Induced Matchings in Cubic Graphs

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ABSTRACT

In this paper, we show that the edge set of a cubic graph can always be partitioned into 10 subsets, each of which induces a matching in the graph. This result is a special case of a general conjecture made by Erdős and Nešetřil: For each \( d \geq 3 \), the edge set of a graph of maximum degree \( d \) can always be partitioned into \([5d^2/4]\) subsets each of which induces a matching. © 1993 John Wiley & Sons, Inc.

1. INTRODUCTION

Throughout this paper, we consider colorings of the edges of a graph with positive integers. Formally, a \( t \)-coloring of a graph \( G = (V, E) \) is a map \( \psi: \rightarrow \{1, 2, \ldots, t\} \). A \( t \)-coloring is proper if \( \psi(e) = \psi(f) \) and \( e \neq f \) imply that the edges \( e \) and \( f \) have no common endpoints. Of course, the chromatic index of a graph \( G \) is the least \( t \) for which \( G \) has a proper \( t \)-coloring. Note that whenever \( \psi \) is a proper \( t \)-coloring of a graph \( G = (V, E) \) and \( \alpha \in \{1, 2, \ldots, t\} \), then the edges in \( M = \{e \in E: \psi(e) = \alpha\} \) form a matching in \( G \).

An induced matching \( M \) in a graph \( G = (V, E) \) is a matching such that no two edges of \( M \) are joined by an edge of \( G \). In other words, an induced matching is an induced subgraph in which every vertex has degree one. A

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**strong t-coloring** of $G$ is a proper $t$-coloring such that edges with the same color form an induced matching of $G$. The **strong chromatic index**, $sq(G)$, is the least $t$ for which $G$ has a strong $t$-coloring. For convenience, two edges of a graph $G$ will be called **neighbors** in $G$ if they do not form an induced matching, i.e., if either they are incident (share an end point), or they are joined by an edge. Also, we will use the abbreviation $[t]$ for $\{1, 2, \ldots, t\}$.

At a seminar in Prague at the end of 1985, Erdős and Nešetřil formulated the following Vising-type problem: Given an upper bound for $sq(G)$ in terms of $\Delta(G)$, the maximum degree of $G$. They also conjectured (see [4]) that

$$sq(G) \leq \frac{5}{4} \Delta^2(G). \quad (EN)$$

When $\Delta(G)$ is even, this conjecture, if true, is best possible. When $\Delta(G)$ is odd, it may be possible to improve it by some term that is linear in $\Delta(G)$. In any case, this conjecture appears to be quite difficult. It is easy to see that $(EN)$ is true when $\Delta(G) \leq 2$. In [8], it is shown that $sq(G) \leq 23$ for any graph $G$ with $\Delta(G) = 4$. The trivial upper bound $sq(G) \leq 2\Delta^2(G) - 2\Delta(G) + 1$ follows from the observations that (1) the color of an edge of $G$ is affected only by the color of its neighbors, and (2) the number of neighbors of any edge of $G$ does not exceed $2\Delta^2(G) - 2\Delta(G)$. But even to improve this inequality to $sq(G) \leq (2 - \epsilon)\Delta^2(G)$, for some absolute constant $\epsilon > 0$, seems to be very hard.

It is possible that these difficulties are connected with a result of K. Cameron who proved [2] that the problem of determining whether there is an induced matching of size at least $k$ in $G$ is NP-complete even when $G$ is a bipartite graph. In [3], the authors solved a problem posed by J. Bond and independently by Erdős and Nešetřil: What is the maximum number of edges in a graph in which any two edges are neighbors? They showed that such a graph has at most $\frac{5}{4} \Delta^2(G)$ edges, with a linear term improvement when $\Delta(G)$ is odd. It is reasonable to view this result as providing some evidence that conjecture $(EN)$ is true.

In this paper, we prove conjecture $(EN)$ when $\Delta(G) \leq 3$. In fact, we show

**Theorem.** If $G$ is a graph with $\Delta(G) \leq 3$, then $sq(G) = 10$. \[\blacksquare\]

This theorem answers a specific question posed to us by A. Gyárfás (see also [5] and [6], where many interesting results and problems on the strong chromatic index are stated). The inequality given in our theorem is best possible as there exist graphs with $\Delta(G) \leq 3$ and $sq(G) = 10$. Two such graphs are (1) an 8-gon with all four diagonals, and (2) a 5-gon in which two consecutive vertices have been multiplied by 2. However, it is not clear whether there are infinitely many cubic graphs with this property.

When preparing this paper, we learned that L. Andersen [1] has also obtained the same theorem. Andersen's proof uses different methods and emphasizes algorithmic aspects.
2. PROOF OF THE THEOREM

The proof requires two lemmas. The arguments for these lemmas and for our theorem involve the construction of a strong 10-coloring \( \psi \) of a graph \( G = (V, E) \) in an inductive manner. In most cases, \( \psi \) will be an extension of a strong 10-coloring \( \psi_0 \) of a subgraph (or suitably modified subgraph) of \( G \). Furthermore, we will take care to ensure that it never happens that two edges of \( G \) belong to the subgraph, are not neighbors in the subgraph, but are neighbors in \( G \). We will let \( F \) denote the edges of \( G \) not already assigned colors by \( \psi_0 \). For each \( e \in F \), we let \( S(e) \) denote the set of colors that have not been assigned by \( \psi_0 \) to edges that are neighbors of \( e \) in the graph \( G \).

Hall's theorem asserts that \( \mathcal{A} \) has an SDR if and only if \( |\bigcup \{ \mathbb{S}(e) : e \in F' \}| \geq |F'| \), for every \( F' \subseteq F \). Whenever \( \mathcal{A} \) has an SDR, the task of extending \( \psi_0 \) to a strong 10-coloring \( \psi \) is easy. We just take \( \psi(e) = \alpha_e \), for every \( e \in F \).

In other cases, for a graph \( G = (V, E) \), we will define a family \( \mathcal{A} = \{S(e): e \in E\} \) and argue directly that there exists a strong 10-coloring \( \psi \) with \( \psi(e) \in S(e) \), for all \( e \in E \). Sometimes, this will be accomplished by describing a linear order on \( F \) so that the First Fit algorithm may be applied. This algorithm chooses \( \psi(e) \) to be the least (first) integer in \( S(e) \) not previously assigned to a neighbor of \( e \). Still other cases will require some ad hoc reasoning.

Lemma 1. If \( G = (V, E) \) is a connected graph with \( \Delta(G) \leq 3 \), and \( G \) has a vertex \( v \) with \( 1 \leq \deg(v) \leq 2 \), then \( sq(G) \leq 10 \).

**Proof.** We proceed by induction on \( |E| \), the number of edges in \( G \). The result is trivially true when \( |E| \leq 10 \). Now consider a graph \( G \), and assume that the conclusion of the lemma holds for any graph with fewer edges. Each nontrivial component of \( G - v \) satisfies the inductive hypothesis, so we may choose a strong 10-coloring \( \psi_0 \) of \( G - v \). Let \( F \) denote the set of edges in \( G \) that are incident with \( v \). Each \( e \in F \) has at most 8 neighbors in \( G - v \), so \( |S(e)| \geq 2 \). As \( |F| = \deg(v) \leq 2 \), the family \( \mathcal{A} = \{S(e): e \in F\} \) has an SDR. It follows that \( \psi_0 \) can be extended to a strong 10-coloring \( \psi \) of \( G \).  

For each \( n \geq 3 \), let \( T_n \) be the tree (actually, a caterpillar) with vertex set \( \{v_0, v_1, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_n\} \) and edge set \( \{e_i = v_{i-1}v_i : 1 \leq i \leq n\} \cup \{f_i = v_iu_i : 1 \leq i \leq n\} \).
Lemma 2. Let \( n \geq 3 \) and suppose that every edge \( e \) of the tree \( T_n \) is assigned a set of colors \( S(e) \), so that

1. \( |S(e_1)| \geq 1 \) and \( |S(e_2)| \geq 3 \);
2. \( |S(e_i)| \geq 5 \), for \( i = 3, 4, \ldots, n \);
3. \( |S(f_1)| \geq 2 \);
4. \( |S(f_i)| \geq 4 \), for \( i = 2, 3, \ldots, n - 1 \); and either
5a. \( |S(f_n)| \geq 4 \), or
5b. \( |S(f_n)| \geq 3 \) and \( |S(e_1) \cup S(f_1)| \geq 3 \).

Then there is such a strong coloring \( \psi \) of \( T_n \) so that \( \psi(e) \in S(e) \), for every edge \( e \) in \( T_n \).

**Proof.** In case (5a) holds, the required strong coloring can be obtained by applying First Fit to color the edges of \( T_n \) in the following order:

\[ e_1, f_1, e_2, f_2, e_3, \ldots, e_{n-1}, f_n, e_n, f_n. \]

Note that each edge \( e \) in the tree \( T_n \) has at most at most \( |S(e)| - 1 \) neighbors preceding it in this list.

In case (5b) holds, First Fit works until the very end, but might possibly fail when coloring \( f_n \). Apparently, the three edges \( e_{n-1}, f_{n-1}, \) and \( e_n \) could be assigned three (necessarily distinct) colors from \( S(f_n) \) leaving no satisfactory choice for \( f_n \). So to complete the proof, we argue by induction on \( n \). Suppose first that \( n = 3 \).

Consider any distinct pair \( \alpha, \beta \) with \( \alpha \in S(e_1) \) and \( \beta \in S(f_1) \). Then consider the remaining four sets \( S'(e_2) = S(e_2) - \{\alpha, \beta\} \), \( S'(f_2) = S(f_2) - \{\alpha, \beta\} \), \( S'(e_3) = S(e_3) - \{\alpha, \beta\} \), and \( S'(f_3) = S(f_3) \). These four sets have an SDR unless \( S(f_3) = S'(e_3) \). We may therefore assume that \( |S(e_3)| = 5 \), \( \alpha, \beta \in S(e_3) \), and \( \alpha, \beta \notin S(f_3) \). That \( S(e_1) \cup S(f_1) \) contains at least three elements allows us to repeat this argument for three distinct pairs \( \{\alpha_i, \beta_i\} : i = 1, 2, 3 \) and conclude that each of the (at least three) elements in the union of these pairs belongs to \( S(e_3) - S(f_3) \). But this requires \( |S(e_3)| \geq 6 \). The contradiction completes the proof of the case \( n = 3 \).

For larger values of \( n \), note that the condition \( |S(e_1) \cup S(f_1)| \geq 3 \) allows us to choose \( \alpha = \psi(e_1) \in S(e_1) \) and \( \beta = \psi(f_1) \in S(f_1) \) so that (5b) holds for the remaining edges when \( \alpha \) and \( \beta \) are removed from \( S(e_2), S(f_2) \), and \( S(e_3) \). Since the remaining edges form a copy of the tree \( T_{n-1} \), the proof is complete. \( \square \)

With the two lemmas in hand, we are now ready to present the central part of the proof of the theorem. We proceed by induction on the number of vertices in the graph. Consider a graph \( G = (V, E) \) with \( \Delta(G) \leq 3 \) and assume that the theorem holds for any graph having fewer vertices than \( G \). We show that \( sq(G) \leq 10 \). In view of Lemma 1, we may assume \( G \) is connected and 3-regular. Now let \( n \) denote the girth of \( G \), i.e., the minimum
number of vertices in a cycle in $G$. Choose a minimum length cycle $C$ in $G$ and label the vertices of $C$ as $v_1, v_2, \ldots, v_n$ so that $e_i = v_{i-1}v_i \in E$, for each $i = 2, 3, \ldots, n$, and $e_1 = v_nv_1 \in E$. In what follows, we interpret subscripts cyclically, so that $v_{n+1} = v_1$, etc.

For each $i = 1, 2, \ldots, n$, let $u_i$ be the unique vertex of $V - C$ adjacent to $v_i$. Also, let $U = \{u_1, u_2, \ldots, u_n\}$. Then let $H = (C \cup U, F)$ be the subgraph of $G$ induced by $C \cup U$. Note that the vertices on the cycle $C$ are distinct, but this may not be the case for the vertices in $U$. We may have $u_i = u_j$ when $i \neq j$. In any case, the edges in $\{e_1, e_2, \ldots, e_n\} \cup \{f_1, f_2, \ldots, f_n\}$ are distinct.

According to Lemma 1, there exists a strong 10-coloring $\psi_0$ of $G - H$. For each edge $e \in F$, let $S(e) = \{\alpha \in [10] :$ there is no neighbor of $e$ in $G$ so that $e$ is an edge of $G - H$ mapped by $\psi_0$ to $\alpha\}$. Observe that for each $i = 1, 2, \ldots, n$, the edge $e_i \in F$ has at most 4 neighbors in $G - H$, the edges incident with $u_{i-1}$ and $u_i$. Similarly, for each $i = 1, 2, \ldots, n$, the edge $f_i \in F$ has at most 6 neighbors in $G - H$. Thus, $|S(e_i)| \geq 6$ and $|S(f_i)| \geq 4$, for $i = 1, 2, \ldots, n$. Note that $|S(e_i) \cap S(f_i)| \geq 2$ and $|S(e_i) \cap S(f_{i-1})| \geq 2$, for each $i = 1, 2, \ldots, n$.

It is easy to see that two edges of $G - H$ are neighbors in $G - H$ if and only if they are neighbors in $G$. However, the subgraph $H$ does not have this property in general. The remainder of the argument is divided into 5 cases according to the relative size of the girth of $G$. The basic idea is that we will extend the strong 10-coloring to a strong 10-coloring $\psi \psi_0$ of $G$.

**Case 1.** $n = 3$.

As $|S(e_i)| \geq 6$ and $|S(f_i)| \geq 4$, for each $i = 1, 2, 3$, the family $\mathcal{A} = \{S(e) : e \in F\}$ has an SDR, which we may label as $\{\psi(e) : e \in F\}$. The map $\psi$ is then extended to all of $E$ by setting $\psi(e) = \psi_0(e)$ when $e \in E - F$.

**Case 2.** $n = 4$.

Note that the edges $f_1$ and $f_3$ are neighbors in $G$ if and only if $u_1u_3 \in E$ or $u_1 = u_3$. An analogous statement holds for $f_2$ and $f_4$. All the other pairs of edges of $H$ are neighbors in $H$ and have to be colored by distinct colors. To obtain the desired extension $\psi$, we observe that

(a) If $f_i$ and $f_{i+2}$ are neighbors in $G$, then $|S(e_{i+1}) \cup S(e_{i+2})| \geq 7$;
(b) If $f_1$ and $f_3$ are neighbors, and $f_2$ and $f_4$ are neighbors in $G$, then $|\bigcup_{i=1}^4 S(e_i)| \geq 8$; and
(c) If $f_i$ and $f_{i+2}$ are not neighbors, then either $S(f_i) \cap S(f_{i+2}) \neq \emptyset$, or $|S(f_i) \cup S(f_{i+2})| \geq 8$.

If $\mathcal{A} = \{S(e) : e \in F\}$ has an SDR, then we are done, so we may suppose that it does not. It follows easily that for each $i = 1, 2, f_i$ and $f_{i+2}$ are not
neighbors and that $S(f_i) \cap S(f_{i+2}) \neq \emptyset$. Choose $\alpha \in S(f_i) \cap S(f_3)$. If 
$S(f_2) \cap S(f_4) - \{\alpha\} = \emptyset$, then $\mathcal{A}' = \{S(e) - \{\alpha\} : e \in F - \{f_1, f_3\}\}$ has an SDR. So we may assume that there exists $\beta \neq \alpha$ with $\beta \in S(f_2) \cap S(f_4)$. Then $\mathcal{A}'' = \{S(e_i) - \{\alpha, \beta\} : i = 1, 2, 3, 4\}$ has an SDR.

**Case 3.** $n = 5$.

In this case, we know that the set $U = \{u_1, u_2, \ldots, u_n\}$ is a collection of $n$ distinct vertices. Otherwise, the girth of $G$ would be less than 5. So for each $i = 1, 2, \ldots, 5$, the edges $f_i$ and $e_{i+2}$ are not neighbors in $G$.

Consider the family $\mathcal{B} = \{B_1, B_2, \ldots, B_5\}$, where $B_i = S(f_i) \cap S(e_{i+2})$, for $i = 1, 2, \ldots, 5$. If $\mathcal{B}$ has an SDR, we obtain the desired strong coloring of $H$. If not, choose a maximum size subset $J \subseteq [5]$ for which the subfamily $\mathcal{B}' = \{B_j : j \in J\}$ has an SDR. Let $|J| = w$, and let $W = \{\alpha_j : j \in J\}$ be an SDR of $\mathcal{B}'$. Now let $I = [5] - J$, $F' = \bigcup\{f_j, e_{j+2} : j \in J\}$, $F'' = F - F'$ and $\mathcal{B}'' = \{S(e) - W : e \in F''\}$. Again, if $\mathcal{B}''$ has an SDR, we get the desired strong 10-coloring of $H$.

Suppose therefore that $\mathcal{B}''$ does not have an SDR. Observe that $\mathcal{B}''$ has a total of $10 - 2w$ sets. Of these, there are $5 - w$ sets of the form $S(f_i) - W$. Each of these sets contains at least $4 - w$ elements. Similarly, $\mathcal{B}''$ contains $5 - w$ sets of the form $S(e_{i+2}) - W$, and each of these sets contains at least $6 - w$ elements. From the maximality of $J$, $B_i - W = \emptyset$, for every $i \in I$, and therefore

$$|(S(f_i) - W) \cup (S(e_{i+2}) - W)| \geq 10 - 2w \text{ for all } i \in I. \quad (1)$$

Thus, the fact that $\mathcal{B}''$ does not have an SDR forces $|\bigcup \{S(f_i) - W : i \in I\}| = |I| - 1 = 4 - w$. This implies

1. $|S(f_i)| = 4$, for each $i \in I$,
2. $S(f_{i_1}) - W = S(f_{i_2}) - W$, for all $i_1, i_2 \in I$, and
3. $W \subseteq S(f_i)$, for each $i \in I$.

Furthermore, for each $i = 1, 2, \ldots, 5$, the edges $f_i$ and $f_{i+2}$ are neighbors in $G$ if and only if $u_iu_{i+2} \in E$. But $u_iu_{i+2} \in E$ implies that $f_i$ has at most 5 neighbors in $G - H$, which contradicts (1). Hence, for each $i \in I$, $f_i$ and $f_{i+2}$ are not neighbors in $G$. Similarly, for each $i \in I$, $f_i$ and $f_{i-2}$ are not neighbors in $G$.

Note that $w > 0$, for $w = 0$ requires $S(f_i) \cap S(e_{i+2}) = \emptyset$, for each $i = 1, 2, \ldots, 5$. Clearly, this would imply that $\mathcal{A} = \{S(e) : e \in F\}$ has an SDR. Since $w < 5$, we may assume without loss of generality that $1 \in J$ and $3 \in I$. For $j \in J - \{1\}$, set $\psi(f_j) = \psi(e_{j+2}) = \alpha_j$. Noting that the edges $f_1$ and $f_3$ are not neighbors, we set $\psi(f_1) = \psi(f_3) = \alpha_1$ and $F'' = (F'' - \{f_3\}) \cup \{e_3\}$. The family $\mathcal{B}''' = \{S(e) - W : e \in F''\}$ has cardinality $10 - 2s$ with $4 - s$ sets of cardinality at least $4 - s$ and
6 - s sets of cardinality at least 6 - s. It is easy to see that $B''$ has an SDR, and the existence of $\psi$ follows.

Case 4. $n \geq 7$.

In this case, $H$ is an induced subgraph of $G$. This means two edges of $H$ are neighbors in $H$ if and only if they are neighbors in $G$. Thus, $g_1 = u_1u_4 \in E$ and $g_2 = u_2u_4 \notin E$. We now start with a strong 10-coloring $\psi_0$ of the graph $G'$ formed by adding the edges $g_1$ and $g_2$ to $G - H$. Note that the strong 10-coloring $\psi_0$ exists by Lemma 1. Let $F'$ denote the set of those edges of $G$ not assigned colors by $\psi_0$, and for each $e \in F'$, let $S'(e)$ denote the set of colors which have not been assigned by $\psi_0$ to edges of $G$ that are neighbors of $e$ in $G$. For each $i = 1, 2, \ldots, n$, we denote by $\alpha_i$ and $\beta_i$ the colors of edges of $G - H$ incident with $u_i$. Thus, for example, $S'(e_2) = \{10\} - \{\alpha_1, \beta_1, \alpha_2, \beta_2\}$. Note that $\alpha_i, \beta_i \notin S'(f_i)$, for each $i = 1, 2, \ldots, n$. Also note that the presence of the edges $g_1$ and $g_2$ in $G'$ imply $\{\alpha_i, \beta_i\} \cap \{\alpha_{i+2}, \beta_{i+2}\} = \emptyset$ for $i = 1, 2$. Furthermore, $S'(f_i) \cap S'(f_{i+2}) \neq \emptyset$ for $i = 1, 2$, because (at least) $\psi_0(g_i) \in S'(f_i) \cap S'(f_{i+2})$.

Now we show how to obtain a strong 10-coloring $\psi$ of $G$. We consider three subcase depending on the value of $t = |\{\alpha_1, \beta_1\} \cap \{\alpha_2, \beta_2\}|$.

Subcase 4a. $t = 2$. Set $\psi(f_1) = \psi(f_3) = \alpha = \psi_0(g_1)$. For each edge $e \in F' - \{f_1, f_3\}$, set $S''(e) = S'(e)$, if $e$ is not a neighbor of either $f_1$ or $f_3$; otherwise, set $S''(e) = S'(e) - \{\alpha\}$. Observe that

$$ |S''(f_2)| \geq 3; |S''(f_4)| \geq 3; $$
$$ |S''(f_i)| \geq 4 \text{ for } i = 5, 6, \ldots, n - 1; $$
$$ |S''(e_i)| \geq 5; \text{ for } i = 1, 3, 4, 5, n; \text{ and } $$
$$ |S''(e_i)| \geq 6 \text{ for } i = 6, 7, \ldots, n - 1. $$

Then we remove all elements of $S(f_2)$ from $S''(e_4)$. The edges in $\{e_4, e_5, \ldots, e_n, f_4, f_5, \ldots, f_n\}$ form a copy of $T_{n-3}$ satisfying the hypothesis of Lemma 2, case (5b). Thus, there is a strong 10-coloring $\psi_0$ of this copy of $T_{n-3}$. To define our strong 10-coloring $\psi$, we will set $\psi(e) = \psi_0(e)$ if $e$ is an edge of $G$ assigned a color by $\psi_0$, and we will set $\psi(e) = \psi_0(e)$ if $e$ is an edge in the copy of $T_{n-3}$ colored by $\psi_0$. It remains only to color $e_1, e_2, e_3$, and $f_2$.

For each $e \in F = \{e_1, e_2, e_3, f_2\}$, let $S(e)$ be the set of colors not already assigned to a neighbor of $e$. Since $\{\alpha_1, \beta_1\} = \{\alpha_2, \beta_2\}$, we know that $|S(e_2)| \geq 8$. So, $|S(e_2)| \geq 4$. Similarly, $|S(e_2)| \geq 2$ and $|S(e_n)| \geq 1$. Finally, $|S(f_2)| \geq 3$, because $f_1$ and $f_3$ have been assigned the same color, and the color assigned to $e_4$ does not belong to $S(f_4)$. Therefore, $A = \{S(e) : e \in F\}$ has an SDR, and the map $\psi$ exists.
Subcase 4b. \( t = 1 \). Let \( \alpha_1 = \alpha_2 \). Because the edges \( g_1 \) and \( g_2 \) belong to \( G' \), we know \( \alpha_i \not\in \{ \alpha_i, \beta_i \} \) for \( i = 3, 4 \). So, we set \( \psi(e_4) = \alpha_1 \). Furthermore, we set \( \psi(f_3) = \psi(g_2) \neq \alpha_1 \). As in the preceding case, we let \( S''(e) \) denote the set of colors available to color an edge \( e \) not already colored. If \( \beta_1 \in S''(e_2) \) we remove it. There are at least three other colors in \( S''(e_2) \). Hence the copy of \( T_{n-3} \) formed by \( \{ e_5, e_6, \ldots, e_1, f_5, f_6, \ldots, f_1 \} \) admits a strong 10-coloring \( \psi_0 \) by Lemma 2, case (5b). It remains only to color the edges in \( F = \{ e_2, e_3, f_3 \} \). Now, for each \( e \in F \), we let \( S(e) \) denote the set of colors available to color an edge \( e \) not already colored. If \( E \in S'(e) \), we remove it. There are at least three other colors in \( S'(e) \). Hence the copy of \( T_{n-3} \) formed by \( \{ e_5, e_6, \ldots, e_1, f_5, f_6, \ldots, f_1 \} \) admits a strong 10-coloring \( \psi_0 \) by Lemma 2, case (5b). It remains only to color the edges in \( F = \{ e_2, e_3, f_3 \} \). Now, for each \( e \in F \), we let \( S(e) \) denote the set of colors available to color an edge \( e \) not already colored. If \( E \in S'(e) \), we remove it. There are at least three other colors in \( S'(e) \). Hence the copy of \( T_{n-3} \) formed by \( \{ e_5, e_6, \ldots, e_1, f_5, f_6, \ldots, f_1 \} \) admits a strong 10-coloring \( \psi_0 \) by Lemma 2, case (5b). It remains only to color the edges in \( F = \{ e_2, e_3, f_3 \} \). Now, for each \( e \in F \), we let \( S(e) \) denote the set of colors available to color an edge \( e \) not already colored. If \( E \in S'(e) \), we remove it. There are at least three other colors in \( S'(e) \). Hence the copy of \( T_{n-3} \) formed by \( \{ e_5, e_6, \ldots, e_1, f_5, f_6, \ldots, f_1 \} \) admits a strong 10-coloring \( \psi_0 \) by Lemma 2, case (5b). It remains only to color the edges in \( F = \{ e_2, e_3, f_3 \} \). Now, for each \( e \in F \), we let \( S(e) \) denote the set of colors available to color an edge \( e \) not already colored. If \( E \in S'(e) \), we remove it. There are at least three other colors in \( S'(e) \). Hence the copy of \( T_{n-3} \) formed by \( \{ e_5, e_6, \ldots, e_1, f_5, f_6, \ldots, f_1 \} \) admits a strong 10-coloring \( \psi_0 \) by Lemma 2, case (5b). It remains only to color the edges in \( F = \{ e_2, e_3, f_3 \} \). Now, for each \( e \in F \), we let \( S(e) \) denote the set of colors available to color an edge \( e \) not already colored. If \( E \in S'(e) \), we remove it. There are at least three other colors in \( S'(e) \). Hence the copy of \( T_{n-3} \) formed by \( \{ e_5, e_6, \ldots, e_1, f_5, f_6, \ldots, f_1 \} \) admits a strong 10-coloring \( \psi_0 \) by Lemma 2, case (5b). It remains only to color the edges in \( F = \{ e_2, e_3, f_3 \} \). Now, for each \( e \in F \), we let \( S(e) \) denote the set of colors available to color an edge \( e \) not already colored. If \( E \in S'(e) \), we remove it.
\[ \psi(f_4) = \beta \in S''(f_4) - \{\alpha, \alpha_2, \beta_1\}. \] Note that there are still three colors available for \( e_5 \), because \( \beta_1 = \beta_4 \not\in S''(e_5) \). It follows that we may color the edges in the copy of \( T_{n-2} \) formed by the edges in \( \{e_5, e_6, \ldots, e_2, f_5, f_6, t_1, \ldots, f_2\} \). When this is done, all edges in \( G \) have been colored, and the definition of \( \psi \) may be completed. We may therefore assume that \( \{\alpha_1, \beta_1\} \cap \{\alpha_4, \beta_4\} = \emptyset \).

Finally, suppose that \( \{\alpha_1, \beta_1\} \cap S(f_2) = \{\alpha_1, \beta_1\} \cap S(f_3) = \{\alpha_1, \beta_1\} \cap \{\alpha_4, \beta_4\} = \emptyset \). Set \( \psi(f_2) = \psi(f_4) = \psi_0(g_2) = \alpha, \psi(e_3) = \alpha_1 \) and \( \psi(e_4) = \beta_1 \). We then color the edges in the copy of \( T_{n-3} \) formed by \( \{e_5, e_6, \ldots, e_1, f_5, f_6, \ldots, f_1\} \) using Lemma 2, case (5b). It remains to color the edges in \( F = \{e_2, e_3\} \). Observe that \( |S(e_2)| \geq 1 \) and \( |S(f_3)| \geq 2 \), so the definition of \( \psi \) can be completed.

**Case 5.** \( n = 6 \).

First, observe that whenever \( u_iu_{i+3} \in E \), the edges \( f_i \) and \( f_{i+3} \) are neighbors and have to be colored by distinct colors. But, when this happens, \( f_i \) and \( f_{i+3} \) have at most 5 neighbors in \( G \), and thus \( |S(f_i)| \geq 5 \) and \( |S(f_{i+3})| \geq 5 \).

When \( u_iu_{i+3} \not\in E \), for \( i = 1, 2, 3, H \) is an induced subgraph of \( G \). Furthermore, the proof for the case \( n \geq 7 \) applies to this case as well. Note that in the construction of the copies of trees, the smallest tree considered is \( T_{n-3} \). Since \( n = 6, \) Lemma 2 can still be applied.

Now suppose that \( \{u_iu_{i+3} : i \in [3]\} \cap E \neq \emptyset \) and consider the individual steps in the proof of the case \( n \geq 7 \). Except for the subcase \( t = 0 \), for each \( i = 1, 2, 3 \), the edges \( f_i \) and \( f_{i+3} \) are not colored at the same stage of the definition of \( \psi \). For example, in the subcase \( t = 2 \), at the first stage, we color \( f_1 \) and \( f_3 \); at the second, we color \( T_{n-3} \); and at the third stage, we color \( e_1, e_2, e_3, \) and \( f_2 \). So, when \( u_iu_{i+3} \in E \) and \( f_i \) is colored before \( f_{i+3} \), we remove the color assigned to \( f_i \) from \( S(f_{i+3}) \). The slack provided by the inequality \( |S(f_{i+3})| \geq 5 \) makes this possible.

Finally, in the subcase \( t = 0 \), we observe that there is a unique value of \( i \) for which \( f_i \) and \( f_{i+2} \) belong to the copy of \( T_{n-2} \) that is colored by appeal to Lemma 2. After relabeling, we may assume that the edges in this copy belong to \( F = \{e_1, e_2, e_3, e_4, f_1, f_2, f_3, f_4\} \). If the edge \( u_1u_4 \not\in E \), the original argument works. If \( u_1u_4 \in E \), then we observe that the following inequalities hold: \( |S(e_1)| \geq 3, |S(e_2)| \geq 4, |S(e_3)| \geq 6, |S(e_4)| \geq 5, |S(f_1)| \geq 3, |S(f_2)| \geq 4, |S(f_3)| \geq 4, \) and \( |S(f_4)| \geq 4 \). It is an easy exercise to show that we can define the strong 10-coloring \( \psi(e) \in S(e), \) for each \( e \in F \), given these inequalities. The proof of our theorem is now complete.

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References