HYBRID APPROACH COMPARED TO GRADIENT METHOD FOR OPTIMAL CONTROL PROBLEM OF HYDRAULIC SYSTEM

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ABSTRACT

In this paper, hybrid approach includes meta-heuristic and conventional method, for optimal control problem of switched systems is proposed. The maximum principal is used to find an optimal continuous input and Particle Swarm Optimization (PSO) to find optimal switching instants minimizing a performance function depends on these instants, in a finite horizon. We demonstrate via hydraulic application the effectiveness of proposed approach where the results are compared to those obtained by gradient method.

Keywords: Nonlinear switched systems, optimal control, switching instants, Particle Swarm Optimization (PSO), gradient method.

1. INTRODUCTION

A switched system is a system that consists of several subsystems and a switching law indicating the active subsystem at each time instant [1]. For an optimal control problem of switched systems, we need to find not only an optimal continuous input but also an optimal switching sequence since the system dynamics vary before and after every switching instant. Several researchers have studied this framework such as Piccoli [2], Lincoln and Bernhardsson [3] Bemporad et al. [4].

In particular, Xu and Antsaklis [1, 5-7], formulate the problem in two stages optimization problem. Stage (a), deals a conventional optimal control problem that finds the optimal cost giving the sequence of active subsystems and the switching instants. Stage (b), deals a nonlinear optimization problem that finds the local optimal switching instants based on gradient method. The conceptual algorithm applied for this optimization methodology, allows at each iteration to find a value of cost \( J \) by solving an optimal control problem (stage (a)), find \( \frac{\partial J}{\partial t} \) and \( \frac{\partial^2 J}{\partial t^2} \) with respect to the switching instants and use the gradient projection method to update switching instant. In fact, for several number of switching instants, the calculation of these derivatives becomes more difficult. Also, gradient method requires much regularity about function to optimize and doesn't necessarily guarantee global optimum solution.

In order to overcome these disadvantages, we changed, only in stage (b), the gradient method by a meta-heuristic algorithm such as Particle Swarms Optimization (PSO) algorithm. So, our approach includes conventional method in stage (a) and meta-heuristic in stage (b) which we call hybrid approach.

In the sequel, an optimal control problem for switched systems is formulated in section 2. Section 3 describes the hybrid approach and presents it implementation for optimal control problem. In the following, we demonstrate via hydraulic application, the effectiveness of the proposed approach compared to gradient method.

2. PROBLEM FORMULATION

2.1. Hybrid System

Considering a switched system described by the following subsystems [7]:

\[
\dot{x} = f_i(x,u), i \in I = \{1,2,\ldots,M\}
\]

(1)

with \( f_i \) is an element of a set of indexed field vectors and \( I \) is a set of finite discrete variables that indicate that the system will be in \( M \) configurations.

2.2. Switching Sequences

For a switched system (1), the inputs of the system consist of both a continuous input \( u \), \( t \in [t_a,t_f] \), and a switching sequence. We define a switching sequence as follows [7]:
\[ \sigma = \left( (t_0, i_0), (t_1, i_1), \ldots, (t_k, i_k) \right) \]  

(2)

with \( 0 \leq K < \infty \), \( t_0 \leq t_1 \leq \ldots \leq t_k \leq t_f \), and \( i_k \in I \) for \( k = 0, 1, \ldots, K \). A switching sequence \( \sigma \) as defined above indicates that, if \( t_k < t_{k+1} \), then subsystem \( i_k \) is active in \([t_k, t_{k+1})\) \( (t_k, t_f] \) if \( k = K \); the system switches from subsystem \( i_k \) to subsystem \( i_{k+1} \).

2.3. Optimal Control Problem

Consider a switched system as described by (1), given an initial continuous state \( x_0 \), a fixed time interval \([t_0, t_f]\) and a prespecified sequence of active subsystems, find a continuous input \( u \) and switching instants \( t_1, \ldots, t_k \) minimizing the cost functional given as follows:

\[
J = \psi\left(x(t_f)\right) + \int_{t_0}^{t_f} L(x(t), u(t)) \, dt
\]

(3)

where \( J_{\text{final}} \) is a component characterizes the contribution of terminal conditions and \( J_{\text{continuous}} \) is component characterizes the effect of temporal accumulation due to continuous evolution of system.

3. HYBRID APPROACH FOR OPTIMAL CONTROL PROBLEM

Two stages can be proposed to solve the optimal control problem of switched systems referring to works of Xu and Antsaklis in 2000, 2002, 2003 and 2004 [1, 5-7].

3.1. Stage (a)

Stage (a) deals with conventional optimal control problem which seeks an optimal continuous control input \( u \). We apply the maximum principal, where an vector function \( p(t) = [p_1(t), \ldots, p_n(t)]^\top \) exists such that the following condition hold:

(i) for almost any \( t \in [t_0, t_f] \), the following state and costate equations hold:

\[
\frac{dx(t)}{dt} = \left( \frac{\partial H}{\partial p}(x(t), p(t), u(t)) \right)^\top = f_i(x(t), u(t))
\]

(4)

\[
\frac{dp(t)}{dt} = -\left( \frac{\partial H}{\partial x}(x(t), p(t), u(t)) \right)^\top = -\left( \frac{\partial f_i}{\partial x} \right)^\top p - \left( \frac{\partial L_i}{\partial x} \right)^\top
\]

(5)

with

\[
H_i(x, p, u) = L(x, u) + p^\top f_i(x, u)
\]

(6)

(ii) for almost any \( t \in [t_k, t_{k+1}] \), the stationary condition hold:

\[
0 = \left( \frac{\partial H}{\partial u}(x(t), p(t), u(t)) \right)^\top = \left( \frac{\partial f_i}{\partial u} \right)^\top p - \left( \frac{\partial L_i}{\partial u} \right)^\top
\]

(7)

(iii) at \( t_i \) the function \( p \) satisfies the boundary condition:

\[
p(t_i) = \left( \frac{\partial \psi}{\partial x}(x(t_i)) \right)^\top
\]

(8)

(iv) at any \( t_k, k = 1, 2, \ldots, K \), we have continuity condition:

\[
p(t_{k+1}) = p(t_k)
\]

(9)

However, the differentials algebraic equations can be solved efficiently by using the function \texttt{bvp4c} of Matlab.
3.2. Stage (b)
Stage (b) deals with non linear optimization problem given by the equation (10) which seeks an optimal switching instants.

$$\tilde{J}(\tilde{t}) = \min J$$

with $\tilde{t} \in T = \{ \tilde{t} = (t_1, ..., t_K)^T | t_0 \leq t_i \leq ... \leq t_K \leq t_f \}$

For optimal switching instants $\tilde{t}_1, ..., \tilde{t}_K$, we use the PSO algorithm. This meta-heuristic, is an evolutionary optimization techniques [8], invented by Russel Eberhart and James Kennedy [9, 10]. The basic PSO is developed from research on swarm such as fish schooling and bird flocking [11]. The population is initialized randomly and the potential solutions, named particles, freely fly across the multidimensional search space [12]. Each particle $s$ is characterized by a position $p_s$ and velocity $v_s$. During flight, each particle updates its own velocity and position by taking benefit from its best experience and the best experience of the entire population.

Let be the $k$ iteration index, the new particle velocity and position are updated according to the move equations [13, 14] given as follow:

$$v_{k+1}^s = \left( w_k v_k^s + (c_1 r_1 (p_{best_k}^s - p_k^s) + c_2 r_2 (p_{gbest_k}^s - p_k^s)) \right)$$

$$p_{k+1}^s = p_k^s + v_{k+1}^s$$

with:
- $p_k^s$: position of each particle $s$ at iteration $k$,
- $v_k^s$: velocity of each particle $s$ at iteration $k$,
- $p_{best_k}^s$: best position discovered by the particle until the iteration $k$,
- $p_{gbest_k}^s$: global best particle position of the entire population.

$c_1$ and $c_2$ are strength of attraction constants and $r_1$ and $r_2$ are uniformly distributed random numbers in $[0,1]$. Inertia weight, $w_k$, is a parameter used to control the impact of the previous velocities on the current velocity. It influences the trade-off between the global and local exploitation abilities of the particles. For initial stages of the search process, large inertia weight to enhance the global exploitation is recommended while for last stages, the inertia weight is reduced for better local exploration. Weight is updated as:

$$w_k = w_{max} - \left( \frac{w_{max} - w_{min}}{k_{max}} \right) k$$

where $w_{min}$, $w_{max}$ and $k_{max}$ are minimum, maximum values of $w_k$ and pre-specified maximum number of iteration cycles, respectively.

A schematic representation of particle flight within the search space is given in figure 1.
In order to have optimal switching instants, we consider that position of particle \( p \) represents a switching sequence. By setting the search space \([t_0, t_f]\), the switching number of system \( n_i \) and the swarm size \( Sw \), the optimization algorithm calculates the numerical value of the cost function at each sequence and selects the one with the minimum cost. In the next, we present the optimization algorithm.

### 3.2. Proposed Optimization Algorithm

Step 1:
- define the parameters of the algorithm: \([t_0, t_f]\), \( Sw \), \( n_i \), \( c_1 \), \( c_2 \), \( w_{\text{min}} \), \( w_{\text{max}} \) and \( k_{\text{max}} \),
- generate randomly \( p_{(i_{\text{random})}} \) and \( v_{(i_{\text{random})}} \).

Step 2:
- by solving an optimal control problem (stage(a)), evaluate each particle by the cost function \( J \),
- identify the switching instants corresponding to the best fitness \( J(i^{'}) \),
- affect the best sequence at \( p_{\text{best}}^{'} \),
- affect \( p_{\text{best}}^{'} \) at \( pg_{\text{best}}^{'} \),
- initialize the iterations index \( k \).

Step 3:
- update \( v_{i}^{'} \) and \( p_{i}^{'} \) by the move equations (11) and (12),
- evaluate each particle by the cost function \( J \) by solving stage (a),
- find the switching instants corresponding to the best fitness and affect it to \( p_{\text{best}}^{'} \),
- compare initial \( pg_{\text{best}}^{'} \) to \( p_{\text{best}}^{'} \),
- if \( p_{\text{best}}^{'} < pg_{\text{best}}^{'} \) then \( pg_{\text{best}}^{'} \leftarrow p_{\text{best}}^{'} \).

Step 4:
- increment \( k \),
- test the fluctuation of the cost value if it tends to \( \varepsilon (\varepsilon \to 0) \) then return to step 3.
else go to step 5,
Step 5:
- give the best switching instants of the minimum cost.

4. ILLUSTRATIVE EXAMPLES
Consider a hydraulic system consisting of two tanks [15] represented in figure 2.

![Hydraulic System](image)

Figure 2. Hydraulic system.

The studied framework consists in obtaining fluid levels $h_1 = 5m$ and $h_2 = 1m$ in the time interval $[0,30s]$ by controlling the input flow $q_e$ and the switching law applied to the valves $V_1$ and $V_2$. These valves switched between two configurations: closed or fully-open. The model equations of the hydraulic system are given by the following equations:

$$S \frac{dh_1}{dt} = q_e - V_1 k_1 \sqrt{h_1}$$

(14)

$$S \frac{dh_2}{dt} = V_1 k_1 \sqrt{h_1} - V_2 k_2 \sqrt{h_2}$$

(15)

where $S$ is the tank section, $k_1$ and $k_2$ are respectively the hydraulic resistances. Their corresponding numerical values are given as follow:

$$S = 1m^2 \quad k_1 = 0.02m^{5/2}/s \quad k_2 = 0.03m^{5/2}/s$$

(16)

To show the effectiveness of the proposed approach, we present two operations cases of hydraulic application.

The parameters of the proposed algorithm for these cases are fixed to:

$$c_1 = c_2 = 0.75 \quad w_{max} = 0.9 \quad w_{max} = 0.4 \quad S_{W} = 40 \quad k_{max} = 15$$

(17)

4.1. Case 1
Initially, we assume that the valve $V_1$ is closed and $V_2$ is open and the initial levels for two tanks are $h_{m1} = 3m$ and $h_{m2} = 0m$. The valve $V_1$ opens at $t = t_1$, thus the hydraulic system has two dynamics described by the following equations:

- for $t \in [0,t_1)$
\[
\begin{align*}
\frac{dh}{dt} &= \frac{1}{S} q_s, \\
\frac{dh_s}{dt} &= -\frac{1}{S} (k_s \sqrt{h_s}) \\
\end{align*}
\]  
(18)

- for \( t \in [t_1, 30s] \)

\[
\begin{align*}
\frac{dh}{dt} &= \frac{1}{S} (q_s - k_s \sqrt{h_s}), \\
\frac{dh_s}{dt} &= -\frac{1}{S} (k_s \sqrt{h_s} - k_s \sqrt{h_s}) \\
\end{align*}
\]  
(19)

Our goal is to find the optimal control input \( q_s \) and the switching instant \( \hat{t} \) minimizing the cost function given as follows:

\[
J = \frac{1}{2} \left( (h_1 (30) - 5)^2 + (h_2 (30) - 1)^2 \right) + \frac{1}{2} \int_0^t \left( (h_1 (t) - 5)^2 + (h_2 (t) - 1)^2 + u(t)^2 \right) dt
\]  
(20)

For optimal input, by applying the necessary optimality conditions, it comes:

- for \( t \in [0, t_1] \), the hamiltonian equation is:

\[
H_1 (x, p, q_s) = \frac{1}{2} \left( (h_1 - 5)^2 + (h_2 - 1)^2 + u^2 \right) + \frac{1}{S} (q_s p - k_s p \sqrt{h_s})
\]  
(21)

and the differential equations are:

\[
\begin{align*}
\frac{dh}{dt} &= \frac{1}{S} q_s, \\
\frac{dh_s}{dt} &= -\frac{1}{S} (k_s \sqrt{h_s}) \\
\frac{dp}{dt} &= -(h_1 - 5) \\
\frac{dp_s}{dt} &= \frac{k_s}{2S \sqrt{h_s}} p_s - (h_s - 1)
\end{align*}
\]  
(22)

with:

\[
q_s = -\frac{1}{S} p
\]  
(23)

- for \( t \in [t_1, 30s] \), the hamiltonian equation is:

\[
H_2 (x, p, q_s) = \frac{1}{2} \left( (h_1 - 5)^2 + (h_2 - 1)^2 + u^2 \right) + \frac{1}{S} (q_s p - k_s \sqrt{h_s}) p_s + \frac{1}{S} (k_s \sqrt{h_s} - k_s \sqrt{h_s}) p_s
\]  
(24)

and the differential equations are:

\[
\begin{align*}
\frac{dh}{dt} &= \frac{1}{S} (q_s - k_s \sqrt{h_s}), \\
\frac{dh_s}{dt} &= \frac{1}{S} (k_s \sqrt{h_s} - k_s \sqrt{h_s}) \\
\frac{dp}{dt} &= \frac{k_s}{2S \sqrt{h_s}} (p_s - p_s) - (h_1 - 5) \\
\frac{dp_s}{dt} &= \frac{k_s}{2S \sqrt{h_s}} p_s - (h_s - 1)
\end{align*}
\]  
(25)

with:
\[
q_\epsilon = -\frac{1}{S}p_i
\]

- the boundary conditions are:

\[
p_i(30) = h_i(30) - 5 \quad p_s(30) = h_s(30) - 1
\]

For optimal switching instant, the proposed algorithm is used when we obtained, as soon the first iteration (figure 3), the optimal switching instant \( t_1^* = 1.5142s \) corresponding to the optimal cost \( \hat{J} = 4.1285 \). The optimal input \( q_\epsilon \) and the fluid levels \( h_1 \) and \( h_2 \) are shown respectively in figure 4, figure 5 and figure 6.

It can be observed in figure 7 that the cost function is nonconvex. However, when we applied the gradient method for a various initial instants, table 1, we note that it ends up with the first solution founded and converges to a global solution only if we have the suitable initial instant.

4.2. Case 2
Initially, we assume that the two valves are closed and the initial levels for two tanks are \( h_0 = 3m \) and \( h_0 = 0.5m \). At \( t = t_1 \), the valve \( V_1 \) opens and \( V_i \) at \( t = t_s \). Each of these discrete switches once at a definite moment, the system then presents three dynamic described, at each interval, given by the following equations:

- for \( t \in [0, t_1) \)

\[
\begin{align*}
\frac{dh_1}{dt} &= \frac{1}{S} q_\epsilon \\
\frac{dh_2}{dt} &= 0
\end{align*}
\]

![Figure 3. Fitness function for case 1.](image-url)
Figure 4. Optimal control input $u$.

Figure 5. Fluid level $h_1$. 
Figure 6. Fluid level $h_2$

![Fluid level graph](image)

Figure 7. Cost function $J(t_1(s))$

![Cost function graph](image)
Table 1. Obtained results by gradient method for case1.

<table>
<thead>
<tr>
<th>Instant initial</th>
<th>Résultats optimaux</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_1 ) (s)</td>
<td>( \hat{t}_1 ) (s)</td>
</tr>
<tr>
<td>23</td>
<td>17.4068</td>
</tr>
<tr>
<td>18</td>
<td>17.04068</td>
</tr>
<tr>
<td>10</td>
<td>17.4068</td>
</tr>
<tr>
<td>2</td>
<td>1.2643</td>
</tr>
<tr>
<td>0.1</td>
<td>0.3367</td>
</tr>
</tbody>
</table>

We conclude that gradient method doesn't guarantee an global solution, for that, we keep the proposed approach.

- for \( t \in [t_1, t_2] \)
  \[
  \begin{cases}
  \frac{dh}{dt} = \frac{1}{S} \left( q_o - k_i \sqrt{h} \right) \\
  \frac{dh_i}{dt} = \frac{1}{S} \left( k_i \sqrt{h} \right)
  \end{cases}
  \tag{29}
  
- for \( t \in [t_2, 30s] \)
  \[
  \begin{cases}
  \frac{dh}{dt} = \frac{1}{S} \left( q_o - k_i \sqrt{h} \right) \\
  \frac{dh_i}{dt} = \frac{1}{S} \left( k_i \sqrt{h} - k_i \sqrt{h} \right)
  \end{cases}
  \tag{30}
  
We seek to find the optimal control input \( q_o \) and the switching instants \( \hat{t}_1 \) and \( \hat{t}_2 \) minimizing the same cost function given as follow:

\[
J = \frac{1}{2} \left( (h_1 (30) - 5)^2 + (h_2 (30) - 1)^2 \right) + \frac{1}{2} \int_0^{30} \left( (h_1 (t) - 5)^2 + (h_2 (t) - 1)^2 + u(t)^2 \right) dt
\tag{31}

Applying the necessary optimality conditions, we obtain:

- for \( t \in [0, t_1] \), the hamiltonian equation is:
  \[
  H_1 (x, p, q_o) = \frac{1}{2} \left( (h_1 - 5)^2 + (h_2 - 1)^2 + u^2 \right) + \frac{1}{S} q_o p_i
  \tag{32}
  
  and the differential equations are:

  \[
  \begin{cases}
  \frac{dh}{dt} = \frac{1}{S} q_o \\
  \frac{dh_i}{dt} = 0 \\
  \frac{dp}{dt} = -(h_i - 5) \\
  \frac{dp_i}{dt} = -(h_i - 1)
  \end{cases}
  \tag{33}
  
  with:
  \[
  q_o = -\frac{1}{S} p_i
  \tag{34}
  
- for \( t \in [t_1, t_2] \), the hamiltonian equation is:
  \[
  H_2 (x, p, q_o) = \frac{1}{2} \left( (h_1 - 5)^2 + (h_2 - 1)^2 + u^2 \right) + \frac{1}{S} (q_o - k_i \sqrt{h}) p_i + \frac{1}{S} (k_i \sqrt{h}) p_2
  \tag{35}
  
  and the differential equations are:
\[
\begin{align*}
\frac{dh}{dt} &= \frac{1}{S} \left( q_n - k_i \sqrt{h} \right), \\
\frac{dh}{dt} &= \frac{1}{S} \left( k_i \sqrt{h} \right), \\
\frac{dp_1}{dt} &= \frac{k_i}{2S \sqrt{h}} \left( p_1 - p_2 \right) - (h_5 - 5), \\
\frac{dp_2}{dt} &= -(h_5 - 1)
\end{align*}
\]

with:
\[ q_n = -\frac{1}{S} p_i \]

- for \( t \in [t_1, 30s] \), the hamiltonian equation is:
\[
H_s(x, p, q_n) = \frac{1}{2} \left( (h_5 - 5)^2 + (h_5 - 1)^2 + u^2 \right) + \frac{1}{S} \left( q_n - k_i \sqrt{h} \right) p_1 + \frac{1}{S} \left( k_i \sqrt{h} - k_i \sqrt{h} \right) p_2
\]

and the differential equations are:
\[
\begin{align*}
\frac{dh}{dt} &= \frac{1}{S} \left( q_n - k_i \sqrt{h} \right), \\
\frac{dh}{dt} &= \frac{1}{S} \left( k_i \sqrt{h} \right), \\
\frac{dp_1}{dt} &= \frac{k_i}{2S \sqrt{h}} \left( p_1 - p_2 \right) - (h_5 - 5), \\
\frac{dp_2}{dt} &= \frac{k_i}{2S \sqrt{h}} p_2 - (h_5 - 1)
\end{align*}
\]

with:
\[ q_n = -\frac{1}{S} p_i \]

- the boundary conditions are:
\[ p_1(30) = h_5(30) - 5 \quad p_2(30) = h_5(30) - 1 \]

By applying the our algorithm and after 5 iteration, figure 8, we obtain the optimal switching instants \( \hat{t}_1 = 2.5602s \) and \( \hat{t}_2 = 8.0592s \) corresponding to the optimal cost \( \hat{J} = 3.0260 \). Gradient method converges to the optimal solution obtained by the proposed approach, as shown in table 2, unless we start by the initial instants belongs to the solution zone.

**Table 2. Obtained results by gradient method for case2.**

<table>
<thead>
<tr>
<th>Instants initiales</th>
<th>Résultats optimaux</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_1(s) )</td>
<td>( t_2(s) )</td>
</tr>
<tr>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>0.5</td>
<td>3.9</td>
</tr>
<tr>
<td>11</td>
<td>18</td>
</tr>
<tr>
<td>2.5</td>
<td>8</td>
</tr>
</tbody>
</table>
The corresponding optimal control input $q_e$ and the fluid levels $h_1$ and $h_2$ are shown respectively in figure 9, figure 10 and figure 11. Figure 12 shows the optimal cost for different switching instants $t_1$ and $t_2$. The optimal solution checked in this plot is a same solution founded by our approach. PSO algorithm can easily be adapted even for several switching instants. In fact, we change the variable $n_c$ and check the active subsystem in his time interval. Contrary for gradient method, this case it's more difficult when we have to find the derivatives of the cost with respect to each switching instants.

5. CONCLUSION
In this paper, we proposed hybrid approach includes a conventional method for optimal continuous input and meta-heuristic such as Particle Swarms Optimization algorithm for switching instants optimization. To shows the effectiveness of our approach, we study a hydraulic application with nonconvex cost function. In the first case we consider a simple configuration with one switching instant where we compared the obtained results by those obtained by gradient method. In fact, gradient method doesn't guarantee the optimal solution for that we conserve our approach. For several switching instants, Hybrid approach can easily be adapted where it considered a general solution for optimal problem of switching instants.
Figure 9. Optimal control input $u$.

Figure 10. Fluid level $h_1$. 
Figure 11. Fluid level $h_2$.

Figure 7. Cost function $J(t_1, t_2)$.
6. REFERENCES


