Graded rough set model based on two universes and its properties

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Abstract

In recent years, much attention has been given to the rough set models based on two universes of discourse and different kinds of rough set models on two universes have been developed from different points of view. In this paper, a novel model, i.e., the graded rough set model on two distinct but related universes (GRSTU) is proposed from the absolute quantitative point of view. We study the basic properties of approximation operators in GRSTU, and introduce a relation matrix based algorithm to compute the lower and upper approximations of a set of objects in GRSTU. Furthermore, the relationships between classical rough set model and GRSTU are discussed and some conclusions related to the GRSTU are given. Finally, several examples are employed to demonstrate the conceptual arguments of GRSTU, and an application of GRSTU is also illuminated in details.

Keywords: Rough sets, Two universes of discourse, Grade, Approximation operators, Relation matrix

1. Introduction

Rough set theory, originally proposed by Pawlak [1,2] as an extension of set theory, is an effective approach to dealing with imprecise, uncertain and incomplete information. It has been successfully used in many research areas, such as pattern recognition, machine learning, knowledge acquisition and data mining [3–7,32–35,38,39,41].

As we know, Pawlak’s rough set model has a basic hypothesis, that is, whether an object belongs to a class or not is completely certain. However, in practice, allowing some extent of uncertainty in the classification process may lead to a deeper understanding and a better utilization of the data being analyzed. In order to deal with the uncertainty in such cases, a lot of models have been proposed. For example, based on the Bayesian decision procedure with minimum cost (risk), Yao [8,9] proposed a decision-theoretic rough set model (DTRSM) which brings new insights into the probabilistic approaches to rough set model. DTRSM not only has good semantic interpretation, but also be beneficial for rule acquisition in the applications involving cost and risk. And Yao and Lin [10,11] presented a graded rough set model (GRS) from the absolute quantitative point of view. Moreover, many other models have also been proposed, such as rough set models based on arbitrary binary relations [12–14,40], rough set models based on incomplete systems [15–17,43], covering rough sets [18–20], rough fuzzy sets and fuzzy rough sets [21,42,44], variable precision rough sets [22,23], etc. Through loosening the strict definition of the approximations in Pawlak’s rough set model, these models enrich the application scope of rough set theory.

In the real world, we often face some situations in which making a decision is difficult. For example, in the process of identifying or determining the nature and cause of a disease, since a certain disease may simultaneously have several symptoms but the same symptom may be shared by diverse diseases, a doctor (or a decision-maker) often finds it is difficult to distinguish whether a person has suffered from the disease or not. In these kinds of situations, more than one universes of discourse are often involved. However, Pawlak’s rough set model and its extensions mentioned above are all based on only one universe, therefore these models may be not suitable to deal with the above problem. Hence, it is meaningful to propose a rough set model based on two universes.
Although the above models can effectively overcome the limitations of rough set models on one universe, they still lack the adaptability in solving uncertainty problems. To solve this problem, Shen et al. [30] proposed a variable precision rough set model on two universes from the relative quantitative point of view. Ma and Sun [37] introduced a probabilistic rough sets over two universes and used it to deal with the problem of Bayesian risk decision. In this paper, from the absolute quantitative point of view, we propose a graded rough set model defined on two distinct but related universes. Our model is not only an extension of the rough set model on two universes but also an extension of Pawlak’s rough set model. To compare with Yao and Lin’s graded rough set model, our model may be more appropriate to handle the problems where more than one universe is involved. Paralleling with Pawlak’s rough set model, the basic properties of our model are discussed. Meanwhile, a relation matrix based algorithm for computing the rough set model, the basic properties of our model are discussed. Finally, Section 6 concludes the paper.

2. Preliminaries

In this section, we outline some basic concepts in rough sets and some current rough set models, such as Pawlak’s rough set model [1], graded rough set model on one universe [10] and rough set model on two universes [24]. Throughout this paper, we suppose that the universe \( U \) or \( V \) is a finite non-empty set.

2.1. Pawlak’s rough set model

Let \( \mathcal{R} \) be a universe of discourse, for any binary relation \( R \) on \( \mathcal{U} \), we call \( \mathcal{R} \) an equivalence relation on \( \mathcal{U} \) if:

1. \( R \) is reflexive if for all \( x \in \mathcal{U}, xRx \);
2. \( R \) is symmetric if for all \( x, y \in \mathcal{U}, xRy \) implies \( yRx \);
3. \( R \) is transitive if for all \( x, y, z \in \mathcal{U}, xRy \) and \( yRz \) implies \( xRz \).

An equivalence relation is a reflexive, symmetric and transitive relation. The equivalence relation \( R \) partitions \( \mathcal{U} \) into disjoint subsets (or equivalence classes). Let \( \mathcal{U}/R \) denote the family of all equivalence classes of \( R \). For every object \( x \in \mathcal{U} \), let \( [x]_R \) denote the equivalence class of \( x \) under relation \( R \).

Let \( \mathcal{U} \) be a universe of discourse, \( \mathcal{R} \) an equivalence relation on \( \mathcal{U} \), for any \( X \subseteq \mathcal{U} \), one can describe \( X \) by a pair of lower and upper approximations defined as follows.

\[
\mathcal{B}(X) = \{ x \in \mathcal{U} | [x]_R \subseteq X \}
\]

\[
\mathcal{R}(X) = \{ x \in \mathcal{U} | [x]_R \cap X \neq \emptyset \}
\]

\( \mathcal{R}(X) \) is called the lower approximation of \( X \), which is the union of all the equivalence classes which contain in \( X \), and \( \mathcal{B}(X) \) is called the upper approximation of \( X \), which is the union of all equivalence classes which have non-empty intersection with \( X \). Then, \( \mathcal{R}(X), \mathcal{B}(X) \) is called the rough sets of \( X \). Accordingly, the positive, negative and boundary regions of \( X \) on the approximation space \( (U, R) \) can be defined as follows: \( pos(X) = \mathcal{B}(X), neg(X) = \mathcal{R}(X) - \mathcal{B}(X), bnd(X) = \mathcal{R}(X) - \mathcal{B}(X) \), where \( \sim \) stands for the complement of a set.

2.2. Generalized rough set operators

The Pawlak rough set model may be extended by using an arbitrary binary relation.

Let \( U \) be a universe of discourse and \( R \) a binary relation on \( U \), the following two operators: \( R(x) = \{ y \in U | xRy \}, \neg R(x) = \{ y \in U | \neg xRy \} \) are called the successor and predecessor neighborhood operator, respectively.

\[
pos_{\neg}(A) = \{ x \in \mathcal{U} | R(x) - A \leq n \} = \{ x \in \mathcal{U} | R(x) - A \leq n \}
\]

\[
neg_{\neg}(A) = \{ x \in \mathcal{U} | R(x) \cap A > n \}
\]

where \( |R(x)| \) denotes the cardinality of set \( R(x) \).

An element of \( U \) belongs to \( pos_{\neg}(A) \) if at most \( n \) of its \( R \)-related elements do not belong to \( A \), and belongs to \( neg_{\neg}(A) \) if more than \( n \) of its \( R \)-related elements belong to \( A \).

2.3. Yao and Lin’s graded rough set model on one universe [10]

Let \( \mathcal{U} \) be a universe and \( R \) a binary relation on \( \mathcal{U}, n \in \mathbb{N} \), where \( \mathbb{N} \) is the set of natural numbers. For any subset \( A \subseteq \mathcal{U} \), the lower and upper approximations of \( A \) with respect to \( n \) (denoted by \( \mathcal{B}_n(A) \) and \( \mathcal{R}_n(A) \), respectively) are defined as follows:

\[
\mathcal{B}_n(A) = \{ x \in \mathcal{U} | R(x) - A \leq n \}
\]

\[
\mathcal{R}_n(A) = \{ x \in \mathcal{U} | R(x) - A \leq n \}
\]

where \( |R(x)| \) denotes the cardinality of set \( R(x) \).

An element of \( U \) belongs to \( \mathcal{B}_n(A) \) if at most \( n \) of its \( R \)-related elements do not belong to \( A \), and belongs to \( \mathcal{R}_n(A) \) if more than \( n \) of its \( R \)-related elements belong to \( A \).

2.4. Rough set model on two universes

Next, we shall review some basic concepts and properties of the rough set model on two universes. Detailed description of the model can be found in [12,26,28].

The above model can be generalized to the case of two universes.

\[
\neg R(Y) = \{ x \in \mathcal{U} | R(x) \subseteq Y \}
\]

\[
\neg R(Y) = \{ x \in \mathcal{U} | R(x) \subseteq Y \}
\]

\( \mathcal{B}(Y) \) is called the lower approximation of \( Y \) and \( \mathcal{R}(Y) \) the upper approximation of \( Y \). (\( \mathcal{B}(Y), \mathcal{R}(Y) \)) is called the rough sets of \( Y \). Accordingly, the positive, negative and boundary regions of \( Y \) over the approximation space \( (U, V, R) \) can be defined as follows: \( pos(Y) = \mathcal{B}(Y), neg(Y) = \mathcal{R}(Y) - \mathcal{B}(Y) \), \( bnd(Y) = \mathcal{R}(Y) - \mathcal{B}(Y) \).

Proposition 1. Given a two-universe approximation space \((U, V, R)\), for any \( Y \subseteq V \), the approximation operators given in Definition 2 have the following properties:

\[
(1) \mathcal{B}(Y) = \mathcal{R}(\sim Y), \mathcal{R}(Y) = \mathcal{R}(\sim Y)
\]

\[
(2) \mathcal{R}(V) = \mathcal{B}(\emptyset), \mathcal{B}(\emptyset) = \mathcal{R}(\emptyset) = \emptyset
\]
For any \( t \) functions with respect to the graded relations from \( a \) to \( b \), we shall define graded rough sets on two universes and Model 4 is the model defined by Definition 2. Each arrow connects two models, where the model located at the beginning of the arrow is the first model, and the model located at the end of the arrow is the second model. For each arrow, the second model is a special case of the first model, and the numbers located on the arrow denotes the conditions that should be satisfied for the first model to degenerate to the second model.

3. Graded rough sets on two universes and their properties

In parallel to graded modal logics, Yao and Lin [10] first introduced graded rough sets on one universe. In this section, based on the work of Yao and Lin, we shall define graded rough sets on two universes and discuss the basic properties of this model.

Definition 3. Let \( U \) and \( V \) be two universes of discourse, \( R \) a binary relation from \( U \) to \( V \), i.e. \( R \subseteq U \times V \), and \( n \in N \), where \( N \) is the set of natural numbers. For any \( Y \subseteq V \), its lower and upper approximations with respect to the graded \( n \) are defined respectively as follows:

\[
\begin{align*}
\mathcal{R}_0(Y) &= \{x \in U | r(x) \cap Y \subseteq \emptyset\} \\
\mathcal{R}_n(Y) &= \{x \in U | r(x) \cap Y \supseteq \emptyset\}
\end{align*}
\]

According to Definition 3, an element of \( U \) belongs to \( \mathcal{R}_0(Y) \) if and only if at most \( n \) of its \( R \)-related elements do not belong to \( Y \), and belongs to \( \mathcal{R}_n(Y) \) if and only if more than \( n \) of its \( R \)-related elements belong to \( Y \). If \( \mathcal{R}_n(Y) \neq \mathcal{R}_0(Y) \), then \( Y \) is called a rough set with respect to the grade \( n \). Otherwise, \( Y \) is called a definable set with respect to the grade \( n \). \( \mathcal{R}_0 \) and \( \mathcal{R}_n \) are called the lower and upper approximations with respect to the grade \( n \), respectively.

Remark 1.

1. If \( n = 0 \), then
   \[
   \begin{align*}
   \mathcal{R}_0(Y) &= \{x \in U | r(x) \cap Y \subseteq \emptyset\} \\
   &= \{x \in U | r(x) \cap Y \subseteq \emptyset\},
   \end{align*}
   \]
   \[
   \begin{align*}
   \mathcal{R}_n(Y) &= \{x \in U | r(x) \cap Y \supseteq \emptyset\} \\
   &= \{x \in U | r(x) \cap Y \supseteq \emptyset\}.
   \end{align*}
   \]
   That is, if \( n = 0 \), then the graded rough set model on two universes degenerates to the rough set model on two universes as defined in Definition 2.

2. If \( U = V \), then the graded rough set model on two universes degenerates into the original graded rough set model proposed by Yao and Lin [10].

3. If \( n = 0 \) and \( U = V \), then
   \[
   \begin{align*}
   \mathcal{R}_0(Y) &= \{x \in U | r(x) \cap Y \subseteq \emptyset\} \\
   &= \{x \in U | r(x) \cap Y \subseteq \emptyset\},
   \end{align*}
   \]
   \[
   \begin{align*}
   \mathcal{R}_n(Y) &= \{x \in U | r(x) \cap Y \supseteq \emptyset\} \\
   &= \{x \in U | r(x) \cap Y \supseteq \emptyset\}.
   \end{align*}
   \]
   In the current case, the graded rough set model on two universes degenerates into the rough set model based on a binary relation over a given universe.

Remark 2. In the graded rough set model on two universes, the positive, negative and boundary regions of set \( Y \subseteq V \) cannot be directly defined as those in Pawlak’s rough set model, since the property \( \mathcal{R}_0(Y) \subseteq \mathcal{R}_n(Y) \) does not always hold (as show in Example 2).

For completeness, we give the definitions of positive, negative and boundary regions of \( Y \subseteq V \) in the graded rough set model on two universes.

Definition 4. Let \( U \) and \( V \) be two universes of discourse, \( R \) a binary relation from \( U \) to \( V \), i.e. \( R \subseteq U \times V \), and \( n \in N \), \( N \) is the set of natural numbers. For any \( Y \subseteq V \), the positive, negative and boundary regions of \( Y \) are respectively defined as follows.

\[
\begin{align*}
\text{POS}_n(Y) &= \{x \in U | r(x) \cap Y \subseteq \emptyset \text{ and } \text{max} | r(x) \cap Y | \leq n\} \\
&= \mathcal{R}_n(Y) \cap \mathcal{R}_0(Y) \\
\text{NEG}_n(Y) &= \{x \in U | r(x) \cap Y \supseteq \emptyset \text{ and } \text{max} | r(x) \cap Y | \leq n\} \\
&= (\mathcal{R}_n(Y) \cup \mathcal{R}_0(Y)) \setminus \text{POS}_n(Y) \\
\text{BND}_n(Y) &= \text{POS}_n(Y) \cup \text{NEG}_n(Y)
\end{align*}
\]

From Definition 4, we can see that the positive, negative and boundary regions of a set in the graded rough set model on two universes have more complex structure than those in Pawlak’s rough set model.

Proposition 2. Given a two-universe approximation space \((U, V, R)\), for any \( Y, Y_1, Y_2 \subseteq V \), the rough set approximation operators with respect to graded \( n \) satisfy the following properties.

1. \( \mathcal{R}(Y) = \mathcal{R}_n(Y) \)
In Definition 5, sum(i) is the cardinality of set \( \{y_j\} \), where \( y_j \) is related to \( x_i \) based on \( R_i \), i.e., \( \text{sum}(i) = |r(x_i)| \), \( 1 \leq i \leq n \). And \( z_i \) is the cardinality of set \( \{y_j\} \), \( 1 \leq i \leq n \). 

**Proposition 3.** In fact, according to Definition 5, \( \text{sum}(i) \) is the cardinality of set \( \{y_j\} \) on which \( y_j \) is related to \( x_i \) based on \( R_i \), i.e., \( \text{sum}(i) = |r(x_i)| \), \( 1 \leq i \leq n \). And \( z_i \) is the cardinality of set \( \{y_j\} \), \( 1 \leq i \leq n \). Proposition 3 is proved.

As a summary, we can describe the algorithm as Algorithm 1.

**Algorithm 1.** A matrix-based algorithm for computing approximations

- **Input:** \( U = \{x_1, x_2, ..., x_n\} \), \( V = \{y_1, y_2, ..., y_m\} \), \( R \subseteq U \times V \)
- **Output:** Approximations of \( Y \) with respect to \( R \);

1. Let \( \mathcal{R}_0(Y) = \emptyset \), \( \mathcal{R}_0(Y) = \emptyset \).
2. Compute \( M_R = (a_{ij})_{m \times n} \) of \( R \), \( \text{sum} \) and \( Z \).
3. for \( i = 1 \) to \( m \) do
   4. if \( \text{sum}(i) - z_i < k \) then
   5. \( \mathcal{R}_0(Y) = \mathcal{R}_0(Y) \cup \{x_i\} \);
   6. end
   7. if \( z_i > k \) then
   8. \( \mathcal{R}_0(Y) = \mathcal{R}_0(Y) \cup \{x_i\} \);
   9. end
10. end
11. return \( \mathcal{R}_0(Y), \mathcal{R}_0(Y) \)

Algorithm 1 involves two closely integrated stages. In the first stage (from lines 1 to 2), it computes \( M_R \), \( \text{sum} \), and \( Z \), according to Definitions 5, 6 and Proposition 3. This stage is the basis for the next stage. In the second stage (from lines 3 to 10), it computes approximations by using Proposition 3.

As shown in Algorithm 1, its major computation lies in the establishment of relation matrix, \( \text{sum} \) and \( Z \). Assume \( |U| = n \) and \( |V| = m \), the time complexity of building a relation matrix is \( O(n \times m) \) and the time complexity for computing \( \text{sum} \) and \( Z \) is also \( O(n \times m) \). The time complexity for calculating approximations is \( O(m) \). Therefore, the total complexity of Algorithm 1 is \( O(n \times m + m) \), which is approximate to \( O(n \times m) \).

In order to show the implicit relations between GRSTU and graded rough set model on one universe, we introduce two binary relations: \( E_U^0 \) and \( E_V^0 \), which are induced by \( R \subseteq U \times V \), where \( U \) and \( V \) are two universes of discourse. The two relations are defined based on only one universe \( U \) or \( V \).

A binary relation \( E_U^0 \subseteq U \times U \) induced by \( R \) can be defined as \( xE_U^0 x' \iff (x, x') \in R \). Usually, \( E_U^0 \) is an equivalence relation on \( U \). Equivalence relation \( E_U^0 \) partitions the universe \( U \) into disjoint subsets. Let \( U/E_U^0 \) denotes the set of all equivalence classes of \( E_U^0 \) and \( |x|_{E_U^0} \) denotes the equivalence class containing \( x \), where \( x \in U \). For convenience, \( |x|_{E_U^0} \) is replaced by \( |x|_U \) in this paper.

**Definition 7.** Let \( U \) and \( V \) be two universes of discourse, \( R \subseteq U \times V \), \( E_U^0 \subseteq U \times U \) be the equivalence relation induced by \( R \), and \( n \in N \).

For any \( x \in U \), its lower and upper approximations with respect to the grade \( n \) under \( E_U^0 \) are defined respectively as follows:

\[
\overline{E_{U}^0}(X) = \{x \in U | |x|_U - n \leq |x|_U \leq |x|_U \}
\]

\[
\underline{E_{U}^0}(X) = \{x \in U | |x|_U \cap |x|_U > n \}
\]

Obviously, \( \overline{E_{U}^0} \) and \( \underline{E_{U}^0} \) are two mappings from \( P(U) \) to \( P(U) \), where \( P(U) \) denotes the power set of \( U \). They are the approximation operators with respect to grade \( n \) over universe \( U \). Similarly, \( \overline{E_{V}^0}, \underline{E_{V}^0} : P(V) \rightarrow P(V) \) can also be defined.
Given any subset $Y \subseteq V$ and a fixed grade $n$, if we apply different approximation operators on $Y$, then we can obtain different results, and there exist some interrelations among these results. Proposition 4 gives a description of such interrelations among the results induced by different approximation operators on $Y$. As shown in (1) of Proposition 4, for any subset $Y \subseteq V$, the result induced by $\overline{\overline{Y}}_n$ and $\overline{\overline{Y}}_n$ on $Y$, i.e., $\overline{\overline{\overline{\overline{Y}}}}_n$, is a subset of that induced by $\overline{\overline{Y}}_n$ and $\overline{\overline{Y}}_n$ on $Y$, and the latter is a subset of the result induced by $\overline{\overline{Y}}_n$ on $Y$. On the contrary, when the lower approximation operators $\overline{\overline{Y}}_n$, $\overline{\overline{Y}}_n$ and $\overline{\overline{Y}}_n$ are concerned, we can obtain a different conclusion, as shown in (2) of Proposition 4. Moreover, when the lower and upper approximation operators are used simultaneously, the outcomes can be found in (3) and (4) of Proposition 4.

Proposition 5. Let $U$ and $V$ be two universes of discourse, $R$ a binary relation from $U$ to $V$, and $n, m \in N, n \geq m$. Let $E^R_m \subseteq U \times U$ and $E^R_m \subseteq V \times V$ be two equivalence relations induced by $R$. For any $Y \subseteq U$, the following properties are satisfied.

$$
(1) \overline{\overline{Y}}_m \subseteq \overline{\overline{Y}}_n \\
(2) \overline{\overline{Y}}_m \supseteq \overline{\overline{Y}}_n \\
(3) \overline{\overline{Y}}_m \supseteq \overline{\overline{Y}}_n \\
(4) \overline{\overline{Y}}_m \supseteq \overline{\overline{Y}}_n
$$

Proof 5. (1) If $n \geq m$, according to (7) of Proposition 2, we have that $\overline{\overline{Y}}_m \supseteq \overline{\overline{Y}}_n$. Thus, we have that $\overline{\overline{\overline{\overline{Y}}}}_m \subseteq \overline{\overline{\overline{\overline{Y}}}}_n$ and $\overline{\overline{\overline{\overline{Y}}}}_n \subseteq \overline{\overline{\overline{\overline{Y}}}}_m$. And from (3) of Proposition 2, we can obtain that $\overline{\overline{\overline{\overline{Y}}}}_n \subseteq \overline{\overline{\overline{\overline{Y}}}}_m$ and $\overline{\overline{\overline{\overline{Y}}}}_m \subseteq \overline{\overline{\overline{\overline{Y}}}}_n$. Therefore, $\overline{\overline{\overline{\overline{Y}}}}_n \subseteq \overline{\overline{\overline{\overline{Y}}}}_m$ and $\overline{\overline{\overline{\overline{Y}}}}_m \subseteq \overline{\overline{\overline{\overline{Y}}}}_n$.

(2) Differing from Proposition 4, given any subset $Y \subseteq V$, Proposition 5 shows the interrelations between the results induced by the approximation operators on $Y$ with different grades. As shown in (1) of Proposition 5, for any subset $Y \subseteq V$, the result induced by $\overline{\overline{Y}}_m$ and $\overline{\overline{Y}}_m$ on $Y$ is a subset of that induced by $\overline{\overline{Y}}_m$ and $\overline{\overline{Y}}_m$ on $Y$, where $n \geq m$.

Definition 8 ([7]). Let $U$ be a universe of discourse, $P$ and $R$ are two equivalence relations on $U$. We say that $P$ is finer than $R$, denoted by $P \preceq R$, if each equivalence class in $U/P$ is a union of some equivalence classes in $U/R$.

Proposition 6. Let $U$ and $V$ be two universes of discourse, $P$ and $R$ are two binary relations from $U$ to $U$, and $n \in N$. If $P \preceq R$, then for any $X \subseteq U, \overline{\overline{\overline{\overline{X}}}}_n \supseteq \overline{\overline{\overline{\overline{X}}}}_n$.

Proof 6. For any $X \subseteq \overline{\overline{Y}}_n$, from Definition 7, $|\overline{\overline{Y}}_n - X| \leq n$ holds. If $P \preceq R$, then $|\overline{\overline{Y}}_n - \overline{\overline{\overline{\overline{X}}}}_n| = \overline{\overline{\overline{\overline{X}}}}_n - \overline{\overline{\overline{\overline{X}}}}_n$, where $N_i$ is the equivalence class with respect to $P$. Hence, we have that $|\overline{\overline{Y}}_n - X| = \sum_{i=1}^{n} |N_i - X| \leq n$, i.e., $\sum_{i=1}^{n} |N_i - X| \leq n$. Obviously, $|N_i - X| \leq n \leq n$, i.e., $x \subseteq \overline{\overline{Y}}_n$. Therefore, we can obtain that $\overline{\overline{\overline{\overline{Y}}}}_n \supseteq \overline{\overline{\overline{\overline{Y}}}}_n$.

Remark 3. $\overline{\overline{\overline{\overline{X}}}}_n$ does not always hold.
Example 1. Let $U$ be the universe of discourse, $P$ and $R$ two equivalence relations on $U$. $U/P$ and $U/R$ are specified as follows:

$U = \{x_1, x_2, \ldots, x_6\}$

$U/P = \{\{x_1\}, \{x_2, x_4\}, \{x_3\}, \{x_5, x_6\}\}$

$U/R = \{\{x_1, x_2, x_3, x_4\}, \{x_5, x_6\}\}$

Obviously, we have that $P \subseteq R$.

Suppose that $X = \{x_2, x_4, x_5\} \subseteq U$ and $n = 1$.

Then we can obtain that

$P_1(X) = \{x_1, x_2, x_3, x_4\}$

$P_2(X) = \{x_1, x_2, x_3, x_4\}$

Therefore $P_1(X) \supseteq P_0(X)$ and $P_0(X) \subseteq P_0(X)$.

4. Two illustrative examples

In this section, two illustrative examples are employed to demonstrate the concepts, method and properties which discussed in Section 3.

Example 2. Let $U$ and $V$ be two universes of discourse, $R$ a binary relation from $U$ to $V$, and $M_R = (a_{ij})_{x \times y}$ be the relation matrix of $R$, which are respectively given as follows:

$U = \{x_1, x_2, \ldots, x_6\}$

$V = \{y_1, y_2, \ldots, y_3\}$

First, let $X \subseteq Y \subseteq V$, and $n = 2$.

Then we can obtain that

$P_1(X) = \{x_1, x_2, x_3, x_4\}$

$P_2(X) = \{x_1, x_2, x_3, x_4\}$

Therefore $P_1(X) \supseteq P_0(X)$ and $P_0(X) \subseteq P_0(X)$.

Example 3. (Continued from Example 2)

From $M_R$, we can obtain that

$P_1(X) = \{x_1, x_2, x_3, x_4\}$

$P_2(X) = \{x_1, x_2, x_3, x_4\}$

Hence, we can further obtain that

$r(x_1) \cap Y = \{y_1\}$

$r(x_2) \cap Y = r(x_3) \cap Y = \{y_2, y_3\}$

$r(x_1) \cap Y = \{y_4\}$

$r(x_2) \cap Y = r(x_3) \cap Y = \{y_2, y_3\}$

We know that $X \subseteq Y$.

First, let $n = 2$.

Then, we have that

$U/E_0 = \{\{x_1\}, \{x_2, x_4\}, \{x_3\}, \{x_5, x_6\}\}$

$V/E_0 = \{\{y_1\}, \{y_2, y_3\}, \{y_4\}, \{y_5, y_6, y_7\}\}$

Suppose $X_1 = \{y_2, y_3, y_4\}$ and $X_2 = \{y_1, y_2, y_3, y_5\}$. Then

$X_2 \subseteq Y_1 \cap Y_2 = \{y_2, y_3, y_5\}, Y_1 \cap Y_2 = \{y_1, y_2, y_3, y_4\}$.

First, let $n = 2$.

Then, we have that

$R_1(Y_1) = \{x_2, x_4\}$, $R_2(Y_2) = \emptyset$

$R_1(Y_3) = \{x_1, x_2, x_4\}$, $R_2(Y_4) = \emptyset$

Suppose $X_1 = \{y_2, y_3, y_4\}$ and $X_2 = \{y_1, y_2, y_3, y_5\}$. Then

$X_2 \subseteq Y_1 \cap Y_2 = \{y_2, y_3, y_5\}, Y_1 \cap Y_2 = \{y_1, y_2, y_3, y_4\}$.

Second, we compute the lower and upper approximations of $Y$ with respect to $n$ using Proposition 3.

From $M_R$, we can obtain that

$X_2 \subseteq Y_1 \cap Y_2 = \{y_2, y_3, y_5\}, Y_1 \cap Y_2 = \{y_1, y_2, y_3, y_4\}$.

Therefore

$X_2 \subseteq Y_1 \cap Y_2 = \{y_2, y_3, y_5\}, Y_1 \cap Y_2 = \{y_1, y_2, y_3, y_4\}$.

According to Definition 3, $R_1(Y) = \{x_1, x_2, x_3, x_4\}$ and $R_2(Y) = \{x_1, x_4\}$.

Second, we compute the lower and upper approximations of $Y$ with respect to $n$ using Proposition 3.

From $M_R$, we can obtain that

$S = (\text{sum}(1), \text{sum}(2), \text{sum}(3), \text{sum}(4), \text{sum}(5), \text{sum}(6))^T$

and

$Z = M_R \cdot Y = (z_1, z_2, \ldots, z_m)^T = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$

Therefore,

$\text{sum}(1) - z_1 = 1; \text{sum}(2) - z_2 = 1; \text{sum}(3) - z_3 = 4;

\text{sum}(4) - z_4 = 1; \text{sum}(5) - z_5 = 3; \text{sum}(6) - z_6 = 3$.
Then, we can conclude that
\[
\tilde{R}_2(E_2(Y_1)) = \emptyset \subseteq \tilde{R}_2(\overline{E}_2(Y_1)) = \emptyset
\]
and
\[
\tilde{E}_2(R_1(Y_1)) = \{x_1, x_2, x_4, x_5, x_6\} \supseteq \tilde{R}_2(\overline{E}_2(Y_1)) = \{x_1, x_2, x_4\}
\]
Second, let \( n = 1 \).

Analogously, we have that
\[
\tilde{R}_1(Y_1) = \{x_2, x_4\}, \quad \tilde{R}_1(Y_2) = \{x_2, x_4\}, \quad \tilde{E}_1(Y) = \emptyset
\]
\[
\tilde{R}_1(Y_1) = \{x_1, x_2, x_4\}, \quad \tilde{R}_1(Y_2) = \emptyset, \quad \tilde{E}_1(Y_1) = \{y_1, y_2, y_3, y_4\}
\]
From the above definitions, we have that
\[
\tilde{R}_1(Y_2) = \{x_2, x_4\} \supseteq \tilde{R}_2(Y_2) = \emptyset
\]
\[
\tilde{R}_1(Y_2) = \emptyset \subseteq \tilde{R}_2(Y_2) = \{x_1, x_2, x_4\}
\]
\[
\tilde{E}_1(R_1(Y_1)) = \{x_2, x_4\}
\]
\[
\tilde{E}_1(R_1(Y_2)) = \{x_2, x_4\} \supseteq \tilde{R}_1(E_1(Y_1)) = \{x_1, x_2, x_4\}
\]
\[
\tilde{E}_1(R_1(Y_2)) = \{x_1, x_2, x_3, x_4\} \supseteq \tilde{R}_1(E_1(Y_1)) = \{x_1, x_2, x_4\}
\]
\[
\tilde{E}_1(Y_1) = \emptyset \subseteq \tilde{E}_1(Y_2) = \{x_1, x_2, x_3, x_4\}
\]
Then, we can conclude that
\[
\tilde{E}_1(Y_1) = \emptyset \subseteq \tilde{E}_1(\overline{E}_1(Y_1)) = \{x_1, x_2, x_3, x_4\}
\]
\[
\tilde{E}_2(R_1(Y_1)) = \{x_1, x_2, x_3, x_4\} \supseteq \tilde{R}_2(\overline{E}_2(Y_1)) = \{x_1, x_2, x_3, x_4\}
\]
\[
\tilde{E}_2(R_1(Y_2)) = \{x_1, x_2, x_3, x_4\} \supseteq \tilde{R}_2(\overline{E}_2(Y_1)) = \{x_1, x_2, x_3, x_4\}
\]
\[
\tilde{R}_2(Y_1) = \{x_1, x_2, x_3, x_4\} \supseteq \tilde{R}_2(Y_1) = \{y_1, y_2, y_3, y_4\}
\]
\[
\tilde{R}_2(Y_1) = \{x_1, x_2, x_3, x_4\} \supseteq \tilde{R}_2(Y_1) = \{y_1, y_2, y_3, y_4\}
\]
\[
\tilde{R}_2(Y_1) = \emptyset \subseteq \tilde{R}_2(Y_2) = \{x_1, x_2, x_3, x_4\}
\]
Let \( U \) denote the set of patients and \( V \) the set of symptoms. Then for any patient \( u \in U \), there exist some symptoms in \( V \) correspond to \( u \). For any \( Y \subseteq V \), a disease which contains some basic symptoms in \( Y \). Then, given a patient, if he or she belongs to \( \text{POS}(Y) \), then he or she is certainly suffered from the disease denoted by \( Y \). Therefore, all of the patients belonging to \( \text{POS}(Y) \) are suitable for the remedy to \( Y \) immediately. On the other hand, if he or she belongs to \( \text{NEG}(Y) \), then his or her disease is connected with \( Y \), but the connection is not certain. Therefore, the doctor should further analyze the pathology for the current patient and adopt an appropriate treatment. Furthermore, if he or she belongs to \( \text{POS}(Y) \), then his or her disease has no connection with \( Y \), and the doctor should adopt other strategies.

We will show the above discussion by the following example.

**Example 4.** Let \( U = \{x_1, x_2, x_3\} \) be the set of patients, \( V = \{y_1, y_2, y_3, y_4\} \) the set of symptoms, and \( n \in \mathbb{N} \). Assume that \( R \subseteq U \times V \) is a binary relation between \( U \) and \( V \), which can be described by the following relation matrix \( M_R = (a_{ij}) \) (where if patient \( i \) has symptom \( j \) then \( a_{ij} = 1 \) else \( a_{ij} = 0 \)):

\[
M_R = \begin{bmatrix}
1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1
\end{bmatrix}
\]

From matrix \( M_R \), we can obtain that
\[
r(x_1) = \{y_1, y_4\}, \quad r(x_2) = \{y_1, y_2\}, \quad r(x_3) = \{y_2, y_3\}.
\]

Let \( Y = \{y_2, y_4\} \subseteq V \) denote a certain disease, that is, \( Y \) shows two symptoms in the clinic. Meanwhile, we suppose that \( n = 1 \).

From Definitions 3 and 4, we can calculate the lower approximation, the upper approximation, the positive region, the negative region and the boundary region of \( Y \) as follows.

\[
R_1(Y) = \{x_1, x_2, x_3\},
\]
\[
R_1(Y) = \{x_1\},
\]
\[
\text{POS}_1(Y) = R_1(Y) \cap \overline{R}_1(Y) = \{x_1\},
\]
\[
\text{NEG}_1(Y) = \{R_1(Y) \cup \overline{R}_1(Y) = \{x_1\},
\]
\[
\text{BDN}_1(Y) = \{\text{POS}_1(Y) \cup \text{NEG}_1(Y) = \{x_1\},
\]
Then, we can obtain the following conclusions:

1. It is certain that patient \( x_1 \) has disease \( Y \) and the doctor should take the corresponding treatment immediately.
2. The doctor cannot decide whether patients \( x_1 \) and \( x_2 \) have disease \( Y \) or not according to the symptoms at present. The patients should be examined further.
3. None of the three patients is healthy after diagnosis.

**6. Conclusions**

Rough set theory based on two universes is a generalization of Pawlak’s rough set theory. In this paper, the graded rough set model based on two distinct but related universes was proposed. We gave some interesting properties and conclusions about the graded rough set model on two universes, which can help us understand the structure of GRSTU. Moreover, an efficient method for calculating the lower and upper approximations of a given set in GRSTU was proposed and several examples were also given. The outcomes of these examples demonstrated that GRSTU is more suitable to the decision of clinical diagnosis than the traditional rough set model. In the future work, we shall further discuss other aspects of GRSTU, for instance, the issue of attribute reduction or rules extraction in GRSTU.
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