PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A

Phil. Trans. R. Soc. A (2011) **369**, 1215–1227
doi:10.1098/rsta.2010.0337

Memories of Hirota's method: application to the reduced Maxwell–Bloch system in the early 1970s†

By P. J. Caudrey*

School of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL, UK

Who remembers 'Hirota's method'? In the early days of solitons, although the Korteweg–de Vries equation had been solved by the 'inverse scattering method' most solutions to integrable non-linear equations were found by simpler more direct methods. Outstanding among these was a method due mainly to Hirota, which involved casting the equation into a 'bilinear form' and then applying intelligent guesswork. In this paper, I shall take a journey down memory lane, looking again at this method.

Keywords: solitons; nonlinear optics; non-linear partial differential equations

1. Introduction

I decided to go down the historical path for this paper for two reasons. The first is because it is fun to see how things were done in the past. The second is because I have been retired for 13 years and all I know is now history. I want to talk about the early 1970s when, although the 'inverse scattering method' [1] had been used, there were only a couple of suitable spectral problems known, which restricted its application somewhat. At that time, the other outstanding method of finding multi-soliton solutions to nonlinear partial differential equations (PDEs) was 'Hirota's method' [2–4]. There was a joke that if you wanted to know whether a PDE was integrable you used the 'postcard test'—you wrote it on a postcard, sent it to Hirota and, if it was integrable, you received the n-soliton solution by return of post!

*philip.caudrey@ntlworld.com

†To the memory of Robin Bullough. I cannot let this occasion pass without sharing at least one memory of Robin, the man not the physicist. It is of him playing with my grandson, David, who was about 18 months old at the time. It was not much of a game, consisting of passing a toy building brick backwards and forwards, but both Robin and David seemed to know the rules and I could not tell who was enjoying it most. Robin was not a religious man and I do not think he believed in an afterlife but I am convinced that he is out there somewhere watching these proceedings with a certain amount of amusement—that is, unless he has just found a new problem to solve, in which case he is totally oblivious to anything that is happening on Earth or in Heaven!

One contribution of 13 to a Theme Issue 'Nonlinear phenomena, optical and quantum solitons'.

2. The Korteweg–de Vries equation

Let us begin, as all soliton theory does, with the Korteweg–de Vries (KdV) equation

$$u_t + u_{xxx} + 6uu_x = 0. \tag{2.1}$$

We can easily find the *one-soliton solution* by putting $u = u(x - vt)$, where v is a constant speed. This reduces equation (2.1) to an ordinary differential equation which, with the boundary conditions that u and its derivatives tend to 0 as x tends to $\pm\infty$, gives the solution

$$u = \tfrac{1}{2}p^2 \operatorname{sech}^2\left(\tfrac{1}{2}p(x - p^2 t - x_0)\right), \tag{2.2}$$

where, to avoid square roots, we have put $v = p^2$. This is a *solitary wave* whose speed is proportional to its amplitude and immediately gives rise to the possibility of choosing initial conditions with two or more of these well separated which, when left to run, will collide.

If we put

$$u = w_x, \tag{2.3}$$

we find that w must satisfy

$$\frac{\partial}{\partial x}(w_t + w_{xxx} + 3w_x^2) = 0, \tag{2.4}$$

and so, if we also assume that w and its derivatives tend to 0 as $x \to \pm\infty$, it satisfies

$$w_t + w_{xxx} + 3w_x^2 = 0. \tag{2.5}$$

Thinking in terms of Padé approximants, we put

$$w = \frac{g}{f}, \tag{2.6}$$

which gives

$$\frac{g_t f - g f_t}{f^2} + \frac{g_{xxx} f - 3g_{xx} f_x - 3g_x f_{xx} - g f_{xxx}}{f^2} + 6\frac{g f f_x f_{xx} + g_x f f_x^2 - g f_x^3}{f^4} + 3\frac{(g_x f - g f_x)^2}{f^4} = 0. \tag{2.7}$$

This can be rewritten as

$$\frac{g_t f - g f_t + g_{xxx} f - 3g_{xx} f_x + 3g_x f_{xx} - g f_{xxx}}{f^2} + 3\frac{(g_x f - g f_x)(f(g - 2f_x)_x - f_x(g - 2f_x))}{f^4} = 0, \tag{2.8}$$

and so, if we now put

$$g = 2f_x, \tag{2.9}$$

we get

$$f f_{xt} - f_x f_t + f f_{xxxx} - 4f_x f_{xxx} + 3f_{xx}^2 = 0, \tag{2.10}$$

with

$$u = w_x = 2\frac{\partial^2}{\partial x^2}\ln(f). \tag{2.11}$$

Equation (2.10) is Hirota's *bilinear form* for the KdV equation. The one-soliton solution can be found by putting

$$f = 1 + \exp(p(x - vt - x_0)). \tag{2.12}$$

Substituting this in equation (2.10) gives $v = p^2$ and equation (2.11) gives the solution already mentioned.

(*a*) *Hirota's D-operators*

At this stage it is convenient to introduce Hirota's *D-operators* [5] to simplify the expression of the bilinear form. If f and g are two functions of x and t, we introduce the operators D_x and D_t that act on the product $f(x,t)g(x',t')$ as follows

$$D_x f \cdot g = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right) f(x,t)g(x',t')\bigg|_{x'=x,t'=t} \tag{2.13}$$

and

$$D_t f \cdot g = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right) f(t,t)g(t',t')\bigg|_{x'=x,t'=t}. \tag{2.14}$$

This notation is extended to products and powers

$$D_t^m D_x^n f \cdot g = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n f(x,t)g(x',t')\bigg|_{x'=x,t'=t}. \tag{2.15}$$

Now, we find that equation (2.10) can be written as

$$D_x(D_t + D_x^3) f \cdot f = 0. \tag{2.16}$$

Hirota found a long list of properties for his D-operators. Here are just some of them.

1.
$$D_x^m D_t^n f \cdot 1 = \frac{\partial^{m+n}}{\partial x^m \partial t^n} f.$$

2.
$$D_x^m D_t^n f \cdot g = (-1)^{m+n} D_x^m g \cdot f.$$

3.
$$D_x^m D_t^n f \cdot f = 0 \quad \text{if } m+n \text{ is odd.}$$

4.
$$\begin{aligned} &D_x^m D_t^n \exp(p_1 x + q_1 t) \cdot \exp(p_2 x + q_2 t) \\ &\quad = (p_1 - p_2)^m (q_1 - q_2)^n \exp((p_1 + p_2)x + (q_1 + q_2)t). \end{aligned}$$

5.
$$P(D_x, D_t)\exp(p_1 x + q_1 t)\cdot\exp(p_2 x + q_2 t) = P(p_1 - p_2, q_1 - q_2)\exp((p_1 + p_2)x + (q_1 + q_2)t),$$
where $P(D_x, D_t)$ is a polynomial in D_x and D_t.

6.
$$\frac{\partial^2}{\partial x \partial t}\ln(f) = \frac{D_x D_t f \cdot f}{f^2}.$$

7.
$$\frac{\partial}{\partial x}\left(\frac{g}{f}\right) = \frac{D_x g \cdot f}{f^2}.$$

8.
$$\frac{\partial^2}{\partial x \partial t}\left(\frac{g}{f}\right) = \frac{D_x D_t g \cdot f}{f^2} - \left(\frac{g}{f}\right)\frac{D_x D_t f \cdot f}{f^2}.$$

Returning to the KdV equation to find the multi-soliton solution, we expand f as a power series in a parameter ε,

$$f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \cdots . \tag{2.17}$$

Substituting this in equation (2.16) and equating powers of ε gives

$$2f_{1xt} + 2f_{1xxxx} = 0, \tag{2.18}$$

$$2f_{2xt} + 2f_{2xxxx} = -D_x(D_t + D_x^3)f_1 \cdot f_1, \tag{2.19}$$

$$2f_{3xt} + 2f_{3xxxx} = -D_x(D_t + D_x^3)(f_1 \cdot f_2 + f_2 \cdot f_1) \tag{2.20}$$

and
$$2f_{4xt} + 2f_{4xxxx} = -D_x(D_t + D_x^3)(f_1 \cdot f_3 + f_2 \cdot f_2 + f_3 \cdot f_1) \tag{2.21}$$

etc.

Choosing

$$f_1 = \sum_{i=1}^{N} \exp\theta_i, \tag{2.22}$$

where

$$\theta_i = p_i(x - v_i t - x_{0i}), \tag{2.23}$$

we find from equation (2.18) that

$$v_i = p_i^2. \tag{2.24}$$

Equation (2.19) now is

$$\begin{aligned} 2f_{2xt} + 2f_{2xxxx} &= -\sum_{i=1}^{N}\sum_{j=1}^{N} D_x(D_t + D_x^3)\exp(\theta_i)\cdot\exp(\theta_j) \\ &= -\sum_{i=1}^{N}\sum_{j=1}^{N}(p_i - p_j)(-(p_i^3 - p_j^3) + (p_i - p_j)^3)\exp(\theta_i + \theta_j) \\ &= 6\sum_{1\le i<j\le N} p_i p_j (p_i - p_j)^2 \exp(\theta_i + \theta_j). \end{aligned} \tag{2.25}$$

So we can choose

$$f_2 = \sum_{1\le i<j\le N} A_{ij}\exp(\theta_i+\theta_j), \tag{2.26}$$

where

$$A_{ij} = \left(\frac{p_i-p_j}{p_i+p_j}\right)^2. \tag{2.27}$$

Similarly,

$$\begin{aligned} 2f_{3xt} + 2f_{3xxxx} &= -\sum_{1\le i<j\le N}\sum_{k=1}^{N} A_{ij}D_x(D_t+D_x^3)(\exp\theta_k\cdot\exp(\theta_i+\theta_j)+\exp(\theta_i+\theta_j)\cdot\exp\theta_k) \\ &= 6\sum_{1\le i<j<k\le N} \frac{(p_i+p_j+p_k)(p_i-p_j)^2(p_j-p_k)^2(p_k-p_i)^2}{(p_i+p_j)(p_j+p_k)(p_k+p_i)}\exp(\theta_i+\theta_j+\theta_k). \end{aligned} \tag{2.28}$$

So we can choose

$$\begin{aligned} f_3 &= \sum_{1\le i<j<k\le N} \frac{(p_i-p_j)^2(p_j-p_k)^2(p_k-p_i)^2}{(p_i+p_j)^2(p_j+p_k)^2(p_k+p_i)^2}\exp(\theta_i+\theta_j+\theta_k) \\ &= \sum_{1\le i<j<k\le N} A_{ij}A_{jk}A_{ki}\exp(\theta_i+\theta_j+\theta_k). \end{aligned} \tag{2.29}$$

Similarly again we can choose

$$f_4 = \sum_{1\le i<j<k<l\le N} A_{ij}A_{jk}A_{ik}A_{jk}A_{jl}A_{kl}\exp(\theta_i+\theta_j+\theta_k+\theta_l). \tag{2.30}$$

And so on. Clearly, this must terminate when we reach f_N.

We can either put $\varepsilon = 1$ or absorb it into the x_{0i}'s. Either way we get

$$f = \sum_{m=0}^{N}\sum_{{}_NC_m} A_{i_1i_2\cdots i_m}\exp(\theta_{i_1}+\theta_{i_2}+\cdots+\theta_{i_m}), \tag{2.31}$$

where the inner sum is over all combinations $i_1, i_2, \ldots, i_m$ chosen from $1, 2, \ldots, N$ and

$$A_{i_1i_2\cdots i_m} = \prod_{1\le j<k\le m}^{m} A_{i_ji_k}. \tag{2.32}$$

This can also be written in the form of a determinant

$$f = \det\begin{vmatrix} 1+\exp(\theta_1) & B_{12}\exp(\theta_2) & B_{13}\exp(\theta_3) & \cdots & B_{1N}\exp(\theta_N) \\ B_{21}\exp(\theta_1) & 1+\exp(\theta_2) & B_{23}\exp(\theta_3) & \cdots & B_{2N}\exp(\theta_N) \\ B_{31}\exp(\theta_1) & B_{32}\exp(\theta_2) & 1+\exp(\theta_3) & \cdots & B_{3N}\exp(\theta_N) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{N1}\exp(\theta_1) & B_{N2}\exp(\theta_2) & B_{N3}\exp(\theta_3) & \cdots & 1+\exp(\theta_N) \end{vmatrix}, \tag{2.33}$$

where

$$B_{ij} = \frac{2\sqrt{p_i p_j}}{p_i + p_j}. \tag{2.34}$$

This solution can be used to show that KdV solitons have the property that distinguishes solitons from mere 'solitary waves', i.e. that they collide elastically. We can expect to find a lone soliton in a region in the xt-plane where x and t are such that one of the θ's is close to zero and all the others are either large and positive or large and negative. We can relabel the θ's so that $\theta_j \ll 0$ for $j = 1, 2, \ldots, i-1$, $\theta_i \approx 0$ and $\theta_j \gg 0$ for $j = i+1, i+2, \ldots, N$. Then the two largest terms in equation (2.31) are the ones for which the exponential contains all the θ_j for $j > i$ and none of those for $j < i$. Neglecting all the other terms

$$\begin{aligned} f &\approx A_{i+1,i+2\cdots N} \exp(\theta_{i+1} + \theta_{i+2} + \cdots + \theta_N) \\ &\quad + A_{i,i+1,i+2\cdots N} \exp(\theta_i + \theta_{i+1} + \theta_{i+2} + \cdots + \theta_N) \\ &= A_{i+1,i+2\cdots N} \exp(\theta_{i+1} + \theta_{i+N} + \cdots + \theta_N) \left(1 + \prod_{j=i+1}^{N} A_{ij} \exp(\theta_i)\right). \end{aligned} \tag{2.35}$$

Thus

$$\begin{aligned} u &= 2\frac{\partial^2}{\partial x^2} \ln(f) \\ &\approx 2\frac{\partial^2}{\partial x^2} \left(\ln(A_{i+1,i+2\cdots N}) + \sum_{j=i+1}^{N} \theta_j + \ln\left(1 + \prod_{j=i+1}^{N} A_{ij} \exp(\theta_i)\right)\right) \\ &= \frac{1}{2} \operatorname{sech}^2 \left(\frac{1}{2} \left(\theta_i + \sum_{j=i+1}^{N} \eta_{ij} \right)\right), \end{aligned} \tag{2.36}$$

where

$$A_{ij} = \exp \eta_{ij}. \tag{2.37}$$

This is the one-soliton solution with its centre shifted by an amount

$$\frac{1}{2p_i} \sum_{j=i+1}^{N} \eta_{ij}. \tag{2.38}$$

If this soliton is involved in a collision some of the θ_j's change sign and so some η_{ij}'s enter or leave the sum and, when the ith soliton is again far from any others, the amount of the shift will have changed accordingly. That is, the effect of a collision is to move the ith soliton by a fixed amount for each of the other solitons involved. This is true even in a complicated collision of several solitons.

In figure 1, we see a two-soliton collision where the 'camera' moves along at the speed of a soliton that is overtaken by a larger, faster soliton. The shift owing to the collision is clearly shown. One interpretation, which figure 1 suggests, is that the two solitons are particles that repel one another. They collide elastically, exchanging momentum and speed. Since the height is proportional to the speed this is exchanged too.

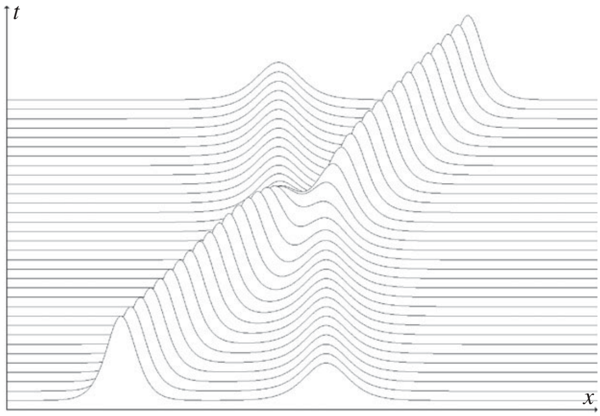

Figure 1. A two-soliton collision.

3. The reduced Maxwell–Bloch equations

In the early 1970s, I and the rest of Robin's group were interested in a set of equations called the *reduced Maxwell–Bloch* (RMB) equations [6]. I would not bore you with the details of the physics but they describe the propagation of an (optical) electromagnetic wave through a medium of two-level atoms in a low-density approximation. In dimensionless form, they can be written as

$$E_x(x,t) + E_t(x,t) = \alpha s(x,t), \tag{3.1}$$

$$r_t(x,t) = -\omega_s s(x,t), \tag{3.2}$$

$$s_t(x,t) = \omega_s r(x,t) + E(x,t)u(x,t) \tag{3.3}$$

and

$$u_t(x,t) = -E(x,t)s(x,t), \tag{3.4}$$

where $E(x,t)$ is the electric field, $r(x,t)$ is the microscopic polarization and $u(x,t)$ is the atomic inversion. The dimensionless constant α is proportional to the atomic density and the atomic dipole matrix element and ω_s is proportional to the atomic resonant frequency. The boundary conditions are $E, r, s \to 0$ and $u \to -1$ as $x \to \pm\infty$ (all the atoms are in their ground state). It follows that the constant of integration $r^2 + s^2 + u^2$ is unity.

If $\omega_s = 0$ these reduce to the *self-induced transparency* (SIT) equations,

$$E_x(x,t) + E_t(x,t) = \alpha s(x,t) \tag{3.5}$$

$$s_t(x,t) = E(x,t)u(x,t) \tag{3.6}$$

and

$$u_t(x,t) = -E(x,t)s(x,t). \tag{3.7}$$

These apply to a similar phenomenon to the RMB equations but $E(x,t)$ now describes the envelope of the optical pulse not the electric field itself. The constant of integration is now $s^2 + u^2$ and is still unity.

Putting $s = -\sin\sigma$ and $u = -\cos\sigma$, we find that $E = \sigma_t$ and

$$\sigma_{xt} + \sigma_{tt} = -\alpha \sin(\sigma). \tag{3.8}$$

Changing the independent variables to

$$x' = \sqrt{\alpha}(t - 2x) \quad \text{and} \quad t' = \sqrt{\alpha}t \tag{3.9}$$

and dropping the primes gives the *sine-Gordon* (SG) equation,

$$\sigma_{xx} - \sigma_{tt} = \sin(\sigma). \tag{3.10}$$

The one-soliton solution of the RMB equations, found by assuming that E is a function of $(\omega t - \kappa x)$ to reduce them to ordinary differential equations, is

$$E = E_0 \operatorname{sech}(\theta), \tag{3.11}$$

$$r = \frac{4E_0\omega_s}{E_0^2 + 4\omega_s^2} \operatorname{sech}(\theta), \tag{3.12}$$

$$s = \frac{2E_0^2}{E_0^2 + 4\omega_s^2} \operatorname{sech}(\theta) \tanh(\theta) \tag{3.13}$$

and

$$u = \frac{2E_0^2}{E_0^2 + 4\omega_s^2} \operatorname{sech}^2(\theta) - 1, \tag{3.14}$$

where

$$\theta = \omega t - \kappa x + \delta \tag{3.15}$$

and

$$\omega = \frac{1}{2}E_0 \quad \text{and} \quad \kappa = \frac{2\alpha E_0}{E_0^2 + 4\omega_s^2} + \frac{1}{2}E_0. \tag{3.16}$$

Comparison with the one-soliton solution of the KdV equation suggests that $u + 1$ should be written as a second derivative of a logarithm and, to get the appropriate coefficients, this must be

$$u = -\frac{2}{\alpha}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\ln(f) - 1 = -\frac{2}{\alpha}\frac{D_t(D_x + D_t)f \cdot f}{f^2} - 1, \tag{3.17}$$

where

$$f = 1 + \exp(2\theta). \tag{3.18}$$

Alternatively, we can use

$$f = \frac{1}{E_1}\cosh(\theta). \tag{3.19}$$

Hopefully, we can obtain the multi-soliton by choosing a more general form for f.

From the first and last of the RMB equations we find that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)E^2 = 2\alpha E s = -2\alpha u_{t,} \tag{3.20}$$

so, using the boundary conditions, we find

$$E^2 = 4\frac{\partial^2}{\partial t^2}\ln(f). \tag{3.21}$$

We also put

$$E = 2\frac{g}{f}, \tag{3.22}$$

which gives immediately

$$2g^2 = D_t^2 f \cdot f. \tag{3.23}$$

And finally we put

$$r = 2\frac{h}{f}. \tag{3.24}$$

Substituting these into the first two RMB equations gives s in terms of f with either g or h,

$$s = \frac{2}{\alpha}\frac{(D_x + D_t)g \cdot f}{f^2} = -\frac{2}{\omega_s}\frac{D_t h \cdot f}{f^2}, \tag{3.25}$$

together with the bilinear equation

$$\omega_s(D_x + D_t)g \cdot f = -\alpha D_t h \cdot f. \tag{3.26}$$

Finally, the third RMB equation gives

$$\alpha(\omega_s h - g)f = D_t(D_x + D_t)g \cdot f. \tag{3.27}$$

We have already taken care of the fourth RMB equation.

Collecting all these together, we have E, r, s and u expressed in terms of f, g and h,

$$E = 2\frac{g}{f}, \tag{3.28}$$

$$r = 2\frac{h}{f}, \tag{3.29}$$

$$s = \frac{2}{\alpha}\frac{(D_x + D_t)g \cdot f}{f^2} \tag{3.30}$$

and

$$u = -\frac{2}{\alpha}\frac{D_t(D_x + D_t)f \cdot f}{f^2} - 1, \tag{3.31}$$

and the system of bilinear equations,

$$2g^2 = D_t^2 f \cdot f, \tag{3.32}$$

$$\omega_s(D_x + D_t)g \cdot f = -\alpha D_t h \cdot f \tag{3.33}$$

and

$$\alpha(\omega_s h - g)f = D_t(D_x + D_t)g \cdot f. \tag{3.34}$$

For the one-soliton solution we already have

$$f = \frac{1}{E_1}\cosh(\theta_1), \tag{3.35}$$

and it easily follows that

$$g = \frac{\omega}{E_1} \tag{3.36}$$

and

$$h = \frac{\omega_s(\kappa - \omega)}{\alpha E_1}. \tag{3.37}$$

At this time we were fairly familiar with the sort of functions which turned up and by looking at, among other things, the two-soliton solution John Gibbon eventually arrived at the following guess for the N-soliton solution:

$$f = \det|M_{ij}|, \tag{3.38}$$

where M_{ij} is an $N \times N$ matrix with ijth element

$$M_{ij} = \frac{2}{E_i + E_j} \cosh \frac{1}{2}(\theta_i + \theta_j), \tag{3.39}$$

$$\theta_i = \omega_i t - \kappa_i x + \delta_i \tag{3.40}$$

and

$$\omega_i = \frac{1}{2} E_i \quad \text{and} \quad \kappa_i = \frac{2\alpha E_i}{E_i^2 + 4\omega_s^2} + \frac{1}{2} E_i. \tag{3.41}$$

Chris Eilbeck provided further confirmation of this by comparing this guess for three solitons with a numerical solution.

Expanding the determinant

$$\begin{aligned} f = \sum_{m=0}^{N} \sum_{{}_N C_m(i)} \sum_{{}_N C_m(j)} & A_{i_1 i_2 \cdots i_m; j_1 j_2 \cdots j_m} \\ & \times \exp \frac{1}{2}(\theta_{i_1} + \theta_{i_2} + \cdots + \theta_{i_m} - \theta_{i_{m+1}} - \theta_{i_{m+2}} - \cdots - \theta_{i_N} \\ & \qquad + \theta_{j_1} + \theta_{j_2} + \cdots + \theta_{j_m} - \theta_{j_{m+1}} - \theta_{j_{m+2}} - \cdots - \theta_{j_N}), \end{aligned} \tag{3.42}$$

where the summations over ${}_N C_m(i)$ and ${}_N C_m(j)$ are over all combinations $i_1, i_2, \ldots, i_m$ and $j_1, j_2, \ldots, j_m$ chosen from $1, 2, \ldots, N$.

The coefficients (given in a more general form for later use) are

$$\begin{aligned} A_{i_1 i_2 \cdots i_m; j_1 j_2 \cdots j_n} = \epsilon(i_1 i_2 \cdots i_N)\epsilon(j_1 j_2 \cdots j_N) & \frac{\prod_{1\le k<l\le m}(E_{i_k} - E_{i_l}) \prod_{1\le k<l\le n}(E_{j_k} - E_{j_l})}{\prod_{k=1}^{m} \prod_{l=1}^{n}(E_{i_k} + E_{j_l})} \\ \times & \frac{\prod_{m<k<l\le N}(E_{i_k} - E_{i_l}) \prod_{n<k<l\le N}(E_{j_k} - E_{j_l})}{\prod_{k=m+1}^{N} \prod_{l=n+1}^{N}(E_{i_k} + E_{j_l})}, \end{aligned} \tag{3.43}$$

where $\epsilon(i_1 i_2 \cdots i_N)$ is $+1$ if the parity of $i_1 i_2 \cdots i_N$ is positive and -1 if it is negative.

Showing that this truly was the n-soliton solution involved a lot of tedious manipulation (this was before algebraic manipulation programs like Maple and Maxima), which is given in Caudrey *et al.* [6]. Just to show how much fun we had in those days I give you the appropriate forms of g and h,

$$\begin{aligned} g = \tfrac{1}{2} \sum_{m=0}^{N-1} \sum_{{}_N C_m(i)} \sum_{{}_N C_{m+1}(j)} & (-1)^m A_{i_1 i_2 \cdots i_m; j_1 j_2 \cdots j_{m+1}} \\ & \times \exp \tfrac{1}{2}(\theta_{i_1} + \theta_{i_2} + \cdots + \theta_{i_m} - \theta_{i_{m+1}} - \theta_{i_{m+2}} - \cdots - \theta_{i_N} \\ & \qquad + \theta_{j_1} + \theta_{j_2} + \cdots + \theta_{j_{m+1}} - \theta_{j_{m+2}} - \theta_{j_{m+3}} - \cdots - \theta_{j_N}) \end{aligned} \tag{3.44}$$

and

$$h = \omega_s \sum_{m=0}^{N-1} \sum_{{}_N C_m(i)} \sum_{{}_N C_{m+1}(j)} (-1)^m B_{i_1 i_2 \cdots i_m; j_1 j_2 \cdots j_{m+1}}$$
$$\times \exp \tfrac{1}{2}(\theta_{i_1} + \theta_{i_2} + \cdots + \theta_{i_m} - \theta_{i_{m+1}} - \theta_{i_{m+2}} - \cdots - \theta_{i_N}$$
$$+ \theta_{j_1} + \theta_{j_2} + \cdots + \theta_{j_{m+1}} - \theta_{j_{m+2}} - \theta_{j_{m+3}} - \cdots - \theta_{j_N}), \qquad (3.45)$$

where

$$B_{i_1 i_2 \cdots i_m; j_1 j_2 \cdots j_n} = A_{i_1 i_2 \cdots i_m; j_1 j_2 \cdots j_n}(-\lambda_{i_1} - \lambda_{i_2} - \cdots - \lambda_{i_m} + \lambda_{i_{m+1}} + \lambda_{i_{m+2}}$$
$$+ \cdots + \lambda_{i_N} + \lambda_{j_1} + \lambda_{j_2} + \cdots + \lambda_{j_n} - \lambda_{j_{n+1}} - \lambda_{j_{n+2}} - \cdots - \lambda_{j_N}) \qquad (3.46)$$

with

$$\lambda_i = \frac{1}{E_i^2 + 4\omega_s^2}. \qquad (3.47)$$

This solution also displays the soliton collision property. As for the KdV equation, we examine a region in the xt-plane where x and t are such that one of the θ's is close to zero and all the others are either large and positive or large and negative and relabel the θ's as before so that $\theta_j \ll 0$ for $j = 1, 2, \ldots, i-1$, $\theta_i \approx 0$ and $\theta_j \gg 0$ for $j = i+1, i+2, \ldots, N$. Neglecting all but the two largest terms gives

$$f \approx A_{i,i+1,\ldots,N;i,i+1,\ldots,N} \exp(2(\theta_i + \theta_{i+1} + \cdots + \theta_N))$$
$$+ A_{i+1,i+2,\ldots,N;i+1,i+2,\ldots,N} \exp(2(\theta_{i+1} + \theta_{i+2} + \cdots + \theta_N))$$
$$= A_{i+1,i+2,\ldots,N;i+1,i+2,\ldots,N} \exp(2(\theta_{i+1} + \theta_{i+2} + \cdots + \theta_n))$$
$$\times \left(1 + \frac{A_{i,i+1,\ldots,N;i,i+1,\ldots,N}}{A_{i+1,i+2,\ldots,N;i+1,i+2,\ldots,N}} \exp(2\theta_i)\right). \qquad (3.48)$$

We can find that

$$\frac{A_{i,i+1,\cdots N;i,i+1,\cdots N}}{A_{i+1,i+2,\cdots N;i+1,i+2,\cdots N}} = \prod_{j=1}^{i-1} \left(\frac{E_i + E_j}{E_i - E_j}\right)^2 \prod_{j=i+1}^{N} \left(\frac{E_i - E_j}{E_i + E_j}\right)^2$$
$$= \exp\left(2\left(\sum_{j=1}^{i-1} \eta_{ij} - \sum_{j=i+1}^{N} \eta_{ij}\right)\right) \qquad (3.49)$$

with

$$\eta_{ij} = \ln\left|\frac{E_i + E_j}{E_i - E_j}\right|. \qquad (3.50)$$

Thus

$$f = A_{1,2,\ldots,i-1;1,2,\ldots,i-1} \exp(2(\theta_1 + \theta_2 + \cdots + \theta_{i-1}))$$
$$\times \left(1 + \exp\left(2\left(\theta_i + \sum_{j=1}^{i-1} \eta_{ij} - \sum_{j=i+1}^{N} \eta_{ij}\right)\right)\right) \qquad (3.51)$$

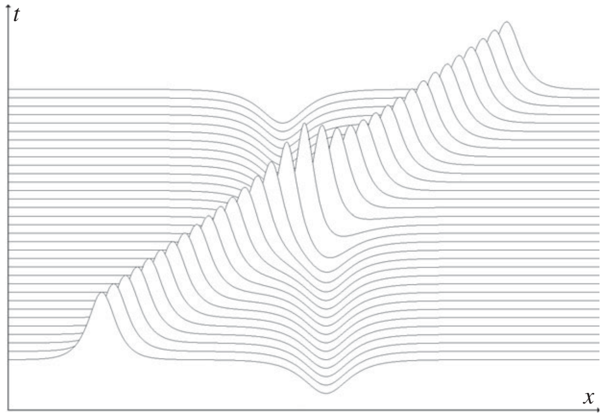

Figure 2. A soliton–anti-soliton collision.

and

$$E = E_i \operatorname{sech}\left(\theta_i + \sum_{j=1}^{i-1} \eta_{ij} - \sum_{j=i+1}^{N} \eta_{ij}\right), \tag{3.52}$$

which is, again, the one-soliton solution with a shift. In this case, during a collision involving both the ith and jth solitons, the sign in front of η_{ij} changes, altering the amount of the shift (for both of them).

Besides solitons, the RMB equations also exhibit anti-solitons. These can collide with each other and with solitons.

Figure 2 shows a soliton–anti-soliton collision with the camera moving at the speed of the anti-soliton. The shift in the anti-soliton is plain and can be interpreted as if the soliton and anti-soliton attract one another.

If particles attract, the possibility of bound states arises. These indeed exist and can be found by making two of the E_n's in the n-soliton formula complex conjugates of one another. Figure 3 shows such a bound state, and its appearance resulted in it being given the name 'breather'.

If the imaginary part of a complex conjugate pair of E_i's is very large compared with the real part, an isolated breather has the approximate form

$$E \approx 2E_R \operatorname{sech}(\theta_R)\cos(\theta_I), \tag{3.53}$$

where

$$\theta_R = \frac{1}{2}E_R\left(t - \left(\frac{4\alpha}{E_R^2 + E_I^2} + 1\right)x\right) \tag{3.54}$$

and

$$\theta_I = \frac{1}{2}E_I\left(t + \left(\frac{4\alpha}{E_R^2 + E_I^2} - 1\right)x\right), \tag{3.55}$$

which is a hyperbolic secant-shaped envelope modulating a carrier wave. This is the solution to the RMB equations, which corresponds (physically not mathematically) to the one-soliton solution of the SIT equation.

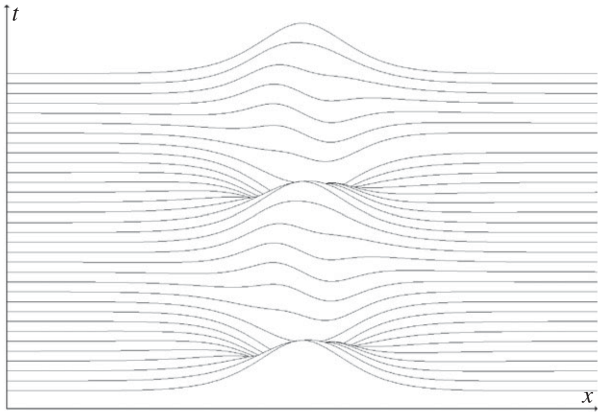

Figure 3. A bound state, the appearance of which resulted in it being called a 'breather'.

Putting $\omega_s = 0$ and carrying out the variable transformation mentioned earlier give the N-soliton solution to the SG equation

$$\sigma_{xx} - \sigma_{tt} = \sin\sigma, \tag{3.56}$$

$$\cos\sigma = 1 - 2\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)\ln(f) \tag{3.57}$$

and

$$f = \det|M_{ij}|, \tag{3.58}$$

where M_{ij} is an $N \times N$ matrix with ijth element

$$M_{ij} = \frac{2}{a_i + a_j}\cosh\frac{1}{2}(\theta_i + \theta_j) \tag{3.59}$$

and

$$\theta_i = \frac{1}{2}\left(a_i + \frac{1}{a_i}\right)x + \frac{1}{2}\left(a_i - \frac{1}{a_i}\right)t + \delta_i. \tag{3.60}$$

References

1 Gardner, C. S., Greene, J. M., Kruskal, M. D. & Miura, R. M. 1967 Method for solving the KdV equation. *Phys. Rev. Lett.* **19**, 1095–1097. (doi:10.1103/PhysRevLett.19.1095)

2 Hirota, R. 1971 Exact solution of the KdV equation for multiple collision of solitons. *Phys. Rev. Lett.* **27**, 1192–1194. (doi:10.1103/PhysRevLett.27.1192)

3 Hirota, R. 1972 Exact solution of the sine-Gordon equation for multiple collision of solitons. *J. Phys. Soc. Jpn* **33**, 1459–1463. (doi:10.1143/JPSJ.33.1459)

4 Caudrey, P. J., Eilbeck, J. C., Gibbon, J. D. & Bullough, R. K. 1973 Multiple soliton and bisoliton bound state solutions of the sine-Gordon equation and related equations in nonlinear optics. *J. Phys. A: Math. Nucl. Gen.* **6**, L112–L115. (doi:10.1088/0305-4470/6/8/002)

5 Hirota, R. 1980 Direct methods in soliton theory. In *Solitons* (eds R. K. Bullough & P. J. Caudrey), pp. 157–176. Berlin, Germany: Springer.

6 Caudrey, P. J., Eilbeck, J. C. & Gibbon, J. D. 1974 Exact multisoliton solution of the reduced Maxwell-Bloch equations of nonlinear optics. *J. Appl. Math.* **14**, 375–386.