THE TOPOLOGICAL ASYMPTOTIC FOR THE
HELMHOLTZ EQUATION WITH DIRICHLET CONDITION ON
THE BOUNDARY OF AN ARBITRARILY SHAPED HOLE

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Abstract. The aim of the topological sensitivity analysis is to obtain an asymptotic expansion of a design functional with respect to the creation of a small hole in the domain. In this paper, such an expansion is obtained for the Helmholtz equation, in two and three space dimensions, with a Dirichlet condition on the boundary of an arbitrarily shaped hole. In this case, the main difficulty is related to the nonhomogeneous symbol of the Helmholtz operator. In the numerical part of this work, we will show that the topological sensitivity method is very promising for solving shape inverse problems in electromagnetic applications.

Key words. topological optimization, topological asymptotic, topological gradient, nonhomogeneous problem, Helmholtz equation, shape inversion, electromagnetic applications, inverse scattering

AMS subject classifications. 49Q10, 49Q12, 78A25, 78A40, 78A45, 78A50, 35J05

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1. Introduction. The same numerical methods are generally used in shape inversion and optimal shape design. There are mainly two categories of shape inversion or shape optimization methods. In the first category we deform continuously the boundary of the object to be optimized in order to decrease a given cost function [5, 20, 25, 28, 31]. The final shape has the same topology as the initial shape given by the designer. Therefore, to reach the optimal geometry, we need a priori knowledge of its topology. However, the topology of the optimal shape is often the main unknown in object detection problems. For example, the knowledge of the number and the locations of buried mines is more important than their accurate shapes. The second category of algorithms allows topology modifications. Many important contributions in this field are concerned with structural mechanics and, in particular, the optimization of the compliance (external work) subject to a volume constraint [4, 16]. In view of the fact that the optimal structure has generally a large number of small holes, most authors [1, 3, 14] have considered composite material optimization. Using the homogenization theory, Allaire and Kohn [1] exhibit a class of laminated materials with an explicit expression for the optimal material at any point of the structure. In this case, the optimal solution is not a classical design—it is a distribution of composite materials. Then penalization methods must be applied in order to retrieve a realistic shape. For all these reasons, global optimization methods are used to solve more general problems [15, 26]. Unfortunately these methods are quite slow.

More recently, Eschenauer and Olhoff [7], Schumacher [27], Céa et al. [6], Garreau, Guillaume, and Masmoudi [8], Sokolowski and Zochowski [29, 30], and Nazarov and Sokolowski [21] presented a method to obtain the optimal topology by calculating the so-called topological gradient (or topological derivative). This gradient is a function
defined in the domain of interest where, at each point, it gives the sensitivity of the cost function when a small hole is created at that point. This approach seems to be general and efficient. To present the basic idea, we consider $\Omega$ a domain of $\mathbb{R}^n$, where $n$ equals 2 or 3, and $j(\Omega) = J(u_\Omega)$ a cost function to be minimized, where $u_\Omega$ is the solution to a given PDE problem defined in $\Omega$. For $\varepsilon > 0$, let $\Omega_\varepsilon = \Omega \setminus \overline{\mathcal{B}(x_0, \varepsilon)}$ be the subset obtained by removing a small part $\mathcal{B}(x_0, \varepsilon)$ from $\Omega$, where $x_0 \in \Omega$ and $\mathcal{B} \subset \mathbb{R}^n$ is a fixed open and bounded subset containing the origin. We can generally prove that the variation of the criterion is given by the asymptotic expansion

$$j(\Omega_\varepsilon) = j(\Omega) + f(\varepsilon)g(x_0) + o(f(\varepsilon)),$$

This expansion is called the topological asymptotic. To minimize the criterion, we have to create holes where $g$ (called the topological gradient) is negative.

In this paper, using the adjoint method and the domain truncation technique introduced in [17], we compute the topological asymptotic expansion for the Helmholtz equation in two and three space dimensions with a Dirichlet condition on the boundary of an arbitrarily shaped hole. The originality of this work is that the symbol of the Helmholtz operator is nonhomogeneous. The basic idea is to say that the leading term of the topological asymptotic expansion is given by the principal part of the operator in the case of a Dirichlet condition on the boundary of the hole. Our work generalizes the contribution of Guillaume and Sid Idris [9] for the Poisson equation and is easily applicable to other problems for which the symbol of the operator is nonhomogeneous, as, for example, the quasi-Stokes problem and the elastic waves problem.

The generalized adjoint method is recalled in section 2. Next, the formulation of the Helmholtz problem is presented in section 3 and its truncated version is described in section 4. Section 5 presents the main results whose proofs are given in section 6. Finally, numerical examples illustrate in section 7 the abilities of the topological sensitivity to solve inverse scattering problems.

2. A generalized adjoint method. In this section, the generalized adjoint method introduced in [17, 8] is slightly modified. The first modification is due to the fact that the cost function is defined in a $C$-Hilbert space and takes values in $\mathbb{R}$; then it is not differentiable. For this reason, the differentiability property is replaced by the formulation (2.5). The second modification is due to the fact that the sesquilinear form associated with our problem is not coercive. For this reason, the coercivity property is replaced by the inf-sup condition (see Hypothesis 2).
Let \( V \) be a fixed complex Hilbert space. For \( \varepsilon \geq 0 \), let \( a_{\varepsilon}(\cdot, \cdot) \) be a sesquilinear and continuous form on \( V \) and let \( l_{\varepsilon} \) be a semilinear and continuous form on \( V \). We consider the following assumptions.

**Hypothesis 1.** There exists a sesquilinear and continuous form \( \delta a_{\varepsilon} \), a semilinear and continuous form \( \delta l_{\varepsilon} \), and a real function \( f(\varepsilon) > 0 \) defined on \( \mathbb{R}^*_{+} \) such that

\[
\begin{align*}
(2.1) & \quad \lim_{\varepsilon \to 0} f(\varepsilon) = 0, \\
(2.2) & \quad \|a_{\varepsilon} - a_0 - f(\varepsilon)\delta a_{L_2(V)} = o(f(\varepsilon)), \\
(2.3) & \quad \|l_{\varepsilon} - l_0 - f(\varepsilon)\delta l_{L(V)} = o(f(\varepsilon)),
\end{align*}
\]

where \( L(V) \) (respectively, \( L_2(V) \)) denotes the space of continuous and semilinear (respectively, sesquilinear) forms on \( V \).

**Hypothesis 2.** There exists a constant \( \alpha > 0 \) such that

\[
\inf_{u \neq 0} \sup_{v \neq 0} \frac{|a_0(u, v)|}{\|u\|\|v\|} \geq \alpha.
\]

We say that \( a_0 \) satisfies the inf-sup condition.

According to (2.2), there exists a constant \( \beta > 0 \) independent of \( \varepsilon \) such that

\[
\inf_{u \neq 0} \sup_{v \neq 0} \frac{|a_{\varepsilon}(u, v)|}{\|u\|\|v\|} \geq \beta.
\]

For \( \varepsilon \geq 0 \), let \( u_{\varepsilon} \) be the solution to the following problem: Find \( u_{\varepsilon} \in V \) such that

\[
(2.4) 
\begin{align*}
\alpha_{\varepsilon}(u_{\varepsilon}, v) = l_{\varepsilon}(v) \quad \forall v \in V.
\end{align*}
\]

We have the following lemma.

**Lemma 2.1.** If Hypotheses 1 and 2 are satisfied, then

\[
\|u_{\varepsilon} - u_0\|_V = \mathcal{O}(f(\varepsilon)).
\]

**Proof.** It follows from Hypothesis 2 that there exists \( v_{\varepsilon} \in V, v_{\varepsilon} \neq 0 \), such that

\[
\beta\|u_{\varepsilon} - u_0\|_V \|v_{\varepsilon}\|_V \leq |a_{\varepsilon}(u_{\varepsilon} - u_0, v_{\varepsilon})|,
\]

which implies

\[
\beta\|u_{\varepsilon} - u_0\|_V \|v_{\varepsilon}\|_V \leq |a_{\varepsilon}(u_0, v_{\varepsilon}) - l_{\varepsilon}(v_{\varepsilon})| = |(a_{\varepsilon}(u_0, v_{\varepsilon}) - a_0(u_0, v_{\varepsilon})) - (l_{\varepsilon} - l_0 - f(\varepsilon)\delta l_{L(V)})(v_{\varepsilon}) - f(\varepsilon)\delta l_{L(V)}(v_{\varepsilon})| \leq |a_{\varepsilon}(u_0, v_{\varepsilon}) - a_0(u_0, v_{\varepsilon}) - f(\varepsilon)\delta a(u_0, v_{\varepsilon})| + |l_{\varepsilon}(v_{\varepsilon}) - l_0(v_{\varepsilon}) - f(\varepsilon)\delta l_{L(V)}(v_{\varepsilon})| + f(\varepsilon)(|\delta a(u_0, v_{\varepsilon})| + |\delta l_{L(V)}(v_{\varepsilon})|).
\]

Using Hypothesis 1 we obtain

\[
\beta\|u_{\varepsilon} - u_0\|_V \|v_{\varepsilon}\|_V \leq (o(f(\varepsilon)) + f(\varepsilon)(|\delta a_{L_2(V)}(u_0) + \|l_0\|_{L(V)})\|v_{\varepsilon}\|_V). \quad \Box
\]

Consider now a cost function \( j(\varepsilon) = J(u_{\varepsilon}) \), where the functional \( J \) satisfies

\[
(2.5) 
\begin{align*}
J(u + h) = J(u) + \Re(L_u(h)) + o(\|h\|) \quad \forall u, h \in V,
\end{align*}
\]

where \( L_u \) is a linear and continuous form on \( V \).
For $\varepsilon \geq 0$, we define the Lagrangian operator $\mathcal{L}_\varepsilon$ by

$$
\mathcal{L}_\varepsilon(u, v) = J(u) + a_\varepsilon(u, v) - l_\varepsilon(v) \quad \forall u, v \in \mathcal{V}.
$$

The next theorem gives the asymptotic expansion of $j(\varepsilon)$.

**Theorem 2.2.** If Hypotheses 1 and 2 are satisfied, then

(2.6) \hspace{1cm} j(\varepsilon) - j(0) = f(\varepsilon)\Re(\delta u_0, p_0) + o(f(\varepsilon)),

where $u_0$ is the solution to (2.4) with $\varepsilon = 0$, and $p_0$ is the solution to the following adjoint problem: Find $p_0 \in \mathcal{V}$ such that

(2.7) \hspace{1cm} a_0(v, p_0) = -L_{u_0}(v) \quad \forall v \in \mathcal{V}

and

$$
\delta u_0 (u, v) = \delta a(u, v) - \delta_l(v) \quad \forall u, v \in \mathcal{V}.
$$

**Proof.** We have that

$$
J(u_\varepsilon) - J(u_0) = \Re(L_{u_0}(u_\varepsilon - u_0)) + o(\|u_\varepsilon - u_0\|).
$$

Next, choosing $v = p_0$, we obtain

$$
j(\varepsilon) - j(0) = \mathcal{L}_\varepsilon(u_\varepsilon, p_0) - \mathcal{L}_0(u_0, p_0)
= J(u_\varepsilon) - J(u_0) + a_\varepsilon(u_\varepsilon, p_0) - a_0(u_0, p_0) + l_\varepsilon(p_0) - l_0(p_0)
= J(u_\varepsilon) - J(u_0) + \Re(a_\varepsilon(u_\varepsilon, p_0) - a_0(u_0, p_0)) - \Re(l_\varepsilon(p_0) - l_0(p_0))
= J(u_\varepsilon) - J(u_0) + \Re(a_\varepsilon(u_\varepsilon, p_0) - a_0(u_\varepsilon, p_0) + a_0(u_\varepsilon - u_0, p_0))
= \Re(l_\varepsilon(p_0) - l_0(p_0) - f(\varepsilon)\delta_l(p_0)) - f(\varepsilon)\Re(\delta_l(p_0)).
$$

Using (2.5), we have that

$$
J(u_\varepsilon) - J(u_0) = \Re(L_{u_0}(u_\varepsilon - u_0)) + o(\|u_\varepsilon - u_0\|).
$$

Hence,

$$
j(\varepsilon) - j(0) = \Re(a_\varepsilon(u_\varepsilon, p_0) - a_0(u_\varepsilon, p_0)) + \Re(a_0(u_\varepsilon - u_0, p_0) + L_{u_0}(u_\varepsilon - u_0))
= \Re(l_\varepsilon(p_0) - l_0(p_0) - f(\varepsilon)\delta_l(p_0)) - f(\varepsilon)\Re(\delta_l(p_0)).
$$

Using that $p_0$ is the adjoint solution, we obtain

$$
j(\varepsilon) - j(0) = \Re(a_\varepsilon(u_\varepsilon, p_0) - a_0(u_\varepsilon, p_0)) + o(\|u_\varepsilon - u_0\|)
= \Re(l_\varepsilon(p_0) - l_0(p_0) - f(\varepsilon)\delta_l(p_0)) - f(\varepsilon)\Re(\delta_l(p_0))
= \Re((a_\varepsilon - a_0)(u_\varepsilon, p_0)) + \Re((a_\varepsilon - a_0)(u_\varepsilon - u_0, p_0)) + o(\|u_\varepsilon - u_0\|)
= \Re(l_\varepsilon(p_0) - l_0(p_0) - f(\varepsilon)\delta_l(p_0)) - f(\varepsilon)\Re(\delta_l(p_0)).
$$

It follows from Hypothesis 1 that

$$
j(\varepsilon) - j(0) = f(\varepsilon)\Re(\delta a(u_0, p_0)) + o(f(\varepsilon)) + f(\varepsilon)\Re(\delta a(u_\varepsilon - u_0, p_0)) + o(f(\varepsilon))\|u_\varepsilon - u_0\|
= f(\varepsilon)\Re(\delta a(u_0, p_0) - \delta_l(p_0)) + o(f(\varepsilon)),
$$

Finally, from Lemma 2.1 and the hypothesis $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$, we have

$$
j(\varepsilon) = j(0) + f(\varepsilon)\Re(\delta a(u_0, p_0) - \delta_l(p_0)) + o(f(\varepsilon)),$$

since $\delta a$ is continuous by assumption. \qed
3. The Helmholtz problem in a domain with a small hole. Let $\Omega$ be an open and bounded subset of $\mathbb{R}^n$ with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, $n = 2$ or 3. The Helmholtz problem is

\[
\begin{cases}
\Delta u_{\Omega} + k^2 u_{\Omega} = 0 & \text{in } \Omega, \\
u_{\Omega} = 0 & \text{on } \Gamma_0, \\
\frac{\partial u_{\Omega}}{\partial n} = \Lambda u_{\Omega} + \Theta & \text{on } \Gamma_1,
\end{cases}
\]

where $k \in \mathbb{R}^*$, $\Theta \in H_{00}^{\frac{1}{2}}(\Gamma_1)'$, and $\Lambda \in \mathcal{L}(H_{00}^{\frac{1}{2}}(\Gamma_1), H_{00}^{\frac{1}{2}}(\Gamma_1)'$).

We define

\[
\begin{align*}
V(\Omega) &= \{ v \in H^1(\Omega), v = 0 \text{ on } \Gamma_0 \}, \\
a(\Omega, u, v) &= \int_{\Omega} \nabla u \cdot \nabla v \, dx - k^2 \int_{\Omega} u v \, dx - \langle \Lambda u, v \rangle, \\
\ell(v) &= \langle \Theta, v \rangle,
\end{align*}
\]

where $\langle \cdot, \cdot \rangle$ is the duality product between $H_{00}^{\frac{1}{2}}(\Gamma_1)'$ and $H_{00}^{\frac{1}{2}}(\Gamma_1)$. The variational formulation associated with (3.1) is the following: Find $u_{\Omega} \in V(\Omega)$ such that

\[
a(\Omega, u_{\Omega}, v) = \ell(v) \quad \forall v \in V(\Omega).
\]

We consider the following assumption.

**Hypothesis 3.** The operator $\Lambda$ is split into $\Lambda_0 + \Lambda_1$, with $\Lambda_1 \in \mathcal{L}(H_{00}^{\frac{1}{2}}(\Gamma_1), H_{00}^{\frac{1}{2}}(\Gamma_1)'$), and satisfies

\[
\Re \langle \Lambda_1 \psi, \overline{\psi} \rangle \leq 0 \quad \forall \psi \in H_{00}^{\frac{1}{2}}(\Gamma_1),
\]

and $\Lambda_2 \in \mathcal{L}(H_{00}^{\frac{1}{2}}(\Gamma_1), H_{00}^{\frac{1}{2}}(\Gamma_1))$. We assume the following property of uniqueness.

**Hypothesis 4.** We have

\[
\begin{align*}
a(\Omega, u, v) &= 0 \quad \forall v \in V(\Omega) \Rightarrow u = 0, \\
a(\Omega, u, v) &= 0 \quad \forall u \in V(\Omega) \Rightarrow v = 0.
\end{align*}
\]

From the Lax–Milgram theorem and the fact that the imbeddings $V(\Omega) \rightarrow L^2(\Omega)$ and $H_{00}^{\frac{1}{2}}(\Gamma_1) \rightarrow L^2(\Gamma_1)$ are compact, and due to the Fredholm alternative, we obtain the following result (see, e.g., [10] for a detailed argument).

**Proposition 3.1.** If Hypotheses 3 and 4 are satisfied, we have the following:

1. Problem (3.3) has one and only one solution.

2. The sesquilinear form $a(\Omega, \cdot, \cdot)$ satisfies the inf-sup condition: There exists a constant $a > 0$ such that

\[
\inf_{u \neq 0} \sup_{v \neq 0} \frac{|a(\Omega, u, v)|}{\|u\|_{V(\Omega)} \|v\|_{V(\Omega)}} \geq a.
\]

For a given $x_0 \in \Omega$, consider the modified open subset $\Omega_\varepsilon = \Omega \setminus \overline{\omega_\varepsilon}$, $\omega_\varepsilon = x_0 + \varepsilon \omega$, where $\omega$ is a fixed open and bounded subset of $\mathbb{R}^n$ containing the origin ($\omega_\varepsilon = \emptyset$ if $\varepsilon = 0$), whose boundary $\partial \omega$ is connected and piecewise of class $C^1$. The modified solution $u_{\Omega_\varepsilon}$ satisfies

\[
\begin{cases}
\Delta u_{\Omega_\varepsilon} + k^2 u_{\Omega_\varepsilon} = 0 & \text{in } \Omega_\varepsilon, \\
u_{\Omega_\varepsilon} = 0 & \text{on } \Gamma_0, \\
\frac{\partial u_{\Omega_\varepsilon}}{\partial n} = \Lambda u_{\Omega_\varepsilon} + \Theta & \text{on } \Gamma_1.
\end{cases}
\]
The function \( u_\Omega \) is defined on the variable open set \( \Omega_\varepsilon \), and thus belongs to a functional space which depends on \( \varepsilon \). Hence, if we want to derive the asymptotic expansion of a function of the form

\[
j(\varepsilon) = J(u_\Omega),
\]

we cannot apply directly the tools of section 2, which require a fixed functional space. For this reason, we use the domain truncation method introduced in [17] to avoid this complication.

4. The truncation method. Let \( R > 0 \) be such that the closed ball \( \overline{B(x_0, R)} \) is included in \( \Omega \). It is supposed throughout this paper that \( \varepsilon \) remains small enough so that \( \omega_\varepsilon \subset B(x_0, R) \). The truncated open subset is defined by

\[
\Omega_R = \Omega \setminus \overline{B(x_0, R)}.
\]

The open subset \( B(x_0, R) \setminus \omega_\varepsilon \) is denoted by \( D_\varepsilon \) (see Figure 4.1). For \( \varphi \in H^{\frac{1}{2}}(\Gamma_R) \) and \( \varepsilon > 0 \), let \( u^\varepsilon \varphi \) be the solution to the following problem: Find \( u^\varepsilon \varphi \) such that

\[
\begin{align*}
\Delta u^\varepsilon \varphi + k^2 u^\varepsilon \varphi &= 0 \quad \text{in } D_\varepsilon, \\
u^\varepsilon \varphi &= 0 \quad \text{on } \partial \omega_\varepsilon, \\
u^\varepsilon &= \varphi \quad \text{on } \Gamma_R,
\end{align*}
\]

(4.2)

where \( \Gamma_R \) is the boundary of the ball \( B(x_0, R) \). For \( \varepsilon = 0 \), \( u^\varepsilon_0 \) is the solution to

\[
\begin{align*}
\Delta u^\varepsilon_0 + k^2 u^\varepsilon_0 &= 0 \quad \text{in } B(x_0, R), \\
u^\varepsilon_0 &= \varphi \quad \text{on } \Gamma_R.
\end{align*}
\]

(4.3)

Using the Poincaré inequality, it can easily be seen that for \( R < \frac{1}{\sqrt{2}|k|}} \), (4.2) has one and only one solution.

For \( \varepsilon \geq 0 \), the Dirichlet-to-Neumann operator \( T_\varepsilon \) is defined by

\[
T_\varepsilon : \ H^{1/2}(\Gamma_R) \quad \longrightarrow \quad H^{-1/2}(\Gamma_R), \\
\varphi \quad \longmapsto \quad T_\varepsilon \varphi = \nabla u^\varepsilon \varphi \cdot n|_{\Gamma_R},
\]

where the normal \( n|_{\Gamma_R} \) is chosen outward to \( D_\varepsilon \) on \( \Gamma_R \) and \( \partial \omega_\varepsilon \).

Fig. 4.1. The truncated domain.
Finally, we define for $\varepsilon \geq 0$ the solution $u_\varepsilon$ to the truncated problem

$$
\begin{align*}
\Delta u_\varepsilon + k^2 u_\varepsilon &= 0 \quad \text{in } \Omega_R, \\
u_\varepsilon &= 0 \quad \text{on } \Gamma_0, \\
\frac{\partial u_\varepsilon}{\partial n} &= \Lambda u_\varepsilon + \Theta \quad \text{on } \Gamma_1, \\
\frac{\partial u_\varepsilon}{\partial n} - T_\varepsilon u_\varepsilon|_{\Gamma_R} &= 0 \quad \text{on } \Gamma_R.
\end{align*}
$$

The variational formulation associated with (4.4) is as follows: Find $u_\varepsilon \in \mathcal{V}_R$ such that

$$
a_\varepsilon (u_\varepsilon, v) = \ell(v) \quad \forall v \in \mathcal{V}_R,
$$

where the functional space $\mathcal{V}_R$ and the sesquilinear form $a_\varepsilon$ are defined by

$$
\mathcal{V}_R = \{ v \in H^1(\Omega_R); v|_{\Gamma_0} = 0 \},
$$

$$
a_\varepsilon (u, v) = \int_{\Omega_R} \nabla u . \nabla v \, dx - k^2 \int_{\Omega_R} u v \, dx - \langle \Lambda u, v \rangle + \int_{\Gamma_R} T_\varepsilon u_{\varepsilon}^{\varepsilon} v \, d\gamma(x).
$$

Here, $\int_{\Gamma_R}$ denotes the duality product between $H^{1/2}(\Gamma_R)$ and $H^{-1/2}(\Gamma_R)$. The following result is standard in PDE theory.

**Proposition 4.1.** Problems (3.8) and (4.4) have a unique solution. Moreover, the restriction to $\Omega_R$ of the solution $u_{\Omega_\varepsilon}$ to (3.8) is the solution $u_\varepsilon$ to (4.4).

We now have at our disposal the fixed Hilbert space $\mathcal{V}_R$ required by section 2. We assume that the following hypothesis holds.

**Hypothesis 5.** The function $J$ introduced in (3.9) is defined in a neighboring part of $\Gamma$ and satisfies

$$
J(u + h) = J(u) + \mathcal{R}(L_u(h)) + o(||h||) \quad \forall u, h \in \mathcal{V}_R,
$$

where $L_u$ is a linear and continuous form on $\mathcal{V}_R$.

Then we obtain that

$$
j(\varepsilon) = J(u_{\Omega_\varepsilon}) = J(u_\varepsilon) \quad \forall \varepsilon \geq 0.
$$

**Remark 1.** We can also consider a more general cost function (see, e.g., [9]); the truncation method does not restrict the choice of the function. In the numerical part of this work, only measurements on the boundary of the domain are used. For this reason and to simplify the presentation, we considered the previous assumption about the cost function.

Let $v_\Omega$ be the solution to the adjoint problem

$$
a(\Omega, w, v_\Omega) = -L_{w_\Omega} (w) \quad \forall w \in \mathcal{V}(\Omega),
$$

where the functional space $\mathcal{V}(\Omega)$ and the sesquilinear form $a(\Omega, \ldots)$ are defined in (3.2). It has been shown in Proposition 4.1 that $u_0$ is the restriction to $\Omega_R$ of $u_{\Omega_\varepsilon}$. Similarly, $v_0$, the solution to

$$
a_0 (w, v_0) = -L_{w_0} (w) \quad \forall w \in \mathcal{V}_R,
$$

is the restriction to $\Omega_R$ of $v_{\Omega_\varepsilon}$. 
5. The main results. This section contains the main results of this paper. All the proofs are reported in section 6. Henceforth, we have to distinguish between the cases \( n = 2 \) and \( n = 3 \). This is due to the fact that the fundamental solutions to the Laplace equation in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) have an essentially different asymptotic expansion at infinity, and (5.1) has generally no solution if \( n = 2 \).

5.1. The three-dimensional case. Possibly changing the coordinate system, we can suppose for convenience that \( x_0 = 0 \). In order to derive the topological sensitivity of the function \( j \), we introduce two auxiliary problems.

The first problem, called the exterior problem, is formulated in \( \mathbb{R}^3 \setminus \omega \) and consists of finding \( v_\omega \), solution to

\[
\begin{cases}
-\Delta v_\omega = 0 & \text{in } \mathbb{R}^3 \setminus \omega, \\
v_\omega = 0 & \text{at } \infty, \\
v_\omega = u_\Omega(x_0) & \text{on } \partial\omega,
\end{cases}
\]

where \( u_\Omega \) is the solution to the direct problem (3.1). Here, one can remark that just the principal part of the Helmholtz operator is used, which was described by the Laplace equation. The function \( v_\omega \) can be expressed by a single layer potential on \( \partial\omega \).

\[
E(y) = \frac{1}{4\pi r}
\]

with \( r = ||y|| \). It is a fundamental solution for the Laplace equation in \( \mathbb{R}^3 \). Then the function \( v_\omega \) reads

\[
v_\omega(y) = \int_{\partial\omega} E(y - x)p_\omega(x) \, d\gamma(x), \quad y \in \mathbb{R}^3 \setminus \omega,
\]

where \( p_\omega \in H^{-\frac{3}{2}}(\partial\omega) \) is the solution to boundary integral equation

\[
\int_{\partial\omega} E(y - x)p_\omega(x) \, d\gamma(x) = u_\Omega(x_0) \quad \forall y \in \partial\omega.
\]

For \( x \) bounded and large \( r = ||y|| \), we have

\[
E(y - x) = E(y) + O\left(\frac{1}{r^2}\right),
\]

and the asymptotic expansion at infinity of the function \( v_\omega \) is given by

\[
v_\omega(y) = P_\omega(y) + W_\omega(y),
\]

\[
P_\omega(y) = A_\omega(u_\Omega(x_0)) E(y),
\]

\[
A_\omega(u_\Omega(x_0)) = \int_{\partial\omega} p_\omega(x) \, d\gamma(x),
\]

\[
W_\omega(y) = O\left(\frac{1}{r^2}\right).
\]

Notice that \( P_\omega \in L^m_{\text{loc}} \) for all \( m < 3 \). Clearly, the function \( \alpha \mapsto A_\omega(\alpha) \) is linear on \( \mathbb{R} \), and the number \( A_\omega(\alpha) \) depends on the shape of \( \omega \).
The second problem, which we call \textit{interior problem}, is formulated in \( D_0 = B(x_0, R) \) and consists of finding \( Q^1_\omega \) solution to

\begin{equation}
\begin{cases}
\Delta Q^1_\omega + k^2 Q^1_\omega = 0 & \text{in } D_0, \\
Q^2_\omega = P_\omega|_{\Gamma_R} & \text{on } \Gamma_R.
\end{cases}
\end{equation}

Here, the idea is to consider an interior and exterior problem that gives a good “first order approximation” of \((u_\varepsilon^ - u_0^0)|_{D_\varepsilon}, \varphi = u_{\Omega_1}|_{\Gamma_R}\), in the form \( f(\varepsilon)(Q^1_\omega - P_\omega)\), in a way which will be stated precisely in section 6. But the given formulation (5.10) of the interior problem, which is the “natural” choice, is not sufficient to get the behavior needed by the adjoint technique described in section 2. More precisely, in this case one can construct the sesquilinear form \( \delta a \) but there is no positive function \( f(\varepsilon) \) such that \( \| a_\varepsilon - a_0 - f(\varepsilon)\delta a \|_{L_2(V_R)} = o(f(\varepsilon)) \). Indeed, one can observe through the proof of Proposition 6.7 that the behavior of \( \| a_\varepsilon - a_0 - f(\varepsilon)\delta a \|_{L_2(V_R)} \) is not of order \( o(\varepsilon) \), but only of order \( O(\varepsilon) \). This is due to the approximation used on the exterior problem (5.1), where just the principal part of the operator is considered. For this reason, a new term \( Q^2_\omega \) is used in order to correct the error caused by this approximation. We construct \( Q^2_\omega \) as the solution to

\begin{equation}
\begin{cases}
\Delta Q^2_\omega + k^2 Q^2_\omega = k^2 P_\omega & \text{in } D_0, \\
Q^2_\omega = 0 & \text{on } \Gamma_R.
\end{cases}
\end{equation}

Setting \( Q_\omega = Q^1_\omega + Q^2_\omega \), then \( Q_\omega \) is the solution to

\begin{equation}
\begin{cases}
\Delta Q_\omega + k^2 Q_\omega = k^2 P_\omega & \text{in } D_0, \\
Q_\omega = P_\omega|_{\Gamma_R} & \text{on } \Gamma_R.
\end{cases}
\end{equation}

Using the corrected interior problem (5.12), one can derive the good approximation of \((u_\varepsilon^ - u_0^0)|_{D_\varepsilon}. \) The main result is the following, which will be proved in section 6.

\textbf{Theorem 5.1.} Let \( j(\varepsilon) = J(u_{\Omega_1}) \) be a cost function satisfying Hypothesis 5. Then the topological asymptotic expansion is given by

\begin{equation}
\begin{aligned}
\big(j(\varepsilon) - j(0) = \varepsilon \Re \left( A_\omega (u_{\Omega_1}(x_0)) v_{\Omega_1}(x_0) \right) + o(\varepsilon),
\end{aligned}
\end{equation}

where \( u_{\Omega_1} \) is the direct state solution to (3.1) and \( v_{\Omega_1} \) is the adjoint state solution to (4.9).

Then the topological gradient is given by

\begin{equation}
g(x) = \Re \left( A_\omega (u_{\Omega_1}(x)) \overline{v_{\Omega_1}(x)} \right) \quad \forall x \in \Omega,
\end{equation}

and only two systems must be solved in order to compute \( g(x) \) for all \( x \in \Omega \).

When \( \omega \) is the unit ball \( B(0, 1) \), then \( v_\omega(y) \), \( P_\omega(y) \), and \( W_\omega(y) \) can be computed explicitly:

\begin{equation}
v_\omega(y) = \frac{u_{\Omega_1}(x_0)}{r} = P_\omega(y), \quad W_\omega(y) = 0, \quad 0 \neq y \in \mathbb{R}^3.
\end{equation}

Then it follows from (5.2) and (5.7) that

\begin{equation}
A_\omega (u_{\Omega_1}(x_0)) = 4\pi u_{\Omega_1}(x_0).
\end{equation}

We have the following result.

\textbf{Corollary 5.2.} \textit{Under the assumptions of Theorem 5.1 and when \( \omega \) is the unit ball \( B(0, 1) \), the topological asymptotic expansion is given by}

\begin{equation}
\big(j(\varepsilon) - j(0) = 4\pi \varepsilon \Re \left( u_{\Omega_1}(x_0) \overline{v_{\Omega_1}(x_0)} \right) + o(\varepsilon).
\end{equation}
5.2. The two-dimensional case. In this section, we intend to derive the asymptotic expansion of the function \( j \) in the two-dimensional case. The technique used is similar to that of the three-dimensional case. We use the principal part of the Helmholtz operator to derive the topological sensitivity expression. Next, we briefly describe the transposition of the previous results to the two-dimensional case. As before, \( u_\Omega \) and the adjoint state \( v_\Omega \) are, respectively, the solutions to (3.1) and (4.9).

The exterior problem must now be defined differently than in (5.1). It consists of finding \( v_\omega \), the solution to
\[
\begin{cases}
-\Delta v_\omega = 0 & \text{in } \mathbb{R}^2 \setminus \omega, \\
v_\omega(y)/\log r = u_\Omega(x_0) & \text{at } \infty, \\
v_\omega = 0 & \text{on } \partial \omega.
\end{cases}
\]

A fundamental solution for the Laplace equation in \( \mathbb{R}^2 \) is given by
\[
E(y) = -\frac{1}{2\pi} \log r.
\]

The function \( v_\omega \) has the form
\[
v_\omega(y) = u_\Omega(x_0)\log \|y\| + P_\omega + W_\omega(y),
\]
where \( P_\omega \) is constant and \( W_\omega(y) = o(1) \) at infinity \([9]\). In the next proposition (where \( \omega \) is not supposed to be a ball), one can observe that in the two-dimensional case the topological sensitivity does not depend on the shape of the hole \( \omega \), in contrast to the three-dimensional case.

**Theorem 5.3.** The assumptions are the same as in Theorem 5.1. The function \( j \) has the asymptotic expansion
\[
j(\varepsilon) = j(0) - \frac{2\pi}{\log \varepsilon} \mathbb{R} \left( u_\Omega(x_0) v_\Omega(x_0) \right) + o \left( \frac{1}{\log \varepsilon} \right).
\]

The proof for the two-dimensional case uses the same tools as the three-dimensional case (see section 6) and will not be repeated.

6. Proofs. This section consists of the proof of Theorem 5.1. The variation of the sesquilinear form \( a_\varepsilon \) reads
\[
a_\varepsilon(u,v) - a_0(u,v) = \int_{\Gamma_R} (T_\varepsilon - T_0) u v \, d\gamma(x).
\]

Hence, the problem reduces to the analysis of \((T_\varepsilon - T_0)\varphi\) for \( \varphi \in H^{\frac{1}{2}}(\Gamma_R) \). More precisely, it will be shown that there exists an operator \( \delta T \in \mathcal{L}(H^{\frac{1}{2}}(\Gamma_R), H^{-\frac{1}{2}}(\Gamma_R)) \) such that
\[
\|T_\varepsilon - T_0 - \varepsilon \delta T\|_{\mathcal{L}(H^{\frac{1}{2}}(\Gamma_R), H^{-\frac{1}{2}}(\Gamma_R))} = O(\varepsilon^{3/2}).
\]

Consequently, defining \( \delta a \) by
\[
\delta a(u,v) = \int_{\Gamma_R} \delta T u v \, d\gamma(x) \quad \forall u,v \in \mathcal{V}_R
\]
will yield straightforwardly
\[
\|a_\varepsilon - a_0 - \varepsilon \delta a\|_{\mathcal{L}(H^{\frac{1}{2}}(\Gamma_R), H^{-\frac{1}{2}}(\Gamma_R))} = O(\varepsilon^{3/2}).
\]

First we need some definitions and preliminary lemmas.
6.1. Definitions. For convenience, the following norms and seminorms are chosen for the functional spaces which will be used.

- For a bounded and open subset \( O \subset \mathbb{R}^3 \) and \( m \geq 0 \), the Sobolev space \( H^m(O) \) is equipped with the norm defined by
  \[
  \|u\|_{m,O}^2 = \sum_{j=0}^{m} |u_j|_{j,O}^2,
  \]
  where the seminorms \( |u_j|_{j,O} \) are given by
  \[
  |u_j|_{j,O}^2 = \sum_{|\alpha|=j} \int_O |\partial^{\alpha} u|^2 \, dx.
  \]

- For a given \( \varepsilon > 0 \), the space \( H^{1/2}(\Gamma_{R/\varepsilon}) \) is equipped with the norm
  \[
  \|u\|_{1/2,\Gamma_{R/\varepsilon}} = \inf\{\|v\|_{1,C(R/2\varepsilon,R/\varepsilon)}; \ v|_{\Gamma_{R/\varepsilon}} = u\},
  \]
  where \( C(r,r') = \{x \in \mathbb{R}^3; r < ||x|| < r'\} \).

- The dual space \( H^{-1/2}(\Gamma_{R/\varepsilon}) \) is equipped with the natural norm
  \[
  \|w\|_{-1/2,\Gamma_{R/\varepsilon}} = \sup\{\langle w, v \rangle_{-1/2,1/2}; \ v \in H^{1/2}(\Gamma_{R/\varepsilon}); \|v\|_{1/2,\Gamma_{R/\varepsilon}} = 1\},
  \]
  where \( \langle , \rangle_{-1/2,1/2} \) is the duality product between \( H^{1/2}(\Gamma_{R/\varepsilon}) \) and \( H^{-1/2}(\Gamma_{R/\varepsilon}) \).

6.2. Preliminary lemmas. Recall that \( x_0 = 0 \). We will use extensively the following change of variable: For a given function \( u \) defined on a subset \( O \), the function \( \tilde{u} \) is defined on \( \tilde{O} = O/\varepsilon \) by

\[
\tilde{u}(y) = u(x), \quad y = \frac{x}{\varepsilon}.
\]

**Lemma 6.1.** We have that

\[
|u|_{1,O} = \varepsilon^{1/2}|\tilde{u}|_{1,\tilde{O}},
\]

\[
\|u\|_{0,O} = \varepsilon^{3/2}\|\tilde{u}\|_{0,\tilde{O}}.
\]

**Proof.** Due to \( \nabla u(x) = \nabla \tilde{u}(y)/\varepsilon \) and to definition (6.5), we have

\[
|u|^2_{1,O} = \int_O |\nabla u|^2 \, dx = \frac{1}{\varepsilon^2} \int_{\tilde{O}} |\nabla \tilde{u}|^2 \varepsilon^3 \, dy.
\]

Similarly, we have

\[
\|u\|_{0,O} = \varepsilon^{3/2}\|\tilde{u}\|_{0,\tilde{O}}.
\]

**Lemma 6.2** (see [9]). For \( \varphi \in H^{1/2}(\partial \omega) \), let \( v \) be the solution to the problem

\[
\begin{aligned}
-\Delta v &= 0 \quad \text{in } \mathbb{R}^3 \setminus \omega, \\
v &= 0 \quad \text{at } \infty, \\
v &= \varphi \quad \text{on } \partial \omega.
\end{aligned}
\]
The function $v$ is split into
\[ v(y) = V(y) + W(y), \]
\[ V(y) = E(y) \int_{\partial\omega} p(x) \, d\gamma(x), \]
where $E(y) = \frac{1}{2n|y|}$ and $p \in H^{-\frac{1}{2}}(\partial\omega)$ is the unique solution to
\[ \int_{\partial\omega} E(y-x)p(x) \, d\gamma(x) = \varphi(y) \quad \forall y \in \partial\omega. \]

There exists a constant $c > 0$ (independent of $\varphi$ and $\varepsilon$) such that
\[
\| V \|_{0,C(R/2\varepsilon,R/\varepsilon)} \leq c \varepsilon^{-1/2} \| \varphi \|_{\frac{1}{2},\partial\omega},
\| V \|_{1,C(R/2\varepsilon,R/\varepsilon)} \leq c \varepsilon^{1/2} \| \varphi \|_{\frac{1}{2},\partial\omega},
\| V \|_{0,D_y/R,\varepsilon} \leq c \varepsilon^{-1/2} \| \varphi \|_{\frac{1}{2},\partial\omega},
\| V \|_{1,D_y/R,\varepsilon} \leq c \| \varphi \|_{\frac{1}{2},\partial\omega},
\| W \|_{0,C(R/2\varepsilon,R/\varepsilon)} \leq c \varepsilon^{1/2} \| \varphi \|_{\frac{1}{2},\partial\omega},
\| W \|_{1,C(R/2\varepsilon,R/\varepsilon)} \leq c \varepsilon^{3/2} \| \varphi \|_{\frac{1}{2},\partial\omega},
\| W \|_{0,D_y/R,\varepsilon} \leq c \| \varphi \|_{\frac{1}{2},\partial\omega}.
\]

Lemma 6.3. We assume that $R < \frac{1}{\sqrt{2|R|}}$. For a given $\varepsilon > 0$, $f_\varepsilon \in L^2(D_\varepsilon)$, and $\varphi \in H^{\frac{1}{2}}(\Gamma_R)$, let $v_\varepsilon$ be the solution to
\[
\begin{cases}
\Delta v_\varepsilon + k^2 v_\varepsilon = f_\varepsilon & \text{in } D_\varepsilon, \\
v_\varepsilon = 0 & \text{on } \partial D_\varepsilon, \\
v_\varepsilon = \varphi & \text{on } \Gamma_R.
\end{cases}
\]

There exists a constant $C(R,k) > 0$ (independent of $\varphi$ and $\varepsilon$) such that
\[ \| v_\varepsilon \|_{1,D_\varepsilon} \leq C(R,k) \left( \| \varphi \|_{\frac{1}{2},\Gamma_R} + \| f_\varepsilon \|_{0,D_\varepsilon} \right). \]

Proof. Let $\mathcal{R}\varphi$ be the lifting of $\varphi$ in the space $H^1(C(R/2,R))$ such that $\mathcal{R}\varphi |_{\Gamma_{R/2}} = 0$. We extend $\mathcal{R}\varphi$ by zero to the domain $D_\varepsilon$. We denote this extension by $\mathcal{R}\varphi$. It belongs to $H^1(D_\varepsilon)$. We introduce
\[
\begin{aligned}
u_\varepsilon &= \mathcal{R}\varphi - v_\varepsilon, \\
g_\varepsilon &= -f_\varepsilon + \Delta \mathcal{R}\varphi + k^2 \mathcal{R}\varphi.
\end{aligned}
\]
The function $g_\varepsilon$ belongs to the space $H^{-1}(D_\varepsilon)$ and the new unknown $u_\varepsilon$ is the solution to
\[
\begin{cases}
\Delta u_\varepsilon + k^2 u_\varepsilon = g_\varepsilon & \text{in } D_\varepsilon, \\
u_\varepsilon = 0 & \text{on } \partial \omega_\varepsilon, \\
u_\varepsilon = 0 & \text{on } \Gamma_R.
\end{cases}
\]
Using the Poincaré inequality and the elliptic regularity, we obtain
\[ \| u_\varepsilon \|_{1,D_\varepsilon} \leq \left( \frac{1 + 2R^2}{1 - 2k^2R^2} \right) \| g_\varepsilon \|_{-1,D_\varepsilon}. \]
Finally, the result follows from (6.12), (6.13), (6.15), and the continuity of the lifting $\mathcal{R}$.  \(\blacksquare\)
Here and in what follows, we assume that $R < \frac{1}{\sqrt{2}|k|}$.

**Lemma 6.4.** For $\varepsilon > 0$ and $\psi \in H^1(D_0)$, let $X_\varepsilon$ be the solution to the problem

\[
\begin{align*}
\Delta X_\varepsilon + k^2 X_\varepsilon &= 0 \quad \text{in } D_\varepsilon, \\
X_\varepsilon &= \psi \quad \text{on } \partial \omega_\varepsilon, \\
X_\varepsilon &= 0 \quad \text{on } \Gamma_R.
\end{align*}
\]

(6.16)

There exists a constant $c > 0$ (independent of $\varphi$ and $\varepsilon$) such that for all $\varepsilon > 0$,

\[
\begin{align*}
|X_\varepsilon|_{1, C(R/2, R)} &\leq c\varepsilon \|\psi(\varepsilon y)\|_{\frac{1}{2}, \partial \omega}, \\
\|X_\varepsilon\|_{0, D_\varepsilon} &\leq c\varepsilon \|\psi(\varepsilon y)\|_{\frac{1}{2}, \partial \omega}, \\
|X_\varepsilon|_{1, D_\varepsilon} &\leq c\varepsilon^{1/2} \|\psi(\varepsilon y)\|_{\frac{1}{2}, \partial \omega}.
\end{align*}
\]

(6.17) \hspace{1cm} (6.18) \hspace{1cm} (6.19)

**Proof.** Let $\tilde{v}_\varepsilon$ be the solution to the exterior problem

\[
\begin{align*}
-\Delta \tilde{v}_\varepsilon &= 0 \quad \text{in } \mathbb{R}^3 \setminus \omega, \\
\tilde{v}_\varepsilon &= 0 \quad \text{at } \infty, \\
\tilde{v}_\varepsilon &= \psi(\varepsilon y) \quad \text{on } \partial \omega.
\end{align*}
\]

(6.20)

The function $X_\varepsilon$ can be written

\[X_\varepsilon = v_\varepsilon - w_\varepsilon,\]

where $v_\varepsilon(x) = \tilde{v}_\varepsilon \left( \frac{x}{\varepsilon} \right)$. The function $w_\varepsilon$ itself is the solution to

\[
\begin{align*}
\Delta w_\varepsilon + k^2 w_\varepsilon &= k^2 v_\varepsilon \quad \text{in } D_\varepsilon, \\
w_\varepsilon &= 0 \quad \text{on } \partial \omega_\varepsilon, \\
w_\varepsilon &= v_\varepsilon \quad \text{on } \Gamma_R.
\end{align*}
\]

(6.21)

It follows from Lemma 6.3 that there exists a constant $c > 0$ such that

\[\|w_\varepsilon\|_{1, D_\varepsilon} \leq c \left(\|v_\varepsilon|_{\Gamma_R}\|_{\frac{1}{2}, \Gamma_R} + k^2 \|v_\varepsilon\|_{0, D_\varepsilon}\right).\]

(6.22)

It follows from Lemmas 6.1 and 6.2 that

\[
\begin{align*}
\|v_\varepsilon|_{1, D_\varepsilon} &\leq c \left(\|v_\varepsilon|_{\Gamma_R}\|_{\frac{1}{2}, \Gamma_R} \right) \\
&\leq c \left(\|v_\varepsilon\|_{0, C(R/2, R)} + |v_\varepsilon|_{1, C(R/2, R)}\right) \\
&= c \left(\varepsilon^{3/2} \|\tilde{v}_\varepsilon\|_{0, C(R/2, \varepsilon)} + \varepsilon^{1/2} |\tilde{v}_\varepsilon|_{1, C(R/2, \varepsilon)}\right) \\
&\leq c \varepsilon \|\psi(\varepsilon y)\|_{\frac{1}{2}, \partial \omega}.
\end{align*}
\]

(6.23) \hspace{1cm} (6.24) \hspace{1cm} (6.25) \hspace{1cm} (6.26)

We have that

\[
\begin{align*}
\|v_\varepsilon\|_{0, D_\varepsilon} &= \varepsilon^{3/2} \|\tilde{v}_\varepsilon\|_{0, D_\varepsilon/\varepsilon} \\
&\leq c \varepsilon \|\psi(\varepsilon y)\|_{\frac{1}{2}, \partial \omega}.
\end{align*}
\]

(6.27) \hspace{1cm} (6.28)

From (6.22), (6.26), and (6.28), we obtain that

\[\|w_\varepsilon\|_{1, D_\varepsilon} \leq c \varepsilon \|\psi(\varepsilon y)\|_{\frac{1}{2}, \partial \omega}.
\]

(6.29)
Then we have
\begin{align*}
\|X_\varepsilon\|_{0,D_\varepsilon} &\leq \|v_\varepsilon\|_{0,D_\varepsilon} + \|w_\varepsilon\|_{1,D_\varepsilon} \\
\leq &\varepsilon \|\psi(\varepsilon y)\|_{\frac{1}{2},\partial\omega} + \|w_\varepsilon\|_{1,D_\varepsilon} \\
\leq &\varepsilon \|\psi(\varepsilon y)\|_{\frac{1}{2},\partial\omega}.
\end{align*}
(6.34)

This completes the proof. \(\square\)

Lemmas 6.3 and 6.4 are summarized in the following lemma.

**Lemma 6.5.** For \(\varepsilon > 0, \varphi \in H^{\frac{1}{2}}(\Gamma_R), \psi \in H^1(D_0), \) and \(f_\varepsilon \in L^2(D_\varepsilon), \) let \(v_\varepsilon\) be the solution to the problem
\begin{equation}
\begin{cases}
\Delta v_\varepsilon + k^2 v_\varepsilon = f_\varepsilon & \text{in } D_\varepsilon, \\
v_\varepsilon = \psi & \text{on } \partial\omega_\varepsilon, \\
v_\varepsilon = \varphi & \text{on } \Gamma_R.
\end{cases}
\end{equation}
(6.40)

There exists a constant \(c > 0\) (independent of \(\varphi, \psi, f_\varepsilon, \) and \(\varepsilon\)) such that for all \(\varepsilon > 0,\)
\begin{align*}
|v_\varepsilon|_{1,C(R/2,R)} &\leq c\left(\varepsilon \|\psi(\varepsilon y)\|_{\frac{1}{2},\partial\omega}, \|\varphi\|_{\frac{1}{2},\Gamma_R} + \|f_\varepsilon\|_{0,D_\varepsilon}\right), \\
\|v_\varepsilon\|_{0,D_\varepsilon} &\leq c\left(\varepsilon \|\psi(\varepsilon y)\|_{\frac{1}{2},\partial\omega}, \|\varphi\|_{\frac{1}{2},\Gamma_R} + \|f_\varepsilon\|_{0,D_\varepsilon}\right), \\
|v_\varepsilon|_{1,D_\varepsilon} &\leq c\left(\varepsilon \|\psi(\varepsilon y)\|_{\frac{1}{2},\partial\omega}, \|\varphi\|_{\frac{1}{2},\Gamma_R} + \|f_\varepsilon\|_{0,D_\varepsilon}\right).
\end{align*}
(6.41 - 6.43)

**Lemma 6.6.** Let \(u\) belong to the space \(H^1(C(R/2,R))\) and satisfy \(\Delta u + k^2 u = 0\) in \(C(R/2,R), u|_{\Gamma_R} = 0.\) Then there exists a constant \(c > 0\) (independent of \(u\)) such that

\begin{equation}
\|\nabla u_n|_{\Gamma_R} - \frac{1}{2} u|_{\Gamma_R}\| \leq c|u|_{1,C(R/2,R)}.
\end{equation}
(6.44)

**Proof.** Let \(\varphi \in H^{\frac{1}{2}}(\Gamma_R).\) We define \(v\) as the solution to the problem
\begin{equation}
\begin{cases}
\Delta v = 0 & \text{in } C(R/2,R), \\
v = 0 & \text{on } \Gamma_{R/2}, \\
v = \varphi & \text{on } \Gamma_R.
\end{cases}
\end{equation}

Using the Green formula, we obtain
\[
\int_{\Gamma_R} \nabla u_n \nabla v d\gamma(x) = \int_{C(R/2,R)} \nabla u \nabla v dx - k^2 \int_{C(R/2,R)} u v dx.
\]

Then we have
\[
\left|\int_{\Gamma_R} \nabla u_n |_{\Gamma_R} v d\gamma(x)\right| \leq |u|_{1,C(R/2,R)} \|v\|_{1,C(R/2,R)} + k^2 |u|_{0,C(R/2,R)} \|v\|_{1,C(R/2,R)}
\leq |u|_{1,C(R/2,R)} \|\varphi\|_{\frac{1}{2},\Gamma_R} + c k^2 |u|_{1,C(R/2,R)} \|\varphi\|_{\frac{1}{2},\Gamma_R}
\leq c|u|_{1,C(R/2,R)} \|\varphi\|_{\frac{1}{2},\Gamma_R}.
\]

This completes the proof. \(\square\)
6.3. Variation of the sesquilinear form. The variation of the sesquilinear form \( a_\varepsilon \) reads

\[
a_\varepsilon(u, v) - a_0(u, v) = \int_{\Gamma_R} (T_\varepsilon - T_0) u v \, d\gamma(x).
\]

For \( \varphi \in H^{1/2}(\Gamma_R) \), recall that \( u_\varepsilon^\varphi \) is the solution to (4.2), or to (4.3) if \( \varepsilon = 0 \). Let \( v_\varepsilon^\varphi \) be the solution to the problem

\[
\begin{cases}
\Delta v_\varepsilon^\varphi = 0 & \text{in } \mathbb{R}^3 \setminus \omega, \\
v_\varepsilon^\varphi = 0 & \text{at } \infty, \\
v_\varepsilon^\varphi = u_0^\varphi(x_0) & \text{on } \partial \omega.
\end{cases}
\]

As in (5.6) and (5.7), let \( P_\varepsilon^\varphi(y) = A_\omega(u_0^\varphi(x_0)) E(y) \) be the dominant part of \( v_\varepsilon^\varphi \), and let \( Q_\varepsilon^\varphi \) be the solution to the associated interior problem

\[
\begin{cases}
\Delta Q_\varepsilon^\varphi + k^2 Q_\varepsilon^\varphi = k^2 P_\varepsilon^\varphi & \text{in } D_0, \\
Q_\varepsilon^\varphi = P_\varepsilon^\varphi & \text{on } \Gamma_R.
\end{cases}
\]

The linear operator \( \delta T \) (independent of \( \varepsilon \)) is defined as follows:

\[
\delta T : H^{1/2}(\Gamma_R) \longrightarrow H^{-1/2}(\Gamma_R), \quad \varphi \mapsto -\delta T \varphi = \nabla (Q_\varepsilon^\varphi - P_\varepsilon^\varphi).n|_{\Gamma_R}.
\]

Proposition 6.7. The operator \( T_\varepsilon \) admits the following asymptotic expansion:

\[
\| T_\varepsilon - T_0 - \varepsilon \delta T \|_{L(H^{1/2}(\Gamma_R), H^{-1/2}(\Gamma_R))} = O(\varepsilon^{3/2}).
\]

Proof. Let \( \varphi \in H^{1/2}(\Gamma_R) \). For simplicity we drop the superscript \( (\,)^\varphi \). For \( y = x/\varepsilon \), we have

\[
v_\omega(y) = P_\omega(y) + W_\omega(y),
\]

with \( P_\omega(x/\varepsilon) = \varepsilon P_\omega(x) \) and \( W_\omega(y) = O(1/|y|^2) \). Let

\[
\psi_\varepsilon(x) = (T_\varepsilon - T_0 - \varepsilon \delta T) \varphi(x).
\]

We have

\[
\psi_\varepsilon(x) = (\nabla u_\varepsilon - \nabla u_0 - \varepsilon(\nabla Q_\omega - \nabla P_\omega)).n|_{\Gamma_R}
\]

\[
= \nabla \left( w_\varepsilon(x) - W_\omega \left( \frac{x}{\varepsilon} \right) \right).n|_{\Gamma_R},
\]

where \( w_\varepsilon \) is defined by

\[
w_\varepsilon(x) = u_\varepsilon(x) - u_0(x) - \varepsilon Q_\omega(x) + v_\omega \left( \frac{x}{\varepsilon} \right).
\]

The function \( w_\varepsilon \) is the solution to

\[
\begin{cases}
\Delta w_\varepsilon + k^2 w_\varepsilon = k^2 W_\omega(x/\varepsilon) & \text{in } D_\varepsilon, \\
w_\varepsilon = W_\omega(x/\varepsilon) & \text{on } \Gamma_R, \\
w_\varepsilon = -u_0(x) + u_0(0) - \varepsilon Q_\omega(x) & \text{on } \partial \omega_\varepsilon.
\end{cases}
\]

In order to apply Lemma 6.5, we have to estimate the right-hand side terms, as follows.
Finally, it follows from Lemmas 6.1 and 6.6 that
\[ \|W_\omega(x/\varepsilon)\|_{0,D_\varepsilon} = \varepsilon^{3/2} \|W_\omega(y)\|_{0,D_\varepsilon/\varepsilon}. \]

Using Lemma 6.2, we obtain
\[ \|W_\omega(y)\|_{0,D_\varepsilon/\varepsilon} \leq c \|u_0(x_0)\|_{\frac{1}{2},\partial \omega} \leq c \|u_0(x_0)\| \leq c \|\varphi\|_{\frac{1}{2},\Gamma_R}. \]

Then we have
\[ \|W_\omega(x/\varepsilon)\|_{0,D_\varepsilon} \leq c \varepsilon^{3/2} \|\varphi\|_{\frac{1}{2},\Gamma_R}. \]

On \( \Gamma_R \), using Lemmas 6.1 and 6.2 and the elliptic regularity, we obtain
\[
\|W_\omega(x/\varepsilon)\|_{\frac{1}{2},\Gamma_R} \leq c \|W_\omega(x/\varepsilon)\|_{1,C(R/2,R)} \leq c \left( \|W_\omega(x/\varepsilon)\|_{0,C(R/2,R)} + \|W_\omega(x/\varepsilon)\|_{1,C(R/2,R)} \right) = c \left( \varepsilon^{3/2} \|W_\omega(y)\|_{0,C(R/2e,R/\varepsilon)} + \varepsilon^{1/2} W_\omega(y) \right)_{1,C(R/2e,R/\varepsilon)} \right).
\]
\[
\leq c \varepsilon^2 \|u_0(x_0)\|_{\frac{1}{2},\partial \omega} \leq c \varepsilon^2 \|u_0(x_0)\| \leq c \varepsilon^2 \|\varphi\|_{\frac{1}{2},\Gamma_R}.
\]

On \( \partial \omega_\varepsilon \), putting
\[ \theta_\varepsilon(x) = -\frac{u_0(x) + u_0(x_0)}{\varepsilon} - \varepsilon Q_\omega(x), \]
we have for small \( \varepsilon \)
\[
\|\theta_\varepsilon(\varepsilon y)\|_{\frac{1}{2},\partial \omega} \leq c \|\theta_\varepsilon(\varepsilon y)\|_{1,\omega} = c \left| \frac{u_0(\varepsilon y) - u_0(x_0)}{\varepsilon} + Q_\omega(\varepsilon y) \right|_{1,\omega} \leq c \left( \|u_0\|_{C^2(B(0,R/2))} + \|Q_\omega\|_{C^1(B(0,R/2))} \right) \leq c \|\varphi\|_{\frac{1}{2},\Gamma_R}.
\]
We can now apply Lemma 6.5, which gives
\[
|w_\varepsilon|_{1,C(R/2,\varepsilon)} \leq c \left( \varepsilon^{3/2} \|\varphi\|_{\frac{1}{2},\Gamma_R} + \varepsilon^2 \|\varphi\|_{\frac{1}{2},\Gamma_R} + \varepsilon \|\varepsilon \theta_\varepsilon(\varepsilon y)\|_{\frac{1}{2},\partial \omega} \right) \leq c \varepsilon^{3/2} \|\varphi\|_{\frac{1}{2},\Gamma_R}.
\]

Finally, it follows from Lemmas 6.1 and 6.6 that
\[
\|\psi\|_{\frac{1}{2},\Gamma_R} = \|\nabla(w_\varepsilon - W_\omega(x/\varepsilon))_{\partial \Gamma_R} \|_{\frac{1}{2},\Gamma_R} \leq c \left( \|w_\varepsilon\|_{1,C(R/2,\varepsilon)} + \|W_\omega(x/\varepsilon)\|_{1,C(R/2,R)} \right) = c \left( \varepsilon^{3/2} \|\varphi\|_{\frac{1}{2},\Gamma_R} + \varepsilon \|\theta_\varepsilon(\varepsilon y)\|_{\frac{1}{2},\partial \omega} \right) \leq c \varepsilon^{3/2} \|\varphi\|_{\frac{1}{2},\Gamma_R}.
\]
Hence,
\[ \| T_\varepsilon - T_0 - \varepsilon \delta T \|_{L(H^{1/2}(\Gamma_R), H^{-1/2}(\Gamma_R))} = O(\varepsilon^{3/2}). \]

The asymptotic expansion of the sesquilinear form \( a_\varepsilon \) follows now straightforwardly.

**Proposition 6.8.** Let
\[ \delta a(u, v) = \int_{\Gamma_R} \delta T u \overline{v} \, d\gamma(x), \quad u, v \in \mathcal{V}_R. \]
Then the asymptotic expansion of the sesquilinear form \( a_\varepsilon \) is given by
\[ \| a_\varepsilon - a_0 - \varepsilon \delta a \|_{L(H^{1/2}(\Gamma_R), H^{-1/2}(\Gamma_R))} = O(\varepsilon^{3/2}). \]

**6.4. Proof of Theorem 5.1.** The proof of this theorem is done in two steps. First, we prove that Hypothesis 2 is satisfied. More precisely, we prove that the sesquilinear form \( a_0 \) satisfies the inf-sup condition. Second, we apply Theorem 2.2 to compute the topological asymptotic expansion.

**6.4.1. The first step: The inf-sup condition.** For all \( u \in \mathcal{V}_R \), we set
\[ \tilde{u} = \begin{cases} u & \text{in } \Omega_R, \\ u_0^\varphi & \text{in } B(x_0, R), \end{cases} \]
where \( \varphi = u|_{\Gamma_R} \) and \( u_0^\varphi \) is the solution to
\[ \begin{cases} \Delta u_0^\varphi + k^2 u_0^\varphi = 0 & \text{in } B(x_0, R), \\ u_0^\varphi = \varphi & \text{on } \Gamma_R. \end{cases} \]
It can easily be proved that
\[ a_0(u, v|_{\Omega_R}) = a(\Omega, \tilde{u}, v) \quad \forall u \in \mathcal{V}_R \quad \forall v \in \mathcal{V}(\Omega), \]
where the functional space \( \mathcal{V}(\Omega) \) and the sesquilinear form \( a(\Omega, \Omega) \) are defined by (3.2). From Proposition 3.1, the sesquilinear form \( a(\Omega, \Omega) \) satisfies the inf-sup condition. As a consequence, there exists \( v \in \mathcal{V}(\Omega), v \neq 0, \) such that
\[ a_0(u, v|_{\Omega_R}) = a(\Omega, \tilde{u}, v) \geq a(\|\tilde{u}\|_{\mathcal{V}(\Omega)}, \|v\|_{\mathcal{V}(\Omega)}) \geq a(\|u\|_{\mathcal{V}_R}, \|v|_{\Omega_R}\|_{\mathcal{V}_R}). \]
Then \( a_0 \) satisfies the inf-sup condition and Hypothesis 2 is satisfied.

**6.4.2. Applying Theorem 2.2.** All the hypotheses of section 2 are satisfied and we can apply Theorem 2.2. We obtain the following asymptotic formula:
\[ j(\varepsilon) - j(0) = \varepsilon \Re(\delta a(u_\Omega, v_\Omega)) + o(\varepsilon) \]
\[ = \varepsilon \Re \left( \int_{\Gamma_R} \nabla(Q_\varphi - P_\varphi) \cdot n|_{\Gamma_R} \overline{v_\Omega} \, d\gamma(x) \right) + o(\varepsilon), \]
where \( \varphi = u_\Omega|_{\Gamma_R} = u_0|_{\Gamma_R} \). Thanks to Green’s formula and (6.46), we obtain that
\[ \int_{\Gamma_R} \nabla(Q_\varphi - P_\varphi) \cdot n|_{\Gamma_R} \overline{v_\Omega} \, d\gamma(x) = k^2 \int_{\Omega_R} P_\varphi \overline{v_\Omega} \, dx + \int_{\Gamma_R} \nabla v_\Omega \cdot n|_{\Gamma_R} P_\varphi \, d\gamma(x) \]
\[ - \int_{\Gamma_R} \nabla P_\varphi \cdot n|_{\Gamma_R} \overline{v_\Omega} \, d\gamma(x). \]
It can be shown that
\[
\int_{\Gamma_R} \nabla \varpi \cdot n_{|\Gamma_R} P_\omega \, d\gamma(x) - \int_{\Gamma_R} \nabla P_\omega \cdot n_{|\Gamma_R} \varpi \, d\gamma(x) = A_\omega (u_\Omega (x_0)) \langle -\Delta E, \varpi \rangle_{D'(D_0), D(D_0)} - k^2 \int_{D_0} P_\omega \varpi \, dx \\
= A_\omega (u_\Omega (x_0)) \langle \delta, \varpi \rangle_{D'(D_0), D(D_0)} - k^2 \int_{D_0} P_\omega \varpi \, dx \\
= A_\omega (u_\Omega (x_0)) \varpi (x_0) - k^2 \int_{D_0} P_\omega \varpi \, dx,
\]

where \( \psi \in D(D_0) \) satisfies \( \psi(x_0) = 1 \). We insert this expression into (6.49) and obtain the desired result.

7. Numerical results: Buried objects detection. We consider a simple problem of detection of metallic objects buried in soil. The aim is to find the number and the positions of metallic objects (supposedly infinite in the \( \vec{e}_z \) direction) using scattered field measurements from a monostatic antenna horizontally translated above the soil. This is a rough model of the facilities described in [19]. The two-dimensional Helmholtz equation is solved with time-domain finite differences (FDTD), the frequency-domain solution obtained with a Fourier transform. The antenna is roughly approximated by a single source point, which will be translated at various locations above the soil. At each point of the mesh, the topological sensitivity will be computed.

Let \( \mathcal{X} = \{x_i\}_{i=1}^{n_x} \) be the set of the successive locations of the source (and sensors, since the antenna is supposed to be monostatic), and let \( \mathcal{F} = \{f_j\}_{j=1}^{n_f} \) be the set of measurement frequencies. Let \( \varepsilon_s \) be the soil permittivity. The set of metallic objects buried in the soil is denoted by \( \Omega \).

We associate with \( \Omega \) a set of “measurements” \( \mathcal{M}(\Omega) \). At each couple \((x_i, f_j) \in \mathcal{X} \times \mathcal{F}\), we first define the field \( u_{\Omega}^{x_i, f_j} \), the solution of

\[
\begin{cases}
\Delta u + k_j^2 u = s_{x_i} & \text{in } \mathbb{R}^2 \setminus \overline{\Omega}, \\
u = 0 & \text{on } \partial \Omega, \\
\lim_{r \to \infty} \sqrt{r}(\partial_r u -iku) = 0,
\end{cases}
\]

where \( s_{x_i} \) represents a source point centered at \( x_i \), and where

\[
k_j^2(x) = \varepsilon(x) \mu \omega_j^2, \\
w_j = 2\pi f_j, \\
\varepsilon(x) = \begin{cases}
\varepsilon_0 & \text{if } x \geq 0, \\
\varepsilon_s & \text{if } x < 0.
\end{cases}
\]

Then the “measurements” are \( \mathcal{M}(\Omega) = \{m_{x_i, f_j}(\Omega)\} \). In our numerical tests, \( m_{x_i, f_j}(\Omega) \) is the value of the scattered field at point \( x_i \).
Fig. 7.1. Repartition of metallic objects in the soil and the corresponding topological sensitivity computed on empty flat soil (dry soil, flat surface $\varepsilon_r = 2.3$, 20 frequencies ranging from $400 MHz$ to $2 GHz$).

Reference measurements $\mathcal{M} = \{\tilde{m}_{x_i,f_j}\}$ are those values obtained from the real objects in the soil. Ideally, these would have been real measurements, but in the following numerical results, we consider only synthetical data obtained via FDTD.

The cost function, which expresses the adequacy between the measurements obtained for a distribution of metallic objects $\Omega$ and the reference data, is

$$j(\Omega) = \|\mathcal{M} - \tilde{\mathcal{M}}\|^2 = \sum_{i,j} j_{x_i,f_j}(\Omega), \quad (7.2)$$

where

$$j_{x_i,f_j}(\Omega) = |m_{x_i,f_j}(\Omega) - \tilde{m}_{x_i,f_j}|^2. \quad (7.3)$$

Applying the expression of the topological asymptotic (see Proposition 5.3), one has

$$j(\Omega \setminus \overline{B(x, \varepsilon)}) - j(\Omega) = \sum_{i,j} -\frac{2\pi}{\log \varepsilon} \Re \left( u_{x_i,f_j}(x) v_{x_i,f_j}^\Omega(x) \right) + o\left( \frac{1}{\log \varepsilon} \right), \quad (7.4)$$

where $v_{x_i,f_j}^\Omega$ is the adjoint state associated with the couple $(x_i,f_j)$.
The first example (see Figure 7.1) shows the topological sensitivity computed on an “ideal” case: There is no noise on the data, and the reference soil is a flat and homogeneous dry sand soil. One can see that the top of the five objects is clearly identified by the negative values of the topological sensitivity. This topological sensitivity can be obtained very quickly since it is evaluated on an empty flat soil, which is invariant by translation: All direct states and adjoint states are just horizontal translations of a “canonical” solution. The computational cost is only 10 seconds on a 300MHz personal computer.

The second example (see Figure 7.2) is a little more realistic: The data is artificially noised since the reference data $\tilde{M}$ was obtained on a nonflat inhomogeneous soil, while the topological sensitivity was still computed on a flat homogeneous soil. One can observe that, although the objects are still located correctly, the image (see Figure 7.2(b)) is a bit distorted.

The third example shows that using an iterative process might give good results at the expense of some computational cost. In this example, the basic iterative algorithm just inserts a metal point at the point where the topological sensitivity is the most negative. Then the topological sensitivity is reevaluated, taking into account the metal points inserted at previous iterations, etc. Figure 7.3 shows the objects and the metal points that were inserted at iterations 10 and 55.
Fig. 7.3. (a) Redistribution of objects. The measures are computed on a dry flat inhomogeneous soil ($\varepsilon_s = 2.3$), 29 measurement points, and 20 frequencies ranging from 490MHz to 3.29GHz. (b) Iteration 10. (c) Iteration 55.

REFERENCES

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