Extrapolating a Band-Limited Function from its Samples Taken in a Finite Interval

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Abstract—A band-limited signal of finite energy can be reconstructed from its samples taken at the Nyquist rate. Moreover, the reconstruction is stable, a feature crucial for implementation: a small error in the sample values generates only a correspondingly small error in the resulting signal. The Nyquist sample values are mutually independent, so that knowledge of them in a given interval $-T \leq t \leq 0$ provides hardly any information about the behavior of the signal outside the interval. However, when the samples are taken at a greater rate—a case referred to as "oversampling"—they are interrelated, and this redundancy can be exploited in various ways to improve the behavior of the reconstruction procedure. A natural question is whether it can also be used to form accurate estimates of the signal outside the interval of observation; this problem is relevant as well to prediction theory. With oversampling, when $T = \infty$, so that the samples are known on the entire half-line $t < 0$, they determine the signal everywhere, although the reconstruction is now no longer stable. Here we examine the case of finite $T$; of course, a finite amount of data can yield only limited accuracy. We prove that the samples can be used to form an approximation to the signal outside the sampling interval, with an error which, asymptotically as $T \to \infty$, decreases exponentially in $T$, over a range which grows linearly with $T$. However, as in the limiting case, these approximations are not useful in practice, since they require the sample values to be known exactly. In the presence of measurement error, the nature of the results changes: good approximations are now available for only a bounded distance outside the interval of observation, regardless of its length, and their accuracy and range of validity can be increased only by improving the precision of sample reading. Since physical measurements are never perfect, it is this conclusion which counts for applications. The author is with Bell Laboratories, 2C-275, Murray Hill, NJ 07974. "IEEE Log Number 8608408."
signal is possible for at most a bounded distance beyond the interval of observation, and that the accuracy of reconstruction cannot be improved by sampling more densely, but only by increasing the precision with which the data are known.

We now focus on what can be done with samples from a finite interval of length $T$, a question which is relevant also to the prediction theory. Since finitely many samples can never determine a band-limited function, we can no longer expect perfect extrapolatability, but we can hope that the information they provide is sufficient (together with some prior bound on size) to approximate the function accurately over a certain region outside the sampling interval. As in the case of the half-line, we will from the outset distinguish two reconstruction problems having very different answers. When the samples are given exactly, we will draw on the theory of analytic functions to construct an approximation over a range that grows linearly with $T$, and with an error that decreases exponentially in $T$. This extends previous results [2], [9], which discuss the approximation only on intervals remaining fixed as $T$ increases, and also improves the corresponding error estimates. On the other hand, when samples are known to merely finite precision, our earlier qualitative considerations apply without change to show that a good approximation is possible for no more than a bounded distance beyond the interval of observation, regardless of the size of $T$ or the rate of oversampling. Since physical measurement can never be perfect, it is this latter result that matters in applications.

Specifically, let $\mathcal{B}_2(\pi)$ denote the set of signals of finite energy

$$\|f\| = \int_{-\infty}^{\infty} |f(t)|^2 \, dt < \infty$$

and bandwidth $[-\pi, \pi]$. For a fixed $\alpha < 1$, we consider as given the sample values

$$S = \{s_k\} = \{f(ka)\}, \quad -T \leq ka \leq 0,$$

of a signal $f(t) \in \mathcal{B}_2(\pi)$, taken from the interval $[-T, 0]$. If we add to $f(t)$ any signal $h(t)$ of $\mathcal{B}_2(\pi)$ which vanishes at all the sampling points in $[-T, 0]$, the sum will have the same samples. Thus the collection of signals having the specified samples $S$ can be described geometrically as the linear subspace of $\mathcal{B}_2(\pi)$ formed by all the possible signals $h$, translated away from the origin by the vector $f$. In turn, this shows that without additional assumptions no extrapolation is possible, since by proper choice of $c$, $f(t) + ch(t)$ can assume any value, wherever $h(t) \neq 0$. To obtain an intelligible problem, we impose from the outset a bound on the size of the signals to be considered, letting $\Gamma(S, E)$ denote the collection of all signals in $\mathcal{B}_2(\pi)$ having the specified samples $S$, and also energy bounded by $E^2$:

$$\Gamma(S, E) = \{ g \in \mathcal{B}_2(\pi) | \|g(ka)\| = s_k, \quad -T \leq ka \leq 0; \|g\|^2 \leq E^2 \}. \quad (2)$$

Thus $\Gamma(S, E)$ consists of all of the permitted signals which are consistent with the sampling data, and, in attempting approximation, we ask how well we can describe the possible values of members of $\Gamma(S, E)$. To answer, the earlier geometric observation suggests that we choose the translating vector $f$ so as to give an orthogonal decomposition of $\Gamma(S, E)$. Accordingly, let $\Gamma(0,1)$ denote the collection $\Gamma(S, 1)$ when $S$ corresponds to vanishing samples. We will show first that the signal $f_5(t)$ of $\Gamma(S, E)$ having least energy consists of a linear combination of the functions $\{\sin \pi(t - ka) / \pi(t - ka)\}$, $-T \leq ka \leq 0$, and that $\Gamma(S, E)$ can be decomposed into orthogonal components as

$$\Gamma(S, E) = f_5(t) + \sqrt{E^2 - \|f_5\|^2} \Gamma(0,1). \quad (3)$$

It follows that the set of values at a point $\tau$ of the functions of $\Gamma(S, E)$ is a disk with center $f_5(\tau)$ and radius $\sqrt{E^2 - \|f_5\|^2} \rho_\tau(\tau)$, where

$$\rho_\tau(\tau) = \sup_{g \in \Gamma(0,1)} |g(\tau)|; \quad (4)$$

the boundary of the disk is traced by the functions $f_\tau + e^{it} \sqrt{E^2 - \|f_5\|^2} g^*$, where $g^*$ attains the supremum in (4), and $\theta$ varies over $[0, 2\pi]$. Thus the best estimate of this set of values, in the sense of minimizing the largest possible deviation, is $f_5(\tau)$; we see that this single function, which depends linearly on the data $S$, serves simultaneously for all points $\tau$. Specifically, to obtain the approximation at any point $\tau$, we evaluate $f_5$ and $g^*$ at $\tau$ and compute

$$\sum_{-T \leq ka \leq 0} a_k \sin \pi(t - ka)/\pi(t - ka),$$

where the coefficients $a_k$ are determined by the linear system

$$\sum_{-T \leq ka \leq 0} a_k \sin \{j - k\} \alpha/\pi(j - k) \alpha = s_j, \quad -T \leq j\alpha \leq 0.$$

From the computational point of view, the solution involves inverting the foregoing Toeplitz matrix; once found, this inverse can be applied to any sequence $\{s_j\}$ of given sample values. The error in this approximation is the radius of the disk and hence is a multiple, independent of $\tau$, of the function $\rho_\tau(\tau)$, independent of $S$. By (3), it is worst when $\|f_5\| = 0$, that is, when extrapolating from vanishing sample values.

To study the error, we consequently examine $\rho_\tau(\tau)$, which describes how large a function of $\mathcal{B}_2(\pi)$ with unit energy can be at $\tau$, when it is constrained by having more zeros in an interval than the Nyquist rate allows. In its dual version, this is a problem of prediction theory, for $\rho_\tau(\tau)$ measures how well the exponential $e^{it\tau}$ is approximable in $L^2[-\pi, \pi]$ by linear combinations of the exponentials $\{e^{ik\alpha}, -T \leq ka \leq 0\}$, or equivalently, in more standard form, how well $e^{i\alpha}$ is approximable in $L^2[-\pi\alpha, \pi\alpha]$ by linear combinations of the exponentials $\{e^{ik\alpha}, -T/\alpha \leq k \leq 0\}$. For values of $\tau$ at a fixed distance from the block of approximating frequencies, we can expect an exponentially good fit. Thus we are led to ask how far away from the interval $[-T, 0]$ will $\rho_\tau$ remain exponentially small in $T$, that is, obey a bound of the form

$$\rho_\tau(\tau) \leq e^{-T + o(T)} \quad (5)$$
for some $\gamma > 0$. To formulate this, we examine $T^{-1} \log \rho_T(\tau)$ and show that this quantity is negative in the range $-(1 + \sigma)T \leq \tau \leq \sigma T$, with $\sigma < \sigma_1(\alpha)$, a function of the sample spacing $\alpha$, and $T$ sufficiently large. Our argument will also show that when $\tau$ is in an interval that does not grow with $T$, the rate $-\gamma$ of (5) can be taken to be $\alpha^{-1} \log \sin \pi \alpha T / 2$, improving on the previously given estimate $\alpha^{-1} \log \sin \pi \alpha T / 2$ valid for $\alpha < 1/2$ only [2]. In the opposite direction, we construct $\sigma_2(\alpha)$ such that when $\tau = \sigma_2(\alpha)T$, $\rho_T(\tau) \geq c/\sqrt{T}$, so that (5) certainly fails to hold. Specifically, we obtain the following result.

**Theorem 1:** There exist $\sigma_1(\alpha)$, $\sigma_2(\alpha)$, and $c > 0$ such that to each $\sigma < \sigma_1$ there corresponds a $\gamma > 0$ with $\limsup_{T \to \infty} T^{-1} \log \rho_T(\tau) \leq -\gamma$, while $\rho_T(\sigma_2(\alpha)T) \geq c/\sqrt{T}$. The function $\sigma_1(\alpha)$ is bounded below by $(\csc(\pi \alpha / 2) - 1)/2$, and $\sigma_2(\alpha)$ is bounded above by the solution of the equation $\alpha^{-1} \log(1 + \alpha^{-1}) + \log(\alpha^{-1} - 1 - 2\alpha) - \log(\alpha^{-1} + 1 + 2\alpha) = 0$. For $\tau$ in a bounded interval $0 \leq \tau \leq B$, $\limsup_{T \to \infty} T^{-1} \log \rho_T(\tau) \leq \alpha^{-1} \lim \sin \pi \alpha T / 2$.

The extrapolation problem we have been considering in the space $B_{2\alpha}(\pi)$ of functions of finite energy and bandwidth $\pi$ can be posed equally well in the (larger) space $B_{\infty}(\pi)$ of bounded functions of bandwidth $\pi$. Specifically, we can ask to extrapolate to $\tau = \tau_0$ the values of a collection of function $g(t) \in B_{\alpha}(\pi)$ which have prescribed samples $\{g(k\alpha)\}$, $-T < k \alpha \leq 0$, and for which $|g(t)| \leq 1$. The best extrapolation no longer has the simple form enjoyed by $f_1(t)$; for a somewhat different norm, it has been determined in the illuminating paper [6]. However, it follows from the results of [7] that, just as in the finite-energy case, the best approximation still depends linearly on the observed samples and that the worst error is produced when these all vanish.

We prove Theorem 1 by using potential theory to estimate the size of a bounded function vanishing at the sampling points; thus, Theorem 1 describes equally well the degree of accuracy with which bounded band-limited functions can be extrapolated. In that context, the conclusion concerning $\sigma_1(\alpha)$ can be strengthened, for our construction shows that a function of $D_{2\alpha}(\pi)$ which vanishes at $\{k\alpha\}$, $-T < k \alpha \leq 0$, can attain its maximum at $\tau = \sigma_1(\alpha)T$. Thus the point $\tau = \sigma_2(\alpha)T$ falls entirely outside the range of influence of the samples, and good extrapolation there is impossible. The bounds for $\sigma_1$ and $\sigma_2$, some of which are given in Table I, evidently do not meet; their role is only to fix roughly the range of accurate extrapolatability. Table I also shows that this range is not very accurately estimated by considerations of the Nyquist rate [8], which would suggest that $\sigma \sim (\alpha^{-1} - 1)/2$.

The approximation we discuss is in the uniform norm, but the bounds are sufficiently strong to apply as well to other measures, such as energy. That is, as $\rho_T(t)$ measures the largest extrapolation error at $t$, the energy of that error over a set $S$ is bounded by $\int_S \rho^2_T(t) \, dt \leq \text{meas}(S) \sup_{t \in \sigma T} \rho_T^2(t)$, so that when $S$ is a subset of $[-(1 + \sigma)T, \sigma T]$, the exponential bound of Theorem 1 persists also for energy. We conclude that, in theory, the data obtained by oversampling allow, asymptotically, good approximation up to a distance beyond the sampling interval that grows proportionally to the length of that interval.

The preceding results concern an approximation constructed from exact knowledge of the samples. If we suppose instead that the samples are known only within some error $\epsilon$, then the characterization (3) of the set of signals which fit the data no longer applies. Instead, we can now allow $f_s(t)$ to be perturbed by any signal $g(t)$ of energy at most $(E - \|f_s\|^2)$, having values bounded by $\epsilon$ at the sampling points. For then $\|f_s + g\| \leq \|f_s\| + \|g\| \leq E$, while the sample values of $f_s + g$ differ from $S$ by at most $\epsilon$, so that $f_s$ and $f_s + g$ are indistinguishable within the level of measurement accuracy. It is easy to produce such permitted deviations; an example is a signal translated sufficiently far to the right, such as $g(t) = K \sin (\pi t - \gamma)/(t - \gamma)$, with $|K| \leq (E - \|f_s\|^2)/\pi$, and $\gamma$ so large that $|K|/\gamma < \epsilon$, for then $|g(t)| < \epsilon$ everywhere in $t < 0$. However, at $t = \gamma$ the function $f_s + g$ has the value $f_s(\gamma) + K\pi$, and as $K$ varies over its allowed range, these points sweep out a disk of radius $(E - \|f_s\|^2)$. This quantity is therefore a lower bound for the error of any extrapolation at $t = \gamma$; we see that it is not decreased by taking more data in $t < 0$. Since $g(t)$ decreases at the rate $1/t$, the order of magnitude of $g$ here is $1/t$. On choosing for $g(t)$ a translate of a function of the form $K \sin (\pi t/k)/t^k$ with a suitable integer $k$, which decreases faster, one can reduce the size of $\gamma$ in the above argument to $-\log \epsilon$. The same example applies when the accuracy of sample readings is measured in a different way, for example, by the energy of the deviations. From this simple observation we can draw the following important conclusion.

**Theorem 2:** When sample measurements are accurate only to within $E > 0$ in amplitude or in total energy, good extrapolation is possible for only a bounded distance (having an order of magnitude $-\log \epsilon$) beyond the interval of observation, regardless of the amount of data used.

Hence the only way to increase the range of extrapolatability is to improve the accuracy with which the samples are determined. Calculations which illustrate this are given in [4]. We thus have here another example of the familiar phenomenon that unstable reconstructions can behave very differently from stable ones.
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APPENDIX

Proofs

As a preliminary observation, we recall that square-integrable functions form a Hilbert space with scalar product

$$(f, g) = \int f(t) \overline{g(t)} \, dt$$

and norm defined correspondingly by \(\|f\|^2 = (f, f)\); thus \(\|f\|^2\) represents the energy of \(f\). By Parseval's theorem, the Fourier transform

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt$$

is a unitary transformation,

$$\|f\| = \|F\|,$$

and so \(\mathcal{B}_2(\pi)\), the collection of function of finite energy whose Fourier transform is supported in \(|\omega| < \pi\), is a closed linear subspace, hence itself a Hilbert space. For \(f \in \mathcal{B}_2(\pi)\), by writing \(f\) as the inverse of its Fourier transform,

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} F(\omega) e^{i\omega t} \, d\omega,$$

and applying Schwarz's inequality, we find

$$|f(t)| \leq \|F\| = \|f\|.$$  \hspace{1cm} (6)

On expanding \(F(\omega)\) in its Fourier series, we obtain

$$F(\omega) = \sum a_k e^{-ik\omega} / \sqrt{2\pi}, \quad |\omega| < \pi,$$

with

$$a_k = \int_{-\pi}^{\pi} F(\omega) e^{ik\omega} / \sqrt{2\pi} \, d\omega = f(k),$$

the series converging in norm, and the Fourier transform of this series yields the sampling expansion (1). In (1) the series converges in norm by (6), hence also uniformly by (8). The construction shows that \((\sin \pi(t - k\omega)/\pi(t - k\omega))\) form a complete orthonormal set in \(\mathcal{B}_2(\pi)\), that

$$f(t) = \int_{-\pi}^{\pi} f(x) \frac{\sin \pi(t - x)}{\pi(t - x)} \, dx$$

and that the values \(\{f(k)\}\) are subject to the single constraint

$$\sum |f(k)|^2 = \|f\|^2.$$

Moreover, by means of (7) any \(f \in \mathcal{B}_2(\pi)\) can be extended from the real axis to an entire function of the complex variable \(t + i\omega\), with growth in the complex plane bounded by

$$|f(t + i\omega)| \leq \frac{1}{\sqrt{2\pi}} \|F\| \left[ \int_{-\pi}^{\pi} |e^{-i\omega \tau}| \, d\omega \right]^{1/2} \leq c\|F\| \theta(1 + |\omega|).$$

Thus \(f(t) \in \mathcal{B}_2(\pi)\) is an entire function of exponential type \(\pi\), restricted to the reals and square-integrable there. By the

Paley–Wiener theorem, \(\mathcal{B}_2(\pi)\) consists of precisely such functions. It follows that if \(f(t) \in \mathcal{B}_2(\pi)\) and \(f(t_0) = 0\), the function

$$g(t) = \frac{f(t)}{t - t_0}(t - t_0)$$

is likewise in \(\mathcal{B}_2(\pi)\) for any \(t_0\), since it remains square-integrable and entire of exponential type \(\pi\).

We now focus on the set of signals of \(\mathcal{B}_2(\pi)\) having specified sample values. Recalling the definitions (2) of \(\Gamma(S, E)\) and of \(\Gamma(0, E)\), we obtain the following decomposition.

Lemma: \(\Gamma(S, E)\) contains a (unique) linear combination \(f_S\) of the functions \(\sin \pi(t - k\omega)/\pi(t - k\omega)\), \(-T \leq k\omega \leq 0\), and has the representation

$$\Gamma(S, E) = f_S + \sqrt{E^2 - \|f_S\|^2} \Gamma(0, 1),$$

with \(f_S\) orthogonal to \(\Gamma(0, 1)\).

Proof: Denote by \(\Sigma\) the subspace spanned by \((\sin \pi(t - k\omega)/\pi(t - k\omega)), -T \leq k\omega \leq 0\). By (9), the linear subspace \(\Gamma(0, \infty)\), consisting of all functions of \(\mathcal{B}_2(\pi)\) that vanish at the sample points, is precisely the orthogonal complement of \(\Sigma\). Let \(f\) be any element of \(\Gamma(S, E)\) and let \(f_S\) be its projection onto \(\Sigma\); as \(f\) and \(f_S\) differ by an element of \(\Gamma(0, \infty)\), \(f\) is unique, \(f_S\) has the same sample values as does \(f\). Moreover, for any \(g \in \Gamma(S, E), g - f_S \in \Gamma(0, \infty)\), while \(f_S\) is orthogonal to \(\Gamma(0, \infty)\). Thus, on writing

$$g = f_S + (g - f_S),$$

we obtain an orthogonal decomposition. Consequently,

$$\|g - f_S\|^2 = \|g\|^2 - \|f_S\|^2,$$

hence \(\|g\|^2 \leq E^2\) if and only if \(\|g - f_S\|^2 \leq E^2 - \|f_S\|^2\). This proves the lemma.

It follows from the lemma that the set of values of the functions of \(\Gamma(S, E)\) at a point \(t = \tau\) is a circle, with center \(f_S(\tau)\) and radius \(\sqrt{E^2 - \|f_S\|^2}\). Hence

$$f_S(\tau) = \sup_{g \in \Gamma(0, 1)} |g(\tau)|.$$

The best approximation to this set, in the sense of minimizing the largest possible deviation, is then \(f_S(\tau)\). The error in the approximation is the radius of the circle. It is, therefore, worst when \(\|f\| = 0\), that is, when extrapolating from vanishing sample values. This conclusion is well-known [7].

We now fix \(\alpha < 1\) and study the asymptotics of \(f_S(\tau)\) as \(T \to \infty\), asking how far away from the interval \([-T, 0]\) is a function of \(\mathcal{B}_2(\pi)\) of unit energy forced to be exponentially small by the requirement that it vanish at all the sampling points in the interval. For this purpose, it is convenient to enlarge the class \(\mathcal{B}_2(\pi)\) by passing to \(\mathcal{B}_2(\pi)\), defined as the space of those bounded functions on the real axis which are extendible to the complex plane as entire functions of exponential type \(\pi\). Analogously to the definition of \(\Gamma(0, 1)\), let

$$\Delta(0, 1) = \{ g \in \mathcal{B}_\infty(\pi) | g(k\omega) = 0, -T \leq k\omega \leq 0, \sup |g(t)| \leq 1 \}. $$

and set

$$r_T(\tau) = \sup_{g \in \Delta(0, 1)} |g(\tau)|.$$
By (7), \( \Gamma(0,1) \subset \Delta(0,1) \); hence
\[
p_f(\tau) \leq r_f(\tau).
\] (13)

Theorem: There exist \( \sigma_1(\alpha) \) and \( \sigma_2(\alpha) \) such that for each \( \sigma < \sigma_1 \),
\[
\limsup_{T \to \infty} \left[ \sup_{-T < \tau \leq T} \frac{r_f(\tau)}{T} \right] < 0.
\] (14)
while
\[
r_f(\sigma T) = 1.
\] (15)
Moreover, \( \sigma_1(\alpha) \geq (\csc(\pi \alpha) / 2) - 1 / 2 \), and \( \sigma_2(\alpha) \) is bounded above by the solution of the equation \( \alpha^{-1} \log(1 + \sigma^{-1}) + \log(\alpha^{-1} - 1 - 2\sigma) - \log(\alpha^{-1} + 1 + 2\sigma) = 0 \). Finally, in any bounded interval \( 0 \leq \tau < B \), \( \lim_{T \to \infty} \log r_f(\tau) \leq \alpha^{-1} \log \sin \pi \alpha / 2 \).

Proof: If \( g \in \mathcal{R}_n(\pi) \) with \( |g(\tau)| \leq 1 \), then [1, p. 82] in the complex plane
\[
|g(t + i\mu)| \leq e^{\pi|\mu|}.
\] (16)
Let \( D_f \) be a closed bounded region of the \( \omega = (t + i\mu) \)-plane which includes the segment \([-T,0]\) in its interior, and on the boundary \( \delta \) of which
\[
e^{\pi|\mu|} \leq c |\sin \pi \omega|,
\] (17)
for some constant \( c \); it suffices here that \( \delta \) avoid a neighborhood of fixed size around each integer. Let \( G_f(\omega, w) \) be the Green's function of \( D_f \) with pole at \( \omega \in D_f \), that is, \( G_f(\omega, w) \) is the function of \( w \in D_f \) which vanishes for \( w \in \delta \), and for which \( G_f(\omega, w) - \log |\omega - w| \) is harmonic in \( D_f \).
If \( g(t) \in \Delta(0,1) \), then \( g(t) \) vanishes at the points \( t = z_k = k \alpha, -T \leq z_k \leq 0 \), and so the function
\[
h(w) = \log |g(w)| - \sum_{\omega \in \delta} G_f(z_k, w)
\] (18)
is subharmonic for \( w \in D_f \). Accordingly, by the Poisson formula, for any \( t \in D_f \),
\[
h(t) \leq \sum_{\xi \in \delta} h(\xi) \frac{\partial G_f(t, \xi)}{\partial n} ds,
\] with the exterior normal. Since \( G_f(z_k, \xi) \) vanishes for \( \xi \in \delta \), we see from (18), (16), and (17) that
\[
h(t) \leq \sum_{\xi \in \delta} \log |c \sin \pi \xi| \frac{\partial G_f(t, \xi)}{\partial n} ds.
\] (19)
However,
\[
\log |c \sin \pi w| = \sum_{m \in D_f} G_f(m, w)
\] is harmonic in \( D_f \), so that
\[
\log |c \sin \pi t| = \sum_{m \in D_f} G_f(m, t) = \int_{t \in \delta} \log |c \sin \pi \xi| \frac{\partial G_f(t, \xi)}{\partial n} ds.
\]
We conclude from (19) that
\[
\log |g(t)| = \sum_k G_f(z_k, t) \leq \log |c \sin \pi t| - \sum_{m \in D_f} G_f(m, t).
\] (20)
Let us now set \( t = \alpha T \) in (20) and use the symmetry of the Green's function to obtain
\[
\frac{\log |g(\alpha T)|}{T} \leq \frac{\log c}{T} + \frac{1}{T} \sum_{k = 0}^{T/\alpha \subset [-T/\alpha]} G_f(\alpha T, \alpha k).
\] (21)
Moreover, \( \sigma_1(\alpha) \geq (\csc(\pi \alpha) / 2) - 1 / 2 \), and \( \sigma_2(\alpha) \) is bounded above by the solution of the equation
\[\alpha^{-1} \log(1 + \sigma^{-1}) + \log(\alpha^{-1} - 1 - 2\sigma) - \log(\alpha^{-1} + 1 + 2\sigma) = 0 \]. Finally, in any bounded interval \( 0 \leq \tau < B \), \( \lim_{T \to \infty} \log r_f(\tau) \leq \alpha^{-1} \log \sin \pi \alpha / 2 \).

The function \( \alpha \) is the Green's function corresponding to the rescaled region \( D_f(\alpha) \), which contains the segment \([-1,0]\) in its interior. The sums in (21) are \( T \sum_{\omega \in \Delta(1/\alpha)} G_f(\sigma \alpha, \sigma \alpha / T) + T \sum_{m \in \Delta(1/\alpha)} G_f(m, m / T) \), that is, they are Riemann sum approximations to the converging integrals \( \alpha^{-1} \int G_f(\sigma, t) dt \) and \( \int_{m \in D_f} G_f(\sigma, t) dt \), respectively. If we choose the regions \( D_f \) so that \( D_f/T \) converge to a region \( D \) as \( T \to \infty \), we obtain
\[
\limsup_{T \to \infty} \frac{\log r_f(\sigma T)}{T} \leq \frac{1}{T} \int_{\omega \in \Delta(1/\alpha)} G_f(\sigma, t) dt - \int_{t \in D} G_f(\sigma, t) dt,
\] (22)
with \( G_f(\alpha \sigma, t) \) the Green's function of \( D_f(\alpha) \). However, now we are free also to vary \( D \). If the \( D \) converge, the corresponding Green's functions converge, uniformly on compact subsets, so in the right side of (22) we can remove the restriction (17), as well as that of boundedness of \( D \). Thus to establish (14) for a given value of \( \tau = T \alpha \sigma \), it suffices to show that the right side of (22) is negative for \( \sigma \leq \sigma_0 \). If \( \alpha \leq 1 \), this can always be done, for on rewriting that quantity as
\[
\left( \frac{1}{\alpha} - 1 \right) \int_{-1}^{0} G_f(\sigma, t) dt - \int_{1 \in D} G_f(\sigma, t) dt
\] and choosing \( D \) so that it intersects the reals in an interval barely larger than \([-1,0]\), the first integral approaches a finite negative value, while the second approaches zero. Of course, in the process \( \alpha \) likewise becomes small.

We can produce specific lower bounds for \( \sigma_1(\alpha) \) by selecting different regions \( D \) and computing the value of \( \sigma \) at which the right side of (22) vanishes. A promising possibility for \( D \) is the entire plane, slit along the real axis from \(-\infty \) to a point \( a \leq -1 \), and from \( 0 < b < \infty \). If we now rescale \( D \) so that \( a \sigma = -1 \), \( b \sigma = 1 \), the interval \([-1,0]\) and the point \( t = \sigma \) are replaced by \([\lambda, \mu]\) and \( v \), and our object is to maximize \((v - \mu)/(\mu - \lambda)\) under the correspondingly rescaled requirement of (22) that
\[
(\mu - \lambda) \left\{ \frac{1}{\alpha} \int_{\lambda}^{\mu} G_f(v, t) dt - \int_{-1}^{0} G_f(v, t) dt \right\} \leq 0.
\] (23)

In the present case, these integrals can be found in closed form, for by noting that the conformal map \( 1/2 \) transforms \( D \) into the plane slit along the reals from \(-1 \) to \( 1 \), that the inverse of \((w + w^{-1})/2 \) maps that region into the unit disk, and that a suitable linear fractional transformation sends an arbitrarily chosen point onto the origin, we find
\[
G_f(v, t) = \frac{1 - \sqrt{1 - t^2} - 1 - \sqrt{1 - v^2}}{v}.
\]
Now, on letting $t = \sin \theta$, $\nu = \sin \eta$, and using the half-angle formulas, we obtain

$$\int_{-\lambda}^{\mu} G_D(\nu, t) \, dt$$

$$= \int_{2^{-\lambda}}^{2^{\lambda}} \frac{2 \sin^2 \theta / 2 - 2 \sin \eta / 2}{2 \sin \theta / 2 \cos \theta / 2 - 2 \sin \eta / 2 \cos \eta / 2} \cos \theta \, d\theta$$

As we have seen, with the present choice of $D$, and $\lambda = -\mu$, the right side of (22) is transformed into $f(\phi, \eta)/4a \sin \phi$. This quantity can therefore also serve as a bound for $\lim_{T \to \infty} r_T(\sigma, T)$ for all values of $\sigma$ smaller than $(\csc \pi a/2 - 1)/2$. Specifically, to such a value of $\sigma$ there corresponds $\nu \leq 1$ such that $\sigma = (\nu - 1)/2\mu$, or $\nu = 2\sigma + 1$. With $\nu$ so determined, $f(\phi, \eta)/4a \sin \phi$ is a function of $\mu$ alone, and its minimum in the range.

$$\sin \pi a/2 \leq \mu \leq 1/(2\sigma + 1)$$

provides a (negative) upper bound for the exponent $-\gamma$ which figures in the estimate $r_T(\tau) \leq e^{-\gamma T + o(T)}$, valid for $-T(1 + \sigma)$.

In particular, for $\tau$ in a bounded interval $\leq B$, the associated $\sigma = 0$, so that $\nu = \mu$, hence $\phi = \eta$, and

$$f(\phi, \phi) = \log \frac{4a \sin \phi}{\alpha} = \frac{(2 \sin^{-1} \mu - \pi a)(1 - \mu^2)}{2a\mu}$$

The minimum of this in the corresponding range $\sin \pi a/2 \leq \mu \leq 1$ is bounded by the value at the left endpoint, which yields the desired estimate $-\gamma \leq a^{-1} \log \pi a/2$.

As (22) goes in only one direction, a lower bound for $-\gamma$ is not available from the present line of reasoning. To obtain (15), we construct a function of $\csc(\pi \eta)$, vanishing at the points $z_i$, and look for its supremum. For this, we apply (12) to $\sin \pi t$ in order to move zeros from outside $[-T, 0]$ into that interval. The zeros in $[-T, 0]$ may need to be shifted, and $T(a^{-1} - 1)$ additional ones must be brought in; we take $T(a^{-1} - 1)/2$ from each end. Specifically, suppose $T(a^{-1} - 1)/2$ to be an integer, we form

$$h(t) = (\sin \pi t)(t + a)(t + 2a) \cdots (t + [T/a])$$

(26)

with

$$Q(t) = \left\{ \begin{array}{l} \{t + 1\} \cdots \{t + [T]\} \\ \\ \{t - 1\} \cdots \{t - (a^{-1} - 1)/2\} \\ \\ \{t + T + 1\} \cdots \{t + T + (a^{-1} - 1)/2\} \end{array} \right.$$

We can gauge the amplitude of this signal at half-integer points, where $\sin \pi t$ has magnitude 1. Again we find that at $t = T\sigma$, for large $T$,

$$\log |h(\sigma T)| = \frac{1}{\alpha} \int_0^1 \log |s + t| \, ds - \int_0^1 \log |s + t| \, ds$$

$$- \int_0^1 \log |s - t| \, ds - \int_t^1 \log |s + t| \, ds, \quad (27)$$

with

$$c = (a^{-1} - 1)/2 \quad d = (a^{-1} + 1)/2.$$
obtain an element of \((0,1);\) hence
\[ r_T(\sigma, T) = 1. \]
This concludes the proof of the theorem.

Corollary: The estimate
\[ r_T(\sigma, T) \leq e^{-\gamma T + o(T)}, \]
with some \(\gamma > 0,\) holds in \(-T(1 + \sigma) \leq \tau \leq \sigma T\) for each \(\sigma \leq \sigma_1,\)
and \(-\gamma \leq \sigma^{-1} \log \sin \pi \sigma / 2\) when \(0 \leq \sigma \leq B,\) but \(r_T(\sigma, T) \geq c/\sqrt{T}.\)

Proof: We can readily adapt the results of the theorem to \(\mathcal{I}_T^2(\sigma, T),\) for by (13) the upper bound (14) holds for \(r_T.\) To estimate the lower bound, we form
\[ k(t) = \frac{h(t)}{|h(\sigma T)|} \frac{1}{|t - cT - 1|}, \]
with \(h(t)\) the function of (24); this belongs to \(\mathcal{I}_T^2(\sigma, T)\) since the first factor is bounded and \(h(cT + 1) = 0.\) For the energy of \(k(t),\) we have
\[ \int_{-\infty}^{\infty} |k(t)|^2 \, dt = \int_{-\infty}^{\infty} \frac{|h(t)|^2}{|h(\sigma T)|^2} \frac{1}{|t - cT - 1|^2} \, dt. \]
For \(|t - \sigma T| > \Theta T,\) \(\Theta\) is fixed, the first factor in the integrand approaches 0 exponentially in \(T\) by (25). While the contribution in \(|t - \sigma T| \leq \Theta T\) is bounded by
\[ \int_{(\sigma - \Theta)T}^{(\sigma + \Theta)T} (t - cT - 1)^{-2} \, dt = \text{const.} / T. \]
Thus \(r_T(\sigma, T) \geq k(\sigma, T)/||k|| = \text{const.} / \sqrt{T},\) as was to be shown. This demonstrates also that, analogously to the case of \(\mathcal{I}_c,\) a neighborhood of \(\sigma T\) accounts for a nonzero fraction of the norm (here energy) of \(k(t).\)

REFERENCES