2} \Re(x_k a_i^* e^{-j\phi}) \right) \]

(41)

\[ \times \sum_{r=1}^{4} \cos \left( \frac{|x_k|^2}{\sigma^2} \Re(x_k a_i^* e^{-j\phi}) \right) \]

(42)

Similarly, we can show that

\[ p(x|\phi) = \prod_{k} \left\{ \frac{1}{32\pi\sigma^2} e^{-\frac{|x_k|^2}{2\sigma^2}} e^{\frac{\Re(x_k a_i^* e^{-j\phi})}{\sigma^2}} \right\} \times \sum_{a_i \in Q_1} \sum_{r=1}^{4} \cos \left( \frac{|x_k|^2}{\sigma^2} \Re(x_k a_i^* e^{-j\phi}) \right) \]

Finally, the pdf is

\[ p(x|\phi) = \prod_{k} \left\{ \frac{1}{16\pi\sigma^2} e^{-\frac{|x_k|^2}{2\sigma^2}} e^{\frac{\Re(x_k a_i^* e^{-j\phi})}{\sigma^2}} \right\} \times \sum_{r=1}^{4} \cos \left( \frac{|x_k|^2}{\sigma^2} \Re(x_k a_i^* e^{-j\phi}) \right) \]

where \( Q_1 = (1/\sqrt{10})\{1 \pm j, 3 \pm 3j, 1 \pm 3j, 3 \pm j\} \) are constellation points in the first quadrant.

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REFERENCES


Bill Moran received B.Sc. (with honors) degree in mathematics from the University of Birmingham, U.K., and the Ph.D. degree from the University of Sheffield, U.K.

He has been Professor of mathematics at Flinders University, Adelaide, South Australia, since 1991. From 1976 to 1991, he was Professor of pure mathematics at the University of Adelaide, where he served two terms as Dean of the Faculty of Mathematical and Computational Sciences. Prior to that, he was with the University of Liverpool, U.K.

From 1991 to 1998, he was responsible for the Medical Signal Processing Program of the Cooperative Research Centre for Sensor Signal and Information Processing. He has done and continues to do research in various areas of mathematics including harmonic analysis, representation theory, and number theory. More recently, he has added research interest in several aspects of signal processing, including image processing, spectral estimation, wavelets, and radar theory. Much of his current work is with the Defence Science and Technology Organization of Australia.

Dr. Moran was elected a Fellow of the Australian Academy of Science in 1984.

Bill Cowley received the B.Sc., B.E., and Ph.D. degrees from the University of Adelaide, Adelaide, Australia, in 1975, 1976, and 1985, respectively.

He led research activities to develop new coding and modulation schemes for mobile satellite systems, including the specifications of the world’s first national L-band digital voice service and the first commercial application of turbo coding to high-speed data transmission. In 1995, he was awarded a Telecommunications Advancement Organization Fellowship from the Japanese Government, sponsored by the Communications Research Laboratory (CRL) of Japan. Since 1996, he has been the Technical Director of DSpace, Mawson Lakes, South Australia, leading the company’s efforts in advanced physical layer design for wireless systems. He is also an Adjunct Associate Professor with the University of South Australia where he participates in turbo coding, digital modem and satellite systems research. He is the co-inventor of two patents in turbo coding and adaptive rate transmission schemes, respectively.

Bill Cowley received the B.Eng. degree with honours in electronic engineering from Shandong University of Technology, China, and the M. Phil. degree in information engineering from City University, London, U.K. She is currently working toward the Ph.D. degree with the Centre for Signal and Information Processing, University of South Australia.

In 1993, she was with Flinders University of South Australia working in the biomedical engineering area. From 1995 to 1996 she worked as a Research Engineer with the Institute for Telecommunications Research, University of South Australia. Since 1996, she has been consulting in the area of simulation and modeling of telecommunications systems for a number of organizations. Her current research interests are performance bounds and algorithms for frequency and phase estimation of digitally modulated signals.


