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DISTRIBUTIONS OF THE EXTREME EIGENVALUES OF BETA–JACOBI RANDOM MATRICES*

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Abstract. We present explicit formulas for the distributions of the extreme eigenvalues of the β–Jacobi random matrix ensemble in terms of the hypergeometric function of a matrix argument. For $\beta = 1, 2, 4$, these formulas specialize to the well-known real, complex, and quaternion Jacobi ensembles, respectively.

Key words. random matrix, Jacobi distribution, MANOVA, multivariate beta, hypergeometric function of a matrix argument, eigenvalue

AMS subject classifications. 15A52, 60E05, 62H10, 65F15

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1. Introduction.

Various multivariate statistical techniques such as canonical correlation analysis, multivariate analysis of variance, etc., are based on the distributions of the extreme eigenvalues of random matrices and, in particular, real and complex Jacobi random matrices [3, 5, 17].

The distribution of the extreme eigenvalues of the real Jacobi random matrices are well known (see section 2.3). The contribution of this paper is to generalize this result by deriving explicit expressions for the extreme eigenvalues of all β–Jacobi matrices in terms of the hypergeometric function of a matrix argument. The real, complex, and quaternion Jacobi random matrices are β–Jacobi distributed for $\beta = 1, 2$, and 4, respectively (see section 2 for the formal definitions of the classical Jacobi and β–Jacobi ensembles).

Other than the classical real case [17], the complex ($\beta = 2$) Jacobi matrices are of interest in wireless communication and signal processing [3, 5]. In general, the β–Jacobi ensembles are prominent in statistical physics in the study of the positions of the particles in a log-Coulomb gas at $2/\beta$ temperature, with Jacobi potentials [9].

The parameter β. Historically, the hypergeometric function of a matrix argument has been defined in terms of a parameter α [2, 10]. Elsewhere in random matrix theory, for example in statistical mechanics, the parameter $\beta = \frac{2}{\alpha}$ is prevalent (and known as the Boltzmann constant). This can sometimes be a source of confusion, thus we emphasize the fact that in this paper we are using the parameter β only.

Organization of the paper. In section 2 we define the Wishart, Jacobi, and β–Jacobi ensembles; we review their matrix models, define the hypergeometric function of a matrix argument, and survey the existing formulas for the distributions of the extreme eigenvalues of the real Jacobi ensemble. We present our main results—

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formulas for the distributions of the extreme eigenvalues of the β–Jacobi ensemble—in section 3. We present numerical experiments in section 4.

2. Background. In this section we recall the definitions of the hypergeometric function of a matrix argument; we define the classical Jacobi random matrix ensembles, the β–Jacobi distributions and ensembles, and survey the classical relationships between them.

2.1. Hypergeometric function of a matrix argument. This function (defined below) is a series of Jack functions. The *Jack function*, $C_\kappa^{(\beta)}(X)$, defined for a partition κ and a symmetric matrix X, is a symmetric, homogeneous polynomial in the eigenvalues $x_1, x_2, \ldots, x_m$ of X. It generalizes the (normalized) Schur function, the zonal polynomial [20, Proposition 1.2], and the quaternion zonal polynomial to which it reduces for $\beta = 1, 2$, and 4, respectively [15].

There are several normalizations of the Jack function; in this paper we use the "C" normalization, i.e., the one for which $\sum_{\kappa\vdash k} C_\kappa^{(\beta)}(X) = (\operatorname{tr} X)^k$. The explicit definition and properties of the Jack function are available from the classical paper by Stanley [20]; we do not need them here.

Definition 2.1 (hypergeometric function of a matrix argument). *Let $p \ge 0$ and $q \ge 0$ be integers, and let X be an $m \times m$ symmetric matrix. The* hypergeometric function of a matrix argument X *and parameter $\beta > 0$ is defined as*

$$
{}_pF_q^{(\beta)}(a_1, \ldots, a_p; b_1, \ldots, b_q; X) \equiv \sum_{k=0}^{\infty} \sum_{\kappa\vdash k} \frac{(a_1)_\kappa^{(\beta)} \cdots (a_p)_\kappa^{(\beta)}}{k!(b_1)_\kappa^{(\beta)} \cdots (b_q)_\kappa^{(\beta)}} \cdot C_\kappa^{(\beta)}(X),
$$

where $\kappa \vdash k$ means $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_m)$, $\kappa_1 \ge \kappa_2 \ge \cdots \ge \kappa_m \ge 0$, is a partition of k and

$$
(a)_\kappa^{(\beta)} \equiv \prod_{i=1}^{m} \prod_{j=1}^{\kappa_i} \left(a - \frac{\beta}{2}(i-1) + j - 1 \right)
$$

is the generalized Pochhammer symbol.

2.2. The classical Jacobi ensembles and the β–Jacobi ensemble. The classical real, complex, and quaternion Jacobi ensembles are defined as "ratios" of real, complex, and quaternion Wishart matrices, respectively. The β–Jacobi ensembles generalize the eigenvalue distributions of the classical Jacobi ensembles, and are defined for any $\beta > 0$.

Definition 2.2 (real, complex, and quaternion Wishart ensembles). *Let Z be an $m \times n$ real, complex, or quaternion Gaussian random matrix distributed as $N(0, I_n \otimes \Sigma)$, $\mathcal{C}N(0, I_n \otimes \Sigma)$, or $\mathcal{H}N(0, I_n \otimes \Sigma)$, respectively.*[1] *Then the matrix $A = Z^D Z$ is a real, complex, or quaternion $m \times m$ central Wishart matrix with n degrees of freedom and covariance matrix Σ.*[2] *We denote the real, complex, and quaternion Wishart distributions as $W_m^{(\beta)}(n, \Sigma)$ for $\beta = 1, 2$, and 4, respectively.*

Definition 2.3 (real, complex, and quaternion Jacobi ensembles). *If $A \sim W_m^{(\beta)}(n_1, \Sigma)$ and $B \sim W_m^{(\beta)}(n_2, \Sigma)$ are independent real, complex, or quaternion Wishart matrices (where $\beta = 1, 2$, or 4, respectively), then $C = A(A+B)^{-1}$ is called a real, complex, or quaternion* Jacobi *matrix, respectively.*

[1] The notation $\mathcal{H}$ stands for William Hamilton, who introduced the quaternions in the 1850s.

[2] The notation Z^D stands for the quaternion conjugate transpose of the matrix Z and reduces to the Hermitian transpose Z^H when Z is complex and the transpose Z^T when Z is real.

The real Jacobi distribution is sometimes called "multivariate Beta distribution" [4, 17] and is closely related to the "MANOVA" (Multivariate ANalysis Of VAriance) distribution [17], which, rather than consider the matrix $A(A+B)^{-1}$, examines the matrix AB^{-1}.

REMARK 2.4. *The eigenvalues of a Jacobi matrix are unaffected by the covariance matrix Σ, which can thus be assumed to be the identity without any loss of generality.*

PROPOSITION 2.5. *Every eigenvalue λ of a real, complex, and quaternion Jacobi matrix C is real; moreover, $\lambda \in [0,1]$.*

Proof. If λ is an eigenvalue of C, then there exists an eigenvalue $\tilde{\lambda}$ of AB^{-1} such that $\lambda = \tilde{\lambda}/(1+\tilde{\lambda})$. Since A and B are positive semidefinite, $\tilde{\lambda}$ is positive as a generalized eigenvalue of the matrix pair (A,B), and the desired conclusion follows. ◻

The following definition is central to this paper.

DEFINITION 2.6 (β–Jacobi ensembles). *For $\beta > 0$ and parameters $a_1, a_2 > \frac{\beta}{2}(m-1)$, a matrix is said to be β–Jacobi distributed if its joint eigenvalue density is*

$$\frac{1}{\mathcal{I}(m,\beta,a_1,a_2)} \prod_{i=1}^{m} \lambda_i^{a_1-\frac{\beta}{2}(m-1)-1}(1-\lambda_i)^{a_2-\frac{\beta}{2}(m-1)-1} \prod_{i<j} |\lambda_i-\lambda_j|^{\beta}, \tag{2.1}$$

where

$$\mathcal{I}(m,\beta,a_1,a_2) = \frac{\Gamma_m^{(\beta)}\left(1+\frac{\beta}{2}m\right)}{\pi^{\frac{m(m-1)\beta}{2}}\left(\Gamma\left(1+\frac{\beta}{2}\right)\right)^m} \cdot \frac{\Gamma_m^{(\beta)}(a_1)\,\Gamma_m^{(\beta)}(a_2)}{\Gamma_m^{(\beta)}(a_1+a_2)}$$

is the Selberg Integral value [19] *and*

$$\Gamma_m^{(\beta)}(c) \equiv \pi^{\frac{m(m-1)\beta}{4}} \prod_{i=1}^{m} \Gamma\left(c - \frac{\beta}{2}(i-1)\right) \qquad \text{for } \Re(c) > \frac{\beta}{2}(m-1)$$

is the multivariate gamma function *of parameter $\beta > 0$.*

From Theorem 3.3.4 in Muirhead [17], the real Jacobi matrices are β–Jacobi distributed for $\beta = 1$. By repeating the same argument for $\beta = 2$ and 4 we obtain the following theorem.

THEOREM 2.7. *With the notation as in Definition* 2.3*, the matrix C is β–Jacobi distributed with parameters $a_1 = \frac{\beta}{2}n_1$ and $a_2 = \frac{\beta}{2}n_2$, for $\beta = 1, 2,$ or 4, respectively.*

Following the methods of Dumitriu and Edelman [7], several matrix models were proposed that have β–Jacobi distributions [11, 14, 21]; here we present the one from Sutton [21].

THEOREM 2.8. *The tridiagonal positive semidefinite matrix $J \equiv Z^T Z$ is β–Jacobi distributed, where*

$$Z \equiv \begin{bmatrix} c_m & -s_m c'_{m-1} & & \\ & c_{m-1}s'_{m-1} & \ddots & \\ & & \ddots & -s_2 c'_1 \\ & & & c_1 s'_1 \end{bmatrix},$$

with

$$c_k \sim \sqrt{\mathrm{Beta}\left(a_1 - \tfrac{\beta}{2}(m-k), a_2 - \tfrac{\beta}{2}(m-k)\right)}, \qquad s_k = \sqrt{1-c_k^2},$$
$$c'_k \sim \sqrt{\mathrm{Beta}\left(\tfrac{\beta}{2}k, a_1+a_2-\tfrac{\beta}{2}(2m-k-1)\right)}, \qquad s'_k = \sqrt{1-c'^2_k},$$

where Beta *stands for the well-known univariate beta distribution.*

2.3. Survey of the real case ($\beta = 1$). We now survey the well-known distribution of the largest eigenvalue of the real Jacobi matrix (which is β–Jacobi distributed for $\beta = 1$). We generalize this result to any β in section 3.

Let $A \sim W_m^{(1)}(n_1, \Sigma)$, $B \sim W_m^{(1)}(n_2, \Sigma)$, where $n_1 \geq m$ and $n_2 \geq m$, be independent real Wishart matrices. For the distribution of the largest eigenvalue $\lambda_{\max}$ of the real Jacobi matrix $A(A+B)^{-1}$ we have [4, equation (61)]:

$$P(\lambda_{\max} < x) = \frac{\Gamma_m^{(1)}(\frac{n_1+n_2}{2})\Gamma_m^{(1)}(\frac{m+1}{2})}{\Gamma_m^{(1)}(\frac{n_1+m+1}{2})\Gamma_m^{(1)}(\frac{n_2}{2})} \cdot x^{\frac{mn_1}{2}} \cdot {}_2F_1^{(1)}(\tfrac{n_1}{2}, \tfrac{-n_2+m+1}{2}; \tfrac{n_1+m+1}{2}; xI).$$

Using [1, equation (9)] we obtain an explicit expression for the *density* of $\lambda_{\max}$:

$$\begin{aligned}\mathrm{dens}(\lambda_{\max}) = \frac{mn_1}{2} \cdot \frac{\Gamma_m^{(1)}(\frac{n_1+n_2}{2})\Gamma_m^{(1)}(\frac{m+1}{2})}{\Gamma_m^{(1)}(\frac{n_1+m+1}{2})\Gamma_m^{(1)}(\frac{n_2}{2})} \cdot (1-x)^{\frac{n_2-m-1}{2}} x^{\frac{mn_1}{2}-1} \\ \times {}_2F_1^{(1)}\left(\tfrac{n_1-1}{2}, \tfrac{m-n_2+1}{2}; \tfrac{n_1+m+1}{2}; xI_{m-1}\right).\end{aligned}$$

3. The extreme eigenvalues of the β–Jacobi ensembles. We obtain our main result in this paper by integrating the joint eigenvalue density of a β–Jacobi matrix and expressing the resulting integral in terms of the hypergeometric function of a matrix argument.

The connection with ${}_2F_1^{(\beta)}$ comes from Kaneko [10, Theorem 5]:

$$\begin{aligned}&{}_2F_1^{(\beta)}(r, a, a+b, tI_m) \\ &= \frac{1}{\mathcal{I}(m,\beta,a,b)} \int_{[0,1]^m} \prod_{i=1}^m x_i^{\lambda_1}(1-x_i)^{\lambda_2}(1-tx_i)^{-r} \prod_{i<j} |x_j - x_i|^\beta dx_1 \cdots dx_m,\end{aligned} \tag{3.1}$$

where $a = \lambda_1 + \frac{\beta}{2}(m-1)+1$ and $b = \lambda_2 + \frac{\beta}{2}(m-1)+1$.

Now let the $m \times m$ matrix J_β be β–Jacobi distributed with parameters a_1, a_2, and let $\lambda_{\max}$ and $\lambda_{\min}$ be its largest and smallest eigenvalues, respectively.

THEOREM 3.1. *The distributions of $\lambda_{\max}$ and $\lambda_{\min}$ are as follows:*

$$\begin{aligned}P(\lambda_{\max} < x) &= \frac{\Gamma_m^{(\beta)}(a_1+a_2) \cdot \Gamma_m^{(\beta)}\left(\frac{\beta}{2}(m-1)+1\right)}{\Gamma_m^{(\beta)}\left(a_1 + \frac{\beta}{2}(m-1)+1\right) \cdot \Gamma_m^{(\beta)}(a_2)} \cdot x^{ma_1} \\ &\quad \times {}_2F_1^{(\beta)}\left(a_1, \frac{\beta}{2}(m-1)+1-a_2; a_1 + \frac{\beta}{2}(m-1)+1; xI_m\right), \\ P(\lambda_{\min} < x) &= 1 - \frac{\Gamma_m^{(\beta)}(a_1+a_2) \cdot \Gamma_m^{(\beta)}\left(\frac{\beta}{2}(m-1)+1\right)}{\Gamma_m^{(\beta)}\left(a_2 + \frac{\beta}{2}(m-1)+1\right) \cdot \Gamma_m^{(\beta)}(a_1)} \cdot (1-x)^{ma_2} \\ &\quad \times {}_2F_1^{(\beta)}\left(a_2, \frac{\beta}{2}(m-1)+1-a_1; a_2 + \frac{\beta}{2}(m-1)+1; (1-x)I_m\right).\end{aligned} \tag{3.2}$$

Proof. We start with the joint eigenvalue density of J_β, (2.1), and note that to compute the distribution of the largest eigenvalue of J_β we need to integrate this density from 0 to xI_m. For $b_i \equiv a_i - \frac{\beta}{2}(m-1) - 1$, $i = 1, 2$, we have

$$P(J_\beta < xI_m) = \frac{1}{\mathcal{I}(m,\beta,a_1,a_2)} \int_{[0,x]^m} \prod_{i=1}^m \lambda_i^{b_1}(1-\lambda_i)^{b_2} \prod_{i<j} |\lambda_i - \lambda_j|^\beta d\lambda_1 \cdots d\lambda_m.$$

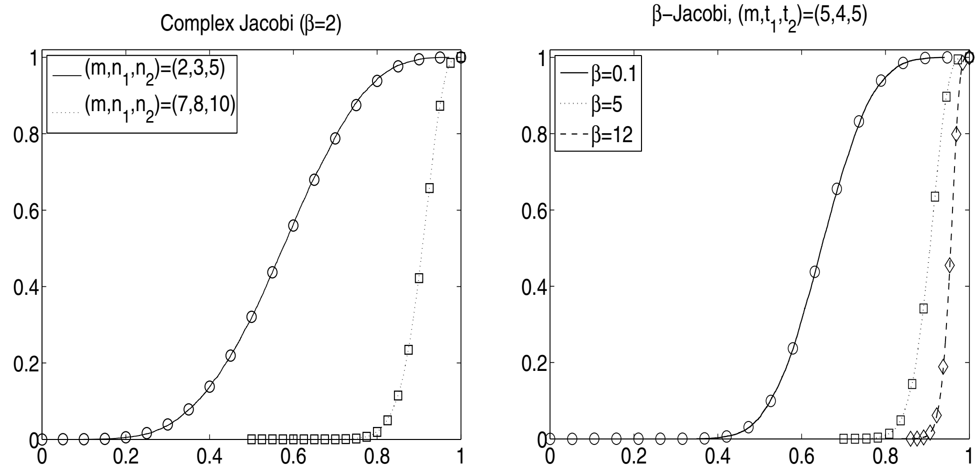

FIG. 1. *Left: Distributions of* $\lambda_{\max}$ *of a complex Jacobi matrix for two sets of parameters* (m, n_1, n_2). *The solid and dotted lines are the empirical distributions with* 10,000 *replications;* "○" *and* "□" *are the corresponding analytical predictions* (3.2). *Right: Distributions of* $\lambda_{\max}$ *of a* 5×5 β*–Jacobi matrix with* $a_i = t_i + \frac{\beta}{2}(m-1) + 1, i = 1, 2$ *for different values of* β. *The solid, dotted, and dashed lines are the empirical distributions with* 10,000 *replications;* "○," "□," *and* "◇" *are the corresponding analytical predictions* (3.3).

We make an m-dimensional change of variables $x\tilde{\lambda}_i = \lambda_i$ to obtain

$$P(J_\beta < xI_m) = \frac{x^{ma_1}}{\mathcal{I}(m, \beta, a_1, a_2)} \int_{[0,1]^m} \prod_{i=1}^{m} \tilde{\lambda}_i^{b_1} (1 - x\tilde{\lambda}_i)^{b_2} \prod_{i<j} |\tilde{\lambda}_i - \tilde{\lambda}_j|^\beta d\tilde{\lambda}_1 \cdots d\tilde{\lambda}_m.$$

We evaluate the last integral using (3.1) and get (3.2).

The result for $\lambda_{\min}$ follows immediately by observing that $1 - \lambda_{\min}$ is the largest eigenvalue of $I - J_\beta$, which is β–Jacobi distributed with parameters a_2 and a_1. □

When $t \equiv a_2 - \frac{\beta}{2}(m-1) - 1$ is a nonnegative integer, the hypergeometric series in (3.2) terminates and becomes a polynomial of degree mt. Then we can use Proposition 11.47 in Forrester [9] to obtain the expression

$$P(\lambda_{\max} < x) = x^{ma_1} \sum_{k=0}^{mt} \sum_{\kappa \vdash k,\, \kappa_1 \le t} \frac{1}{k!} (a_1)_\kappa^{(\beta)} C_\kappa^{(\beta)}((1-x)I), \tag{3.3}$$

which numerically is often much more feasible than (3.2).

Analogous results extending the well-known distributions of the extreme eigenvalues of real random matrices [4, 17] were obtained for β–Laguerre ensembles in [6, section 10.2] and complex Wishart ensembles in [18].

4. Numerical experiments. We performed extensive numerical tests against Monte-Carlo experiments to confirm the correctness of Theorem 3.2. We report the results of two tests whose results were typical.

In our first experiment we tested formula (3.2) against the empirical distribution of the largest eigenvalue of a complex Jacobi matrix. The more samples we used for the empirical distribution, the better approximation it was to the analytical prediction, with 10,000 samples sufficing for a perfect visual match. We generated our sample matrices in MATLAB [16] as C=A/(A+B), where A and B were complex

Wishart matrices generated as A=Z'*Z/2, where Z=randn(n_1, m)+i*randn(n_1, m), with B generated analogously. We evaluated the analytical formula (3.2), also in MATLAB, using the algorithms for computing ${}_pF_q^{(\beta)}$ from [12, 13]. We plot the results in Figure 1, left.

In our second experiment we demonstrate the β dependence of the largest eigenvalue of a 5×5 β–Jacobi matrix with parameters $a_i = t_i + \frac{\beta}{2}(m-1)+1, i = 1, 2$, where we fixed $t_1 = 4$ and $t_2 = 5$. In Figure 1, right, we plotted the empirical distribution from 10,000 replications which, again, matched the theoretical prediction (3.3) (which we could use since $a_2 - \frac{\beta}{2}(m-1) - 1 = t_2 = 5$ was a nonnegative integer).

In line with the results of [8], this experiment supports a conjecture that as β increases (and so do a_1 and a_2), the largest eigenvalue of the β–Jacobi ensemble approaches $1 = \lim_{a,b\to -1} \lambda^{a,b}$, where $\lambda^{a,b}$ is the largest root of the mth orthogonal Jacobi polynomial $J_m^{a,b}(x)$.

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