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SINGULAR INTEGRALS ON SYMMETRIC SPACES OF REAL RANK ONE

ALEXANDRU D. IONESCU

Abstract
In this paper we prove a new variant of the Herz majorizing principle for operators defined by $\mathbb{K}$-bi-invariant kernels with certain large-scale cancellation properties. As an application, we prove L^p-boundedness of operators defined by Fourier multipliers which satisfy singular differential inequalities of the Hörmander-Michlin type. We also find sharp bounds on the L^p-norm of large imaginary powers of the critical L^p-Laplacian.

1. Introduction

Classical singular integrals on Euclidean spaces are operators defined by Fourier multipliers $m : \mathbb{R}^n \to \mathbb{C}$ which satisfy the Hörmander-Michlin differential inequalities

$$\partial_\xi^j m(\xi) \leq A_j |\xi|^{-|j|} \tag{1.1}$$

for all $\xi \in \mathbb{R}^n \setminus \{0\}$ and any index $j = (j_1, j_2, \ldots, j_n)$. The operator T_m associated to the multiplier m is defined by $\widehat{T_m f} = m \cdot \widehat{f}$ for any Schwartz function f, where $\widehat{f}$ denotes the Fourier transform of f. Alternatively, one can define singular integrals as convolution operators $Tf = f * K$, where the distributions K are appropriate generalizations of the classical Calderon-Zygmund kernels. The operator T_m extends to a bounded operator on $L^p(\mathbb{R}^n)$ for any $p \in (1, \infty)$. The theory of singular integrals extends naturally to the setting of nilpotent Lie groups equipped with nonisotropic dilations (see [22, Chap. 13] and the reference given there).

In this paper we introduce and study an analogue of the singular integral operators in the setting of noncompact symmetric spaces. Let $\mathbb{G}$ be a noncompact connected semisimple Lie group with finite center, let $\mathbb{K}$ be a maximal compact subgroup of $\mathbb{G}$, and let $\mathbb{X} = \mathbb{G}/\mathbb{K}$ be an associated symmetric space. Our notation is standard and is recalled in Section 2. By Plancherel theorem, any bounded W-invariant function m on $\mathfrak{a}^*$ defines a bounded operator on $L^2(\mathbb{X})$ given by $\widetilde{T_m f} = m \cdot \widetilde{f}$. Here $\widetilde{f}$ denotes the

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Fourier transform of the function $f : \mathbb{X} \to \mathbb{C}$, and W is the Weyl group associated to the pair $(\mathfrak{g}, \mathfrak{a})$. The question of L^p-boundedness of operators defined by multipliers is more delicate if $p \neq 2$. A necessary condition for boundedness on L^p of the operator T_m is that the multiplier m extend to a bounded W-invariant holomorphic function in the interior of the tube $\mathscr{T}_p = \mathfrak{a}^* + i\,\mathrm{co}(W \cdot \rho_p)$. Here $\rho_p = |2/p - 1|\rho$, and $\mathrm{co}(W \cdot \rho_p)$ denotes the interior of the convex hull in $\mathfrak{a}^*$ of the set of points $\{w \cdot \rho_p : w \in W\}$. This necessary condition was noticed by J.-L. Clerc and E. Stein [5], who also proved a sufficient condition when the group $\mathbb{G}$ is complex. By analogy with the Euclidean case, a natural theorem is the following: assume that $p \in (1, 2) \cup (2, \infty)$ and that the multiplier m extends to a holomorphic function in the interior of the tube $\mathscr{T}_p$. Assume, in addition, that m satisfies differential inequalities of the form

$$\partial_\xi^j m(\xi) \leq A_j \big(1 + |\xi|\big)^{-|j|} \tag{1.2}$$

for all $\xi \in \mathscr{T}_p$ and any index j. Then the operator T_m extends to a bounded operator on $L^p(\mathbb{X})$ (see [5], [20], [3], [24], [1]). More refined statements (i.e., considering multipliers m with boundary values in certain Sobolev spaces or subject to minimal regularity assumptions) can be found in [1] and [11].

Notice, however, that there is an important difference between the differential inequalities (1.1) and (1.2). The multiplier in (1.1) is not assumed to be smooth at the origin, and this translates into a singularity of the corresponding kernel at infinity (in the sense that this kernel is not absolutely integrable at infinity and its large-scale cancellation plays an essential role). On the other hand, if a multiplier m on a symmetric space satisfies the differential inequalities (1.2), then the large-scale cancellation of the corresponding kernel is irrelevant. In this case, one uses the Herz majorizing principle in [13] or the Kunze-Stein phenomenon to control the L^p-norm of the induced operator. One of our main theorems (Theorem 8) addresses this problem. We consider a certain class of multipliers m that have singularities at the points $w \cdot \rho_p$, $w \in W$, and we prove that the induced operators are still bounded on $L^p(\mathbb{X})$. (This possibility was noticed in [1, Sec. 5].) Our symbols satisfy the differential inequalities (4.1) (for groups $\mathbb{G}$ of real rank one), which are the natural analogues on symmetric spaces of the Hörmander-Michlin differential inequalities (1.1). As an application, we prove sharp estimates on the L^p-norm of large imaginary powers of the critical Laplacian (Theorem 9). The only previously known estimates of this type were established in [9, Cor. 4.2] by a transference technique; we improve slightly on the polynomial power of the exponent and show that this improvement yields the sharp estimate.

This paper is organized as follows. In Section 2 we recall some basic facts related to harmonic analysis on semisimple Lie groups and symmetric spaces. In Section 3 we prove a new transference principle (Theorem 1) and use it to establish a new variant of the Herz criterion for operators defined by certain $\mathbb{K}$-bi-invariant kernels with large scale cancellation (Corollary 2). These kernels can be thought of as analogues of the

Calderon-Zygmund kernels on symmetric spaces. In Sections 4 and 5 we state and prove the two theorems described in the previous paragraph (Theorems 8 and 9).

For simplicity we assume in this paper that the group $\mathbb{G}$ has real rank one. Suitable generalizations of the theorems to symmetric spaces of arbitrary real rank will be discussed in a future work.

2. Preliminaries

Most of our notation related to semisimple Lie groups and symmetric spaces is standard and can be found, for example, in [12]. Let $\mathbb{G}$ be a noncompact connected semisimple Lie group with finite center, let $\mathfrak{g}$ be the Lie algebra of $\mathbb{G}$, let θ be a Cartan involution of $\mathfrak{g}$, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the associated Cartan decomposition. Let $\mathbb{K} = \exp \mathfrak{k}$ be a maximal compact subgroup of $\mathbb{G}$, and let $\mathbb{X} = \mathbb{G}/\mathbb{K}$ be an associated symmetric space. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$; recall that we assumed that the group $\mathbb{G}$ has real rank one, that is, that $\dim \mathfrak{a} = 1$. Let $\mathbb{M}$ be the centralizer of $\exp \mathfrak{a}$ in $\mathbb{K}$, let Σ be the set of nonzero roots of the pair $(\mathfrak{g}, \mathfrak{a})$, and let W be the associated Weyl group. Since $\mathbb{G}$ has real rank one, it is well known that Σ is of the form $\Sigma = \{-\alpha, \alpha\}$ or of the form $\Sigma = \{-2\alpha, -\alpha, \alpha, 2\alpha\}$. Let $\mathfrak{a}^+ = \{H \in \mathfrak{a} : \alpha(H) > 0\}$ be a positive Weyl chamber, and let Σ^+ be the corresponding set of positive roots. (In our case, $\Sigma^+ = \{\alpha\}$ or $\Sigma^+ = \{\alpha, 2\alpha\}$.) For any root $\beta \in \Sigma$, let $\mathfrak{g}_\beta$ be the root space associated to β, let $\mathfrak{n} = \sum_{\beta \in \Sigma^+} \mathfrak{g}_\beta$, and let $\overline{\mathfrak{n}} = \theta(\mathfrak{n})$. Finally, let $\mathbb{N} = \exp \mathfrak{n}$, and let $\overline{\mathbb{N}} = \exp \overline{\mathfrak{n}}$.

The group $\mathbb{G}$ has an Iwasawa decomposition $\mathbb{G} = \mathbb{K}(\exp \mathfrak{a})\mathbb{N}$ and a Cartan decomposition $\mathbb{G} = \mathbb{K}(\exp \overline{\mathfrak{a}^+})\mathbb{K}$. For each $g \in \mathbb{G}$, denote by $H(g) \in \mathfrak{a}$ and $g^+ \in \overline{\mathfrak{a}^+}$ the middle components of g in these decompositions. We also use the Iwasawa decomposition $\mathbb{G} = \overline{\mathbb{N}}(\exp \mathfrak{a})\mathbb{K}$.

Let H_0 be the unique element of $\mathfrak{a}$ with the property that $\alpha(H_0) = 1$, and normalize the Killing form on $\mathfrak{g}$ such that $|H_0| = B(H_0, H_0)^{1/2} = 1$. To simplify the notation, we often identify the Lie subgroup $\exp \mathfrak{a}$ with the real line $\mathbb{R}$ using the map $r \to \exp(r H_0)$. Notice that $\mathbb{R}_+$ is identified with $\exp \overline{\mathfrak{a}^+}$. Let $\rho = (1/2)(m_1 \cdot \alpha + m_2 \cdot 2\alpha)$ and $|\rho| = (1/2)(m_1 + 2m_2)$, where m_1 and m_2 are the dimensions of the root spaces $\mathfrak{g}_{-\alpha}$ and $\mathfrak{g}_{-2\alpha}$, respectively. Let dg, dk, and $d\overline{n}$ be Haar measures on the groups $\mathbb{G}$, $\mathbb{K}$, and $\overline{\mathbb{N}}$. Normalize dk such that $\int_{\mathbb{K}} 1\, dk = 1$, and let dz be the translation invariant measure on the symmetric space $\mathbb{X}$ induced by the measure dg on the group $\mathbb{G}$. After suitable normalizations of the Haar measures dg and $d\overline{n}$, one has the following integral formulae:

$$\int_{\mathbb{G}} F(g)\, dg = c_1 \int_{\mathbb{K}} \int_{\mathbb{K}} \int_{\mathbb{R}_+} F\big(k_1 \exp(t H_0) k_2\big) (\sinh t)^{m_1} (\sinh 2t)^{m_2}\, dt\, dk_1\, dk_2 \tag{2.1}$$

and

$$\int_{\mathbb{G}} F(g)\,dg = \int_{\overline{\mathbb{N}}}\int_{\mathbb{R}}\int_{\mathbb{K}} F\big(\overline{n}\exp(tH_0)k\big)e^{2|\rho|t}\,dk\,dt\,d\overline{n},$$

for any continuous compactly supported function $F : \mathbb{G} \to \mathbb{C}$.

3. A transference theorem

Let $\phi : \mathbb{R} \to \mathbb{C}$ be a function supported in $\mathbb{R}_+$, and let

$$A(r) = \sup_{|r'-r|\leq 1} |\phi(r) - \phi(r')|. \tag{3.1}$$

Assume that ϕ satisfies the following basic assumptions:

$$\lim_{r\to\infty} \phi(r) = 0 \tag{3.2}$$

and

$$\int_{\mathbb{R}} A(r) = A < \infty. \tag{3.3}$$

One should think of ϕ as a Calderon-Zygmund kernel on $\mathbb{R}$ with the singularity at the origin removed. Let p be a fixed exponent in the interval $(1, 2)$, and let $K_{p,\phi} : \mathbb{G} \to \mathbb{C}$ be the $\mathbb{K}$-bi-invariant kernel given by

$$K_{p,\phi}\big(k_1\exp(rH_0)k_2\big) = e^{-2|\rho|r/p}\phi(r) \tag{3.4}$$

for any $r \geq 0$ and $k_1, k_2 \in \mathbb{K}$. Let $|||{*K_{p,\phi}}|||_{L^p(\mathbb{X})}$ denote the norm of the convolution operator defined by the kernel $K_{p,\phi}$ on $L^p(\mathbb{X})$, and let $|||*\phi|||_{L^p(\mathbb{R})}$ denote the norm of the convolution operator defined by the kernel ϕ on $L^p(\mathbb{R})$. Our first main theorem is the following.

THEOREM 1

(i) *There is a constant C_p such that*

$$|||*K_{p,\phi}|||_{L^p(\mathbb{X})} \leq C_p\big(A + |||*\phi|||_{L^p(\mathbb{R})}\big).$$

(ii) *Conversely, one has*

$$|||*\phi|||_{L^p(\mathbb{R})} \leq C_p\big(A + |||*K_{p,\phi}|||_{L^p(\mathbb{X})}\big).$$

Remark. The point of this theorem is to be able to conclude that the two operator norms $|||*K_{p,\phi}|||_{L^p(\mathbb{X})}$ and $|||*\phi|||_{L^p(\mathbb{R})}$ are essentially proportional. This theorem is sharper than the classical transference principle of R. Coifman and G. Weiss [6, Th. 8.7] because the factor that makes the transition between the kernels ϕ and $K_{p,\phi}$ is $e^{2\rho(H)/p}$. Notice that the transition factor in [6, Th. 8.7] is $\Delta(H)$, which is proportional to $e^{2\rho(H)}$ for $\rho(H) \geq 1$.

Throughout this section the constants C_p depend only on p and the group $\mathbb{G}$. The letters c and C are used to denote universal constants depending only on the group $\mathbb{G}$.

As an application, we have a new variant of the Herz majorizing principle for operators defined by $\mathbb{K}$-bi-invariant kernels. Assume that the function $\phi : \mathbb{R} \to \mathbb{C}$ is supported in $\mathbb{R}_+$ and satisfies the differential inequalities

$$\begin{cases} |\phi(r)| \le A'(1+r)^{-1} & \text{for any } r \ge 0, \\ |\phi'(r)| \le A'(1+r)^{-2} & \text{for any } r \ge 0 \end{cases} \tag{3.5}$$

and the cancellation condition

$$\left| \int_0^N \phi(r)\, dr \right| \le A' \tag{3.6}$$

for any $N > 0$. In this case, the function ϕ is a kernel of the Calderon-Zygmund type on the real line, and one has the following consequence of Theorem 1.

COROLLARY 2
If $p \in (1,2)$ and if ϕ satisfies (3.5) and (3.6), then

$$||| * K_{p,\phi} |||_{L^p(\mathbb{X})} \le C_p \cdot A'.$$

This corollary should be compared with the following important criterion known as the Herz majorizing principle (see [13]). Assume that $1 \le p < 2$ and that $p' = p/(p-1)$ is the conjugate exponent of p. Let K be a $\mathbb{K}$-bi-invariant kernel on $\mathbb{G}$. Then

$$||| * K |||_{L^p(\mathbb{G})} = ||| * K |||_{L^p(\mathbb{X})} \le C_p \int_{\mathbb{R}_+} \left| K\big(\exp(rH_0)\big)\right| \delta(r) e^{-2|\rho| r/p'}\, dr, \tag{3.7}$$

where $\delta(r) = (\sinh r)^{m_1} (\sinh 2r)^{m_2}$ is the factor that appears in the integral formula (2.1). For comparison, the L^1-norm of the kernel K is

$$||K||_{L^1(\mathbb{G})} = c_1 \int_{\mathbb{R}_+} \left| K\big(\exp(rH_0)\big)\right| \delta(r)\, dr.$$

The inequality (3.7) is the best possible if the kernel K is positive. On the other hand, Corollary 2 cannot be obtained as a consequence of this inequality since the large-scale cancellation of the kernel ϕ plays a crucial role.

We also remark that both Theorem 1 and Corollary 2 are false if $p = 2$. The rest of this section is devoted to proving Theorem 1.

Proof of Theorem 1
Recall that for any locally integrable $\mathbb{K}$-bi-invariant function K and any smooth compactly supported function $f : \mathbb{X} \to \mathbb{C}$, the convolution $f * K$ is defined by the

formula

$$f * K(z) = \int_{\mathbb{G}} f(h\cdot\mathbf{0})K(h^{-1}\cdot z)\,dh,$$

where $\mathbf{0} = \mathbb{G}/\mathbb{K}$ is the origin of the symmetric space $\mathbb{X}$. Notice that

$$|||*K_{p,\phi}|||_{L^p(\mathbb{X})} = \sup_{||f||_p=||g||_{p'}=1}\Big|\int_{\mathbb{G}}\int_{\mathbb{G}} f(h\cdot\mathbf{0})K_{p,\phi}(h^{-1}h')g(h'\cdot\mathbf{0})\,dh\,dh'\Big|,$$

where the supremum is taken over all smooth compactly supported functions $f, g : \mathbb{X} \to \mathbb{C}$. As usual, $p' = p/(p-1)$ is the conjugate exponent of p. We identify the group $\mathbb{G}$ with $\overline{\mathbb{N}} \times \mathbb{R} \times \mathbb{K}$ using the Iwasawa decomposition $\mathbb{G} = \overline{\mathbb{N}}(\exp\mathfrak{a})\mathbb{K}$ and the identification of $\exp\mathfrak{a}$ with $\mathbb{R}$ described in Section 2. This identifies the symmetric space $\mathbb{X}$ with $\overline{\mathbb{N}} \times \mathbb{R}$. The relevant measure on $\overline{\mathbb{N}} \times \mathbb{R}$ corresponding to this identification is $d\mu = e^{2|\rho|s}\,d\overline{n}\,ds$. To prove part (i) of the theorem, it suffices to prove that for any smooth compactly supported functions $f, g : \overline{\mathbb{N}} \times \mathbb{R} \to \mathbb{C}$ one has

$$\begin{aligned}\Big|\int_{\overline{\mathbb{N}}}\int_{\overline{\mathbb{N}}}\int_{\mathbb{R}}\int_{\mathbb{R}} & f(\overline{m}, s) \\ &\cdot K_{p,\phi}\big(\delta_{-s}(\overline{m}^{-1}\overline{n})\exp((t-s)H_0)\big)g(\overline{n}, t)e^{2|\rho|(s+t)}\,ds\,dt\,d\overline{m}\,d\overline{n}\Big| \\ &\le C_p\big(A + |||*\phi|||_{L^p(\mathbb{R})}\big)||f||_{L^p(\overline{\mathbb{N}}\times\mathbb{R},d\mu)}||g||_{L^{p'}(\overline{\mathbb{N}}\times\mathbb{R},d\mu)}. \end{aligned} \tag{3.8}$$

By definition, $\delta_r(\overline{v}) = (\exp(rH_0))\overline{v}(\exp(-rH_0))$ for any $r \in \mathbb{R}$ and $\overline{v} \in \overline{\mathbb{N}}$. It is well known that δ_r is a dilation of the group $\overline{\mathbb{N}}$. Denote by $I_{p,\phi}(f, g)$ the integral on the left-hand side of (3.8); in order to estimate $|I_{p,\phi}(f, g)|$, we need to understand the connection between the Iwasawa decomposition and the Cartan decomposition of the group $\mathbb{G}$. This idea was used by J.-O. Strömberg in [23].

LEMMA 3
If $\overline{v} \in \overline{\mathbb{N}}$ *and* $r \ge 0$, *then*

$$\big[\overline{v}\exp(rH_0)\big]^+ = rH_0 + H(\overline{v}) + E(\overline{v}, r)H_0, \tag{3.9}$$

where

$$0 \le E(\overline{v}, r) \le 2e^{-2r}. \tag{3.10}$$

Proof
Recall that g^+ and $H(g)$ denote the $\mathfrak{a}$-components of the element $g \in \mathbb{G}$ in the Cartan decomposition and the Iwasawa decomposition of the group $\mathbb{G}$. It follows from Kostant's convexity theorem that

$$\rho\big([\overline{v}\exp(rH_0)]^+ - rH_0 - H(\overline{v})\big) \ge 0.$$

Thus $E(\overline{v}, r) \geq 0$. To prove the upper estimate in (3.10), one can use the explicit formulae in [12, Chap. II, Th. 6.1]:

$$\cosh^2\left(\alpha((\overline{v}\exp(rH_0))^+)\right) = \left[\cosh r + \frac{c_0}{2}e^r|X|^2\right]^2 + c_0e^{2r}|Y|^2$$

and

$$e^{2\alpha(H(\overline{v}))} = \left[1 + c_0|X|^2\right]^2 + 4c_0|Y|^2,$$

where X and Y are the coordinates of $\overline{v}$ in $\overline{\mathbb{N}}$ corresponding to the root spaces $\mathfrak{g}_{-\alpha}$ and $\mathfrak{g}_{-2\alpha}$, and c_0 is a constant. The upper estimate in (3.10) follows from these two equalities and the observation that

$$e^{2[r+\alpha(H(\overline{v}))+E(\overline{v},r)]}/4 \leq \cosh^2\left(\alpha((\overline{v}\exp(rH_0))^+)\right). \qquad \square$$

We use this lemma to estimate the function $(\overline{v}, r) \to K_{p,\phi}(\overline{v}\exp(rH_0))$ for any $\overline{v} \in \overline{\mathbb{N}}$ and $r \in \mathbb{R}_+$. Let $P(\overline{v}) = e^{-\rho(H(\overline{v}))}$; it is well known that for any $\varepsilon > 0$,

$$\int_{\overline{\mathbb{N}}} P(\overline{v})^{(1+\varepsilon)}\, d\overline{v} = C_\varepsilon < \infty. \tag{3.11}$$

For any $r, x \geq 0$, let

$$D\phi(r, x) = \int_r^{r+x} A(y)\, dy,$$

where A is the function defined in (3.1).

PROPOSITION 4

(i) *If* $r \geq 0$ *and* $\overline{v} \in \overline{\mathbb{N}}$, *then*

$$\left|K_{p,\phi}\big(\overline{v}(\exp(rH_0))\big)\right| \leq CAe^{-2|\rho|r/p}P(\overline{v})^{2/p}. \tag{3.12}$$

(ii) *If* $r \geq 0$ *and* $\overline{v} \in \overline{\mathbb{N}}$, *then*

$$K_{p,\phi}\big(\overline{v}(\exp(rH_0))\big) = e^{-2|\rho|r/p}\phi(r)P(\overline{v})^{2/p} + E_{p,\phi}(\overline{v}, r), \tag{3.13}$$

where

$$|E_{p,\phi}(\overline{v}, r)| \leq Ce^{-2|\rho|r/p}\left[Ae^{-2r} + D\phi\big(r, 2 + \alpha(H(\overline{v}))\big)\right]P(\overline{v})^{2/p}. \tag{3.14}$$

Proof

Part (i) follows immediately from definition (3.4) of the kernel $K_{p,\phi}$, Lemma 3, and the observation that $|\phi(r')| \leq 2A$ for any $r' \geq 0$. To prove part (ii) of the proposition, notice that it follows from Lemma 3 that the difference between $K_{p,\phi}\big(\overline{v}(\exp(rH_0))\big)$ and $e^{-2|\rho|r/p}\phi(r)P(\overline{v})^{2/p}$ is equal to

$$e^{-2|\rho|r/p}P(\overline{v})^{2/p}\left[e^{-2|\rho|E(\overline{v},r)/p}\phi\big(r + \alpha(H(\overline{v})) + E(\overline{v}, r)\big) - \phi(r)\right].$$

Estimate (3.14) on this error term follows from estimate (3.10) and the definition of the function $D\phi$. □

Let χ be the characteristic function of the interval $[0,\infty)$. Recall that ϕ is supported in the interval $[0,\infty)$, and decompose the kernel $K_{p,\phi}\big(\delta_{-s}(\overline{m}^{-1}\overline{n})\exp((t-s)H_0)\big)$ in (3.8) into three parts:

$$\begin{aligned}\big(1-\chi(t-s)\big)&K_{p,\phi}\big(\delta_{-s}(\overline{m}^{-1}\overline{n})\exp((t-s)H_0)\big)\\&+\chi(t-s)E_{p,\phi}\big(\delta_{-s}(\overline{m}^{-1}\overline{n}),(t-s)\big)\\&+\phi(t-s)P\big(\delta_{-s}(\overline{m}^{-1}\overline{n})\big)^{2/p}e^{-2|\rho|(t-s)/p}.\end{aligned}$$

This induces a decomposition of the integral $I_{p,\phi}(f,g)$ on the left-hand side of (3.8) as $I_{p,\phi}(f,g)=I^1_{p,\phi}(f,g)+I^2_{p,\phi}(f,g)+I^3_{p,\phi}(f,g)$, where

$$\begin{aligned}I^1_{p,\phi}(f,g)&=\int_{\mathbb{N}}\int_{\mathbb{N}}\int_{\mathbb{R}}\int_{\mathbb{R}}f(\overline{m},s)K_{p,\phi}\big(\delta_{-s}(\overline{m}^{-1}\overline{n})\exp((t-s)H_0)\big)g(\overline{n},t)\\&\quad\cdot e^{2|\rho|(s+t)}\big(1-\chi(t-s)\big)\,ds\,dt\,d\overline{m}\,d\overline{n},\\I^2_{p,\phi}(f,g)&=\int_{\mathbb{N}}\int_{\mathbb{N}}\int_{\mathbb{R}}\int_{\mathbb{R}}f(\overline{m},s)E_{p,\phi}\big(\delta_{-s}(\overline{m}^{-1}\overline{n}),(t-s)\big)g(\overline{n},t)\\&\quad\cdot e^{2|\rho|(s+t)}\chi(t-s)\,ds\,dt\,d\overline{m}\,d\overline{n},\end{aligned}$$

and

$$\begin{aligned}I^3_{p,\phi}(f,g)=\int_{\mathbb{N}}\int_{\mathbb{N}}\int_{\mathbb{R}}\int_{\mathbb{R}}&f(\overline{m},s)P\big(\delta_{-s}(\overline{m}^{-1}\overline{n})\big)^{2/p}e^{-2|\rho|(t-s)/p}\phi(t-s)g(\overline{n},t)\\&\cdot e^{2|\rho|(s+t)}\,ds\,dt\,d\overline{m}\,d\overline{n}.\end{aligned}\tag{3.15}$$

LEMMA 5
One has

$$|I^1_{p,\phi}(f,g)|\le C_p\cdot A||f||_{L^p(\overline{\mathbb{N}}\times\mathbb{R},d\mu)}||g||_{L^{p'}(\overline{\mathbb{N}}\times\mathbb{R},d\mu)}.\tag{3.16}$$

Proof
Let

$$\begin{cases}F_1(s)=\big[\int_{\overline{\mathbb{N}}}|f(\overline{m},s)|^p\,d\overline{m}\big]^{1/p},\\G_1(t)=\big[\int_{\overline{\mathbb{N}}}|g(\overline{n},t)|^{p'}\,d\overline{n}\big]^{1/p'}.\end{cases}$$

By Hölder inequality, the absolute value of $I^1_{p,\phi}(f,g)$ is dominated by

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}}&F_1(s)G_1(t)e^{2|\rho|(s+t)}\big(1-\chi(t-s)\big)\\&\cdot\Big(\int_{\overline{\mathbb{N}}}\big|K_{p,\phi}\big(\delta_{-s}(\overline{v})\exp((t-s)H_0)\big)\big|\,d\overline{v}\Big)\,ds\,dt.\end{aligned}\tag{3.17}$$

Recall that the map $\overline{n}_1 \to \overline{n}_2 = \delta_r(\overline{n}_1)$ is a dilation of $\overline{\mathbb{N}}$ with $d\overline{n}_2 = e^{-2|\rho|r} d\overline{n}_1$. In addition, the Abel transform

$$\mathscr{A}K(H) = e^{\rho(H)} \int_{\overline{\mathbb{N}}} K\big(\overline{v}(\exp H)\big)\, d\overline{v}$$

takes $\mathbb{K}$-bi-invariant functions K to W-invariant functions on $\mathfrak{a}$ (see [12, p. 396]). Thus if $t \leq s$, one has

$$\begin{aligned}
&\int_{\overline{\mathbb{N}}} \big| K_{p,\phi}\big(\delta_{-s}(\overline{v}) \exp((t-s)H_0)\big)\big|\, d\overline{v} \\
&\quad = e^{-2|\rho|s} \int_{\overline{\mathbb{N}}} \big| K_{p,\phi}\big(\overline{v} \exp((t-s)H_0)\big)\big|\, d\overline{v} \\
&\quad = e^{-2|\rho|t} \int_{\overline{\mathbb{N}}} \big| K_{p,\phi}\big(\overline{v} \exp((s-t)H_0)\big)\big|\, d\overline{v} \leq C_p \cdot A e^{-2|\rho|t} e^{-2|\rho|(s-t)/p}.
\end{aligned}$$

The last estimate is a consequence of (3.11) and (3.12). Therefore the absolute value of the expression in (3.17) is dominated by

$$C_p \cdot A \int_{\mathbb{R}} \int_{\mathbb{R}} F_1(s) G_1(t) e^{2|\rho|s} e^{-2|\rho|(s-t)/p} \big(1 - \chi(t-s)\big)\, ds\, dt,$$

which is equal to

$$C_p \cdot A \int_0^\infty \Big(\int_{\mathbb{R}} \big(F_1(s) e^{2|\rho|s/p}\big)\big(G_1(s-r) e^{2|\rho|(s-r)/p'}\big)\, ds \Big) e^{2|\rho|r(1/p'-1/p)}\, dr.$$

By Hölder inequality, the inner integral is dominated by

$$\Big(\int_{\mathbb{R}} |F_1(s)|^p e^{2|\rho|s}\, ds \Big)^{1/p} \Big(\int_{\mathbb{R}} |G_1(t)|^{p'} e^{2|\rho|t}\, dt \Big)^{1/p'}.$$

Since $p < 2 < p'$, the estimate (3.16) follows. □

LEMMA 6

One has

$$|I^2_{p,\phi}(f,g)| \leq C_p \cdot A \|f\|_{L^p(\overline{\mathbb{N}} \times \mathbb{R}, d\mu)} \|g\|_{L^{p'}(\overline{\mathbb{N}} \times \mathbb{R}, d\mu)}. \tag{3.18}$$

Proof

By Hölder inequality, the absolute value of $I^2_{p,\phi}(f,g)$ is dominated by

$$\int_{\mathbb{R}} \int_{\mathbb{R}} F_1(s) G_1(t) e^{2|\rho|(s+t)} \chi(t-s) \Big(\int_{\overline{\mathbb{N}}} \big| E_{p,\phi}\big(\delta_{-s}(\overline{v}), (t-s)\big)\big|\, d\overline{v} \Big)\, ds\, dt. \tag{3.19}$$

It follows from (3.11) and (3.14) that

$$\begin{aligned}
&\int_{\overline{\mathbb{N}}} \big| E_p\big(\delta_{-s}(\overline{v}), (t-s)\big)\big|\, d\overline{v} \\
&\qquad \leq C_p e^{-2|\rho|s} e^{-2|\rho|(t-s)/p} \Big[A e^{-2(t-s)} + \int_0^\infty A(t-s+x) k_p(x)\, dx \Big],
\end{aligned}$$

where

$$k_p(x) = \int_{\overline{\mathbb{N}}} P(\overline{v})^{2/p} \chi\big(2 + \alpha(H(\overline{v})) - x\big)\, d\overline{v}.$$

We substitute this last estimate into (3.19) and make the change of variable $t = s + r$. It follows that the absolute value of $I^2_{p,\phi}(f, g)$ is dominated by

$$C_p \int_0^\infty \Big(\int_{\mathbb{R}} \big(F_1(s) e^{2|\rho| s/p}\big)\big(G_1(s+r) e^{2|\rho|(s+r)/p'}\big)\, ds \Big) \cdot \Big[A e^{-2r} + \int_0^\infty A(r+x) k_p(x)\, dx \Big]\, dr.$$

Since $p < 2$, it follows from (3.11) that $\int_0^\infty k_p(x)\, dx = C_p < \infty$. Estimate (3.18) follows from Hölder inequality and the estimate above. □

LEMMA 7

One has

$$|I^3_{p,\phi}(f, g)| \le C_p ||| * \phi |||_{L^p(\mathbb{R})} || f ||_{L^p(\overline{\mathbb{N}} \times \mathbb{R}, d\mu)} || g ||_{L^{p'}(\overline{\mathbb{N}} \times \mathbb{R}, d\mu)}. \tag{3.20}$$

Proof

The change of variable $t = s + r$ shows that the integral $I^3_{p,\phi}(f, g)$ is equal to

$$\int_{\overline{\mathbb{N}}} \int_{\overline{\mathbb{N}}} \int_{\mathbb{R}} f(\overline{m}, s) G_2(\overline{n}, s) P\big(\delta_{-s}(\overline{m}^{-1}\overline{n})\big)^{2/p} e^{2|\rho| s} e^{2|\rho| s/p}\, ds\, d\overline{m}\, d\overline{n}, \tag{3.21}$$

where

$$G_2(\overline{n}, s) = \int_{\mathbb{R}} g(\overline{n}, s+r) e^{2|\rho|(s+r)/p'} \phi(r)\, dr.$$

One has

$$\int_{\mathbb{R}} |G_2(\overline{n}, s)|^{p'}\, ds \le ||| * \phi |||^{p'}_{L^p(\mathbb{R})} \int_{\mathbb{R}} |g(\overline{n}, t)|^{p'} e^{2|\rho| t}\, dt. \tag{3.22}$$

Let

$$G_3(s) = \Big(\int_{\overline{\mathbb{N}}} |G_2(\overline{n}, s)|^{p'} d\overline{n} \Big)^{1/p'}.$$

It follows by Hölder inequality and (3.11) that the absolute value of the integral (3.21) is dominated by

$$\int_{\mathbb{R}} F_1(s) G_3(s) e^{2|\rho| s/p}\, ds,$$

which, again by Hölder inequality, is dominated by

$$\Big(\int_{\mathbb{R}} G_3(s)^{p'}\, ds \Big)^{1/p'} \Big(\int_{\mathbb{R}} F_1(s)^p e^{2|\rho| s}\, ds \Big)^{1/p}.$$

Estimate (3.20) follows from (3.22). □

Part (i) of the theorem now follows from (3.16), (3.18), and (3.20). For the converse, notice that it suffices to show that for any smooth compactly supported functions $a, b : \mathbb{R} \to \mathbb{C}$, one has

$$\left| \int_{\mathbb{R}} \int_{\mathbb{R}} a(s)\phi(t-s)b(t)\,ds\,dt \right| \leq C_p\big(A + |||\ast K_{p,\phi}|||_{L^p(\mathbb{X})}\big)||a||_{L^p(\mathbb{R})}||b||_{L^{p'}(\mathbb{R})}. \tag{3.23}$$

Let $J_\phi(a,b)$ denote the left-hand side of (3.23). We construct two functions $f_a, g_b : \overline{\mathbb{N}} \times \mathbb{R} \to \mathbb{C}$ such that $J_\phi(a,b) = I^3_{p,\phi}(f_a, g_b)$. Assume that the functions a and b are supported inside the interval $[-N, N]$. Let B_N be a large ball in $\overline{\mathbb{N}}$, for example, $B_N = \{\overline{n}(X,Y) : |X| \leq e^N \text{ and } |Y| \leq e^{2N}\}$, where $X \in \mathfrak{g}_{-\alpha}$, $Y \in \mathfrak{g}_{-2\alpha}$, and $\overline{n}(X,Y) = \exp(X+Y)$ (see [12, Chap. II, Sec. 6]). It is well known that $\delta_s(\overline{n}(X,Y)) = \overline{n}(e^{-s}X, e^{-2s}Y)$. Let $\psi_N : \overline{\mathbb{N}} \to \{0,1\}$ be the characteristic function of the ball B_N, and notice that $\int_{\overline{\mathbb{N}}} \psi_N(\overline{v})\,d\overline{v} = Ce^{2|\rho|N}$. For any $s \in [-N, N]$, let

$$\begin{aligned} q(s) &= e^{-2|\rho|N} \int_{\overline{\mathbb{N}}} \int_{\overline{\mathbb{N}}} \psi_N(\overline{m}) P(\overline{v})^{2/p} \psi_N\big(\overline{m} \cdot \delta_s(\overline{v})\big)\,d\overline{m}\,d\overline{v} \\ &= e^{-2|\rho|N} e^{2|\rho|s} \int_{\overline{\mathbb{N}}} \int_{\overline{\mathbb{N}}} \psi_N(\overline{m}) P\big(\delta_{-s}(\overline{m}^{-1}\overline{n})\big)^{2/p} \psi_N(\overline{n})\,d\overline{m}\,d\overline{n}. \end{aligned} \tag{3.24}$$

Notice that $c \leq q(s) \leq C_p$ for all $s \in [-N, N]$. Let

$$f_a(\overline{m}, s) = e^{-2|\rho|N/p} \psi_N(\overline{m}) a(s) e^{-2|\rho|s/p} q(s)^{-1}$$

and

$$g_b(\overline{n}, t) = e^{-2|\rho|N/p'} \psi_N(\overline{n}) b(t) e^{-2|\rho|t/p'}.$$

It follows from (3.15) and (3.24) that

$$J_\phi(a,b) = I^3_{p,\phi}(f_a, g_b);$$

therefore

$$|J_\phi(a,b)| \leq |I_{p,\phi}(f_a, g_b)| + |I^1_{p,\phi}(f_a, g_b)| + |I^2_{p,\phi}(f_a, g_b)|. \tag{3.25}$$

Finally, notice that

$$\begin{cases} ||f_a||_{L^p(\overline{\mathbb{N}} \times \mathbb{R}, d\mu)} \approx ||a||_{L^p(\mathbb{R})}, \\ ||g_b||_{L^{p'}(\overline{\mathbb{N}} \times \mathbb{R}, d\mu)} \approx ||b||_{L^{p'}(\mathbb{R})}, \end{cases}$$

and (3.23) follows from (3.16), (3.18), and (3.25). □

4. L^p-Fourier multipliers

The Fourier transform on the symmetric space $\mathbb{X}$ associates to any smooth compactly supported function f on $\mathbb{X}$ a function $\tilde{f} : \mathfrak{a}^*_{\mathbb{C}} \times \mathbb{K}/\mathbb{M} \to \mathbb{C}$, where $\mathfrak{a}^*_{\mathbb{C}}$ is the complex dual of $\mathfrak{a}$. By definition, one has

$$\widetilde{f}(\lambda, b) = \int_{\mathbb{X}} f(z) e^{(-i\lambda+\rho)A(z,b)}\, dx,$$

where $A(z, b)$ is an $\mathfrak{a}$-valued analogue of the usual scalar product on Euclidean spaces (see [12, Chap. III]). For any $g \in \mathbb{G}$ and $k \in \mathbb{K}$, one has, by definition, $A(g\mathbb{K}, k\mathbb{M}) = -H(g^{-1}k)$. The Fourier transform extends to an isometry of $L^2(\mathbb{X})$ onto $L^2(\mathfrak{a}^*_+ \times \mathbb{K}/\mathbb{M}, |\mathbf{c}(\lambda)|^{-2}\, d\lambda\, db)$, where, if $\mathbb{G}$ has real rank one, $\mathfrak{a}^*_+ = \{c \cdot \alpha : c \in \mathbb{R}_+\}$ and $\mathbf{c}$ is the Harish-Chandra function (Plancherel theorem). Let $\mathfrak{a}^*$ be the real dual of $\mathfrak{a}$. By Plancherel theorem, any bounded even multiplier $m : \mathfrak{a}^* \to \mathbb{C}$ defines a bounded operator T_m on $L^2(\mathbb{X})$ given by $\widetilde{T_m f}(\lambda, b) = m(\lambda)\, \tilde{f}(\lambda, b)$. Assume that $p \in (1, 2) \cup (2, \infty)$ is a fixed exponent, and let $\rho_p = |2/p - 1|\rho$. For any $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, let, by definition, $|\lambda| = |\lambda(H_0)|$, and let $\mathscr{T}_p = \mathfrak{a}^* + i(-\rho_p, \rho_p) = \{\lambda \in \mathfrak{a}^*_{\mathbb{C}} : |\Im(\lambda(H_0))| < |\rho_p|\}$. Assume that the multiplier $m : \mathfrak{a}^* \to \mathbb{C}$ extends to an even holomorphic function in the interior of the tube $\mathscr{T}_p$ and satisfies the differential inequalities

$$\Big|\frac{\partial^j}{\partial\lambda^j} m(\lambda)\Big| \le A_j\big(|\lambda + i\rho_p|^{-j} + |\lambda - i\rho_p|^{-j}\big) \tag{4.1}$$

for any $j = 0, 1, \ldots$ and $\lambda \in \mathscr{T}_p$.

THEOREM 8
If $p \in (1, 2) \cup (2, \infty)$ and if m satisfies the differential inequalities (4.1), then the operator T_m extends to a bounded operator on $L^p(\mathbb{X})$.

An operator T_m defined by a multiplier m that satisfies (4.1) can be thought of as a singular integral operator on the symmetric space $\mathbb{X}$. In this section the constants may depend on finitely many of the constants A_j in (4.1).

Proof
One can clearly assume that $p \in (1, 2)$. In order to ensure the convergence of the integrals throughout this section, we also assume that the multiplier $m(\lambda)$ is premultiplied with a factor of the form $e^{-\delta\lambda(H_0)^2}$, where $0 < \delta \le 1$. Our estimates are uniform in δ; once one proves suitable uniform estimates, standard limiting arguments allow one to pass to the general theorem. The multiplier m is holomorphic inside the tube $\mathscr{T}_p$; therefore we can assume that for any $\xi \in (-\rho_p, \rho_p)$ the function $\lambda \to m(\lambda + i\xi)$ is a Schwartz function on $\mathfrak{a}^*$.

To simplify the notation, we identify the complex plane $\mathbb{C}$ with $\mathfrak{a}^*_{\mathbb{C}}$ using the map $\lambda \to \lambda \cdot \alpha$. Notice that this map also gives an identification of $\mathbb{R}$ with $\mathfrak{a}^*$. By the Fourier inversion formula, the kernel K of the operator T_m is given by

$$K\big(k_1(\exp(rH_0))k_2\big) = C \int_{\mathbb{R}} m(\lambda)\Phi_\lambda\big(\exp(rH_0)\big)|\mathbf{c}(\lambda)|^{-2}\, d\lambda$$

for any $k_1, k_2 \in \mathbb{K}$ and $r \geq 0$. The functions Φ_λ are the elementary spherical functions on $\mathbb{X}$, and $\mathbf{c}(\lambda)$ is the Harish-Chandra function. Let $\chi : [0,\infty) \to [0,1]$ be a smooth cutoff function supported in the interval $[1,\infty)$ and equal to 1 on the interval $[2,\infty)$. It is shown in both [20, Secs. 4, 5] and [1, Sec. 4] that the kernel $(1-\chi(r))K(k_1(\exp(rH_0))k_2)$ (i.e., the local part of our kernel) defines a bounded operator on $L^p(\mathbb{X})$. It remains therefore to prove a similar statement for the kernel $\chi(r)K(k_1(\exp(rH_0))k_2)$.

It is well known (see, e.g., [14, App. A]) that the function $\lambda \to \mathbf{c}(-\lambda)^{-1}$ is holomorphic inside the region $\Im\lambda \geq 0$ and satisfies the estimates

$$\Big|\frac{\partial^k}{\partial\lambda^k}\big(\mathbf{c}(-\lambda)^{-1}\big)\Big| \leq C_k\big(1+|\lambda|\big)^{d-k} \tag{4.2}$$

for all integers $k = 0, 1, \ldots$ and for all λ with the property that $0 \leq \Im\lambda \leq |\rho|$. The number d in (4.2) is equal to $(\dim\mathbb{X}-1)/2 = (m_1+m_2)/2$. It is also shown in [14, App. A] that if $r \geq 1/2$, then the spherical functions Φ_λ can be written in the form

$$\begin{aligned}&\Phi_\lambda\big(\exp(rH_0)\big)\\ &\quad= e^{-|\rho|r}\big[e^{i\lambda r}\mathbf{c}(\lambda)(1+e^{-2r}a(\lambda,r)) + e^{-i\lambda r}\mathbf{c}(-\lambda)(1+e^{-2r}a(-\lambda,r))\big].\end{aligned} \tag{4.3}$$

The function a in (4.3) satisfies the inequalities

$$\Big|\frac{\partial^j}{\partial\lambda^j}\frac{\partial^l}{\partial r^l}a(\lambda,r)\Big| \leq C_j\big[(1+|\lambda|)\big]^{-j} \tag{4.4}$$

for all integers $j = 0, 1, \ldots$ and $l \in \{0,1\}$, and for all $r \geq 1/2$ and λ with the property $0 \leq \Im\lambda \leq |\rho|$. One also has $|\mathbf{c}(\lambda)|^2 = \mathbf{c}(\lambda)\mathbf{c}(-\lambda)$ for any $\lambda \in \mathbb{R}$. Since the symbol m is even, it follows from Corollary 2 and (4.3) that it suffices to prove that the function

$$\phi_0(r) = \chi(r)e^{|\rho_p|r}\int_{\mathbb{R}} e^{i\lambda r}m(\lambda)\big(1+e^{-2r}a(\lambda,r)\big)\mathbf{c}(-\lambda)^{-1}\,d\lambda$$

satisfies inequalities (3.5) and (3.6). An easy argument based on (4.4) shows that the error term introduced by the factor $a(\lambda,r)$ in the definition above and its derivative are dominated by Ce^{-r}. Therefore it remains to prove that the function

$$\phi(r) = \chi(r)e^{|\rho_p|r}\int_{\mathbb{R}} e^{i\lambda r}m_1(\lambda)\,d\lambda \tag{4.5}$$

satisfies inequalities (3.5) and (3.6), where $m_1(\lambda) = m(\lambda)\mathbf{c}(-\lambda)^{-1}$. It follows from (4.1) and (4.2) that the function m_1 is holomorphic in the interior of the region $0 \leq \Im\lambda < |\rho_p|$ and satisfies the inequalities

$$\Big|\frac{\partial^j}{\partial\lambda^j}m_1(\lambda)\Big| \leq C'_j\big|\lambda - i|\rho_p|\big|^{-j}\big(1+|\lambda|\big)^d \tag{4.6}$$

for any $j = 0, 1, \dots$ and λ in the region $0 \leq \Im\lambda < |\rho_p|$. To prove (3.5), one moves the contour of integration in (4.5) to the line $\mathbb{R} + i|\rho_p|(1 - r^{-1})$ (notice that there is nothing to prove if $r \leq 1$), and the function ϕ in (4.5) becomes

$$\phi(r) = C\chi(r) \int_{\mathbb{R}} e^{i\lambda r} m_1\big(\lambda + i|\rho_p|(1 - r^{-1})\big)\, d\lambda. \tag{4.7}$$

To prove the first inequality in (3.5), we integrate by parts N times and notice that if $r \geq 1$, then

$$r^{-N} \int_{\mathbb{R}} \big(1 + |\lambda|\big)^d |\lambda - i|\rho_p| r^{-1}|^{-N}\, d\lambda \leq C/r$$

if $N > d + 1$. The first inequality in (3.5) now follows from (4.6). The proof of the second inequality in (3.5) is similar, the only difference being that differentiation with respect to r brings down an extra factor of λ in the integral in (4.7) (or a factor of r^{-2}) that weakens the singularity at $\lambda = 0$, and a similar integration by parts allows one to obtain the desired estimate.

Thus it remains to prove that the function ϕ in (4.5) satisfies the cancellation condition (3.6). Let ψ be a smooth function supported in the interval $[-1, 1]$ and equal to 1 in the interval $[-1/2, 1/2]$. Since the function ϕ already satisfies the first inequality in (3.5), it suffices to prove that for any $\varepsilon \leq 1$ one has

$$\Big| \int_0^\infty \phi(r)\psi(\varepsilon r)\, dr \Big| \leq C.$$

We move the integration in (4.5) to the line $\mathbb{R} + i|\rho_p|(1 - \varepsilon)$, and we have to prove that

$$\Big| \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} e^{i\lambda r} \chi(r)\psi(\varepsilon r) e^{|\rho_p|\varepsilon r}\, dr \Big) m_1\big(\lambda + i|\rho_p|(1 - \varepsilon)\big)\, d\lambda \Big| \leq C \tag{4.8}$$

for any $\varepsilon \in (0, 1]$. Let $\psi_1(r) = \psi(r) e^{|\rho_p| r}$, and notice that

$$\Big| \frac{\partial^j}{\partial r^j} \psi_1(r) \Big| \leq C \tag{4.9}$$

for any integer $j \in [0, d + 2]$, where $d = (\dim \mathbb{X} - 1)/2$. Let $\mathscr{F}$ denote the Euclidean Fourier transform on $\mathbb{R}$ given by $\mathscr{F}(a)(\xi) = \int_{\mathbb{R}} a(x) e^{-ix\cdot\xi}\, dx$ for Schwartz functions $a : \mathbb{R} \to \mathbb{C}$. Let L be the distribution on $\mathbb{R}$ with the property that $\mathscr{F}(L) = \chi$. Then the inner integral in (4.8) can be written as

$$L\Big(\xi \to \frac{1}{\varepsilon}(\mathscr{F}\psi_1)\Big(\frac{\xi - \lambda}{\varepsilon}\Big)\Big).$$

It follows from (4.9) that the function $\mathscr{F}\psi_1$ is sufficiently rapidly decreasing at infinity; thus (4.8) is established once we prove that for any $x \in \mathbb{R}$,

$$\big| L\big(\xi \to m_1(\xi + x + i|\rho_p|(1 - \varepsilon))\big) \big| \leq C\big(1 + |x|\big)^d \tag{4.10}$$

with a constant C independent of x and ε. Notice that $\mathscr{F}((-i\xi)L) = \chi'$. Let η be the inverse Fourier transform of the smooth compactly supported function χ'. Thus $(-i\xi)L = \eta(\xi)$, and one has

$$L(a) = \int_{\mathbb{R}} \frac{\eta(\xi)(a(\xi) - a(0)e^{-\xi^2})}{-i\xi}\, d\xi + c_1 a(0)$$

for any Schwartz function $a : \mathbb{R} \to \mathbb{C}$, where $c_1 = L(\xi \to e^{-\xi^2})$. Therefore the term on the left-hand side of (4.10) is equal to

$$\int_{\mathbb{R}} \frac{\eta(\xi)[m_1(\xi + x + i|\rho_p|(1-\varepsilon)) - m_1(x + i|\rho_p|(1-\varepsilon))e^{-\xi^2}]}{-i\xi}\, d\xi \\ + c_1 m_1\big(x + i|\rho_p|(1-\varepsilon)\big).$$

One first moves the contour of integration in the integral above to the line $-i|\rho_p|(1-\varepsilon)+\mathbb{R}$ (this is possible since η is the inverse Fourier transform of a smooth compactly supported function; thus η is holomorphic in $\mathbb{C}$), and (4.10) follows from (4.6) and the observation that the function $\xi \to \eta(\xi - i|\rho_p|(1-\varepsilon))$ is rapidly decreasing at ∞. This completes the proof of the cancellation condition (3.6), and the theorem follows from Corollary 2. □

5. Imaginary powers of Laplacians

Natural examples of multipliers satisfying the differential inequalities (4.1) are provided by exponential powers of modified Laplacians on $\mathbb{X}$. Let Δ be the Laplace-Beltrami operator on $\mathbb{X}$. One has

$$\widetilde{\Delta f}(\lambda, b) = -\big(\lambda(H_0)^2 + |\rho|^2\big)\, \tilde{f}(\lambda, b)$$

for any smooth compactly supported function $f : \mathbb{X} \to \mathbb{C}$ and any $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. For any $p \in [1, \infty]$ and $u \in \mathbb{R}$, let

$$m_{p,u}(\lambda) = \big(\lambda(H_0)^2 + |\rho_p|^2\big)^{-iu}.$$

Notice that $m_{p,u}$ is bounded and holomorphic in the interior of the tube $\mathscr{T}_p$ and has singularities at the points $\lambda = i\rho_p$ and $\lambda = -i\rho_p$ (if $p \neq 2$). Let $L_{p,u}$ be the operator defined by the Fourier multiplier $m_{p,u}$. Notice that $L_{p,u} = [-\Delta - (|\rho|^2 - |\rho_p|^2)I]^{-iu}$.

THEOREM 9

If $p \in (1,2) \cup (2,\infty)$ and $u \in \mathbb{R}$, then there exist constants c_p and C_p such that

$$c_p e^{\pi|u|/2}(1+|u|)^{|1/p-1/2|} \le |||L_{p,u}|||_{L^p(\mathbb{X})\to L^p(\mathbb{X})} \le C_p e^{\pi|u|/2}(1+|u|)^{|1/p-1/2|}. \tag{5.1}$$

The standard notation

$$|||L_{p,u}|||_{L^p(\mathbb{X})\to L^p(\mathbb{X})} \approx_p e^{\pi|u|/2}\big(1+|u|\big)^{|1/p-1/2|}$$

is used in this section to denote the double inequality (5.1). $L^p \to L^q$ boundedness properties of operators of the form $(zI-\Delta)^s$ for suitable values of z and s have been studied extensively in various settings (see [2, Sec. 4] for a detailed discussion of the problem and appropriate references). The upper estimate in (5.1) is a slight improvement over the estimate in [9, Cor. 4.2]. It is not hard to see, however, that this improvement can also be obtained as a consequence of Lemma 10 at the end of this paper and the method used in [9]. (This was observed by Andreas Seeger.) On the other hand, the lower estimate in (5.1) is new and cannot be obtained by the argument in [5].

It is shown in [4] that if $q \in (1,\infty)$ is such that $|2/q-1| < |2/p-1|$, then the operator $L_{p,u}$ is bounded on $L^q(\mathbb{X})$ and

$$|||L_{p,u}|||_{L^q(\mathbb{X})\to L^q(\mathbb{X})} \approx_{p,q} e^{|u|\arcsin(|2/q-1|/|2/p-1|)}. \tag{5.2}$$

Notice that the expression on the right-hand side of (5.2) agrees with the best possible lower bound given by the Clerc-Stein condition

$$|||L_{p,u}|||_{L^q(\mathbb{X})\to L^q(\mathbb{X})} \geq \sup_{\lambda\in\mathscr{T}_q} |m_{p,u}(\lambda)| \approx e^{|u|\arcsin(|2/q-1|/|2/p-1|)}.$$

However, the L^p-analysis of the operator $L_{p,u}$ is more delicate (mainly because size estimates on the kernel of the operator are not sufficient), and the L^p-norm estimate in (5.1) does not agree with the lower bound predicted by the Clerc-Stein condition. Indeed, one can easily check that

$$\sup_{\lambda\in\mathscr{T}_p} |m_{p,u}(\lambda)| = e^{\pi|u|/2},$$

which is not proportional to the bound on the right-hand side of (5.1) if $|u|$ is large.

Proof of Theorem 9

Assume as in Section 4 that $p \in (1,2)$ and that the multiplier $m_{p,u}$ is premultiplied with a factor of the form $e^{-\delta\lambda(H_0)^2}$ in order to ensure the convergence of all the integrals throughout this section. Also, identify $\mathbb{C}$ with $\mathfrak{a}_{\mathbb{C}}$ via the map $\lambda \to \lambda\cdot\alpha$. The kernel K of the operator $L_{p,u}$ is given by the Fourier inversion formula

$$K\big(k_1(\exp(rH_0))k_2\big) = C\int_{\mathbb{R}} m_{p,u}(\lambda)\Phi_\lambda\big(\exp(rH_0)\big)|\mathbf{c}(\lambda)|^{-2}\,d\lambda.$$

The basic idea of the proof is to apply Theorem 1 and control the L^p-norm of the operator $L_{p,u}$ using the norm of a convolution operator on $L^p(\mathbb{R})$. Notice first that one can assume that $|u|$ is large, say, $|u| \geq C_p$ for some suitable constant C_p. (The theorem for $|u|$ small follows from Th. 8 and the observation that, by the Clerc-Stein condition, $|||L_{p,u}|||_{L^p(\mathbb{X})\to L^p(\mathbb{X})} \geq 1$ for any $u \in \mathbb{R}$.) We divide the proof of the theorem into three steps.

Step 1: Reduction of the problem. We assume from now on that u is positive. As in Section 4, let $\chi : [0,\infty) \to [0,1]$ be a smooth cutoff function supported in the interval $[1,\infty)$ and equal to 1 in the interval $[2,\infty)$. It follows from [20, Secs. 4 and 5] that the kernel $(1-\chi(r))K(k_1(\exp(rH_0))k_2)$ (i.e., the local part of our kernel) defines a bounded operator on $L^p(\mathbb{X})$ with norm at most $C_p(1+|u|)^N$, where N is a large fixed integer that depends only on the group $\mathbb{G}$. It remains therefore to prove that the kernel $\chi(r)K\big(k_1(\exp(rH_0))k_2\big)$ defines a bounded operator on $L^p(\mathbb{X})$ with norm proportional to $e^{\pi|u|/2}(1+|u|)^{|1/p-1/2|}$. Let c_2 be such that

$$e^{c_2} = \sup \left|(z^2+|\rho_p|^2)^{-i}\right|,$$

where the supremum is taken over $z \in \mathscr{T}_p \setminus [B(i|\rho_p|,1/2) \cup B(-i|\rho_p|,1/2)]$. Notice that $c_2 < \pi/2$. Expand the function $\Phi_\lambda(\exp(rH_0))$ as in (4.3), and notice that the part of the kernel $\chi(r)K(k_1(\exp(rH_0))k_2)$ corresponding to the error term $e^{-2r}a(\lambda,r)$ induces a bounded operator on $L^p(\mathbb{X})$ with norm at most $C_p e^{c_2 u}(1+|u|)^N$. Thus it remains to prove that if

$$K^1\big(k_1(\exp(rH_0))k_2\big) = \chi(r)e^{-|\rho|r}\int_{\mathbb{R}} e^{i\lambda r} m_{p,u}(\lambda)\mathbf{c}(-\lambda)^{-1}\,d\lambda \tag{5.3}$$

for any $k_1, k_2 \in \mathbb{K}$ and $r > 0$, then

$$|||*K^1|||_{L^p(\mathbb{X})} \approx_p e^{\pi|u|/2}\big(1+|u|\big)^{|1/p-1/2|}.$$

We move the contour of integration in (5.3) to the line $i|\rho_p|+\mathbb{R}$ and remove the large frequencies using a smooth cutoff function. One can check easily that the error term (corresponding to large frequencies) induces a bounded operator on $L^p(\mathbb{X})$ with norm at most $C_p e^{c_2 u}(1+|u|)^N$. Thus it remains to prove that the kernel

$$K^2\big(k_1(\exp(rH_0))k_2\big) = \chi(r)e^{-2|\rho|r/p}\int_{\mathbb{R}} e^{i\lambda r}\big(\lambda^2+2i|\rho_p|\lambda\big)^{-iu}\eta(\lambda)\,d\lambda \tag{5.4}$$

has the property that

$$|||*K^2|||_{L^p(\mathbb{X})} \approx_p e^{\pi|u|/2}(1+|u|)^{|1/p-1/2|}. \tag{5.5}$$

The function η in (5.4) is smooth, supported in the interval $[-1,1]$, and $\eta(0) = \mathbf{c}(-i|\rho_p|)^{-1} \neq 0$. The estimate (5.5) is a consequence of Theorem 1. Let

$$\phi_{p,u}(r) = \chi(r)\int_{\mathbb{R}} e^{i\lambda r}\big(\lambda^2+2i|\rho_p|\lambda\big)^{-iu}\eta(\lambda)\,d\lambda. \tag{5.6}$$

Recall that we assume that u is positive (i.e., $u \geq C_p$), and let $Y_p : \mathbb{R} \to [0, \pi]$ be the function defined implicitly by $\tan(Y(\lambda)) = \lambda/(2|\rho_p|)$. An elementary computation shows that

$$\phi_{p,u}(r) = e^{\pi u/2} \chi(r) \int_{\mathbb{R}} e^{i[\lambda r - u \log(\lambda^4 + 4|\rho_p|^2 \lambda^2)/2]} \eta(\lambda) e^{-uY_p(\lambda)}\, d\lambda. \tag{5.7}$$

We first prove a rough estimate on $|\phi'_{p,u}(r)|$. Notice that $Y(\lambda) > \pi/2$ if $\lambda < 0$ and $Y_p(\lambda) \geq c\lambda$ if $\lambda \in [0, 1]$. It follows from (5.7) and the assumption that u is positive that $|\phi'_{p,u}(r)| \leq Ce^{\pi|u|/2}/|u|$ if $r \leq 2$ and $|\phi'_{p,u}(r)| \leq Ce^{\pi|u|/2}/|u|^2$ if $2 \leq r \leq 10|u|^2$. In order to estimate $|\phi'_{p,u}(r)|$ when $r \geq 10|u|^2$, notice that the derivative of the phase function in (5.7) is equal to

$$r - \frac{u}{\lambda} \frac{4|\rho_p|^2 + 2\lambda^2}{4|\rho_p|^2 + \lambda^2}.$$

An integration-by-parts argument shows that $|\phi'_{p,u}(r)| \leq Ce^{\pi|u|/2}|u|^2/r^2$ if $r \geq 10|u|^2$. To summarize, one has

$$|\phi'_{p,u}(r)| \leq \begin{cases} Ce^{\pi|u|/2}/|u| & \text{if } r \leq 2, \\ Ce^{\pi|u|/2}/|u|^2 & \text{if } 2 \leq r \leq 10|u|^2, \\ Ce^{\pi|u|/2}|u|^2/r^2 & \text{if } r \geq 10|u|^2. \end{cases}$$

Therefore

$$\int_0^\infty |\phi'_{p,u}(r)|\, dr \leq Ce^{\pi|u|/2},$$

and it follows from Theorem 1 that (5.5) and the theorem follow once we prove that

$$|||\ast \phi_{p,u}|||_{L^p(\mathbb{R})} \approx_p e^{\pi|u|/2}\big(1 + |u|\big)^{|1/p-1/2|}.$$

Notice also that the contribution of the integral over $\lambda < 0$ in formula (5.7) defining $\phi_{p,u}$ is negligible (since the absolute value of the derivative of the phase function is greater than or equal to r and $e^{-uY_p(\lambda)} \leq e^{-\pi|u|/2}$ if $\lambda < 0$). Thus one can replace integrals (5.6) and (5.7) with the corresponding integrals on $\mathbb{R}_+$. A similar argument shows that one can also remove the factor $\chi(r)$ in (5.6) and (5.7) at the expense of a negligible error. Finally, it remains to prove that the operator $S_{p,u}$ defined by the Fourier multiplier $q_{p,u} : \mathbb{R} \to \mathbb{C}$,

$$q_{p,u}(\xi) = \begin{cases} 0 & \text{if } \xi \leq 0, \\ e^{-iu \log(\xi^4 + 4|\rho_p|^2\xi^2)/2} e^{-uY_p(\xi)} \eta(\xi) & \text{if } \xi > 0, \end{cases} \tag{5.8}$$

extends to a bounded operator on $L^p(\mathbb{R})$ and

$$|||S_{p,u}|||_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} \approx_p \big(1 + |u|\big)^{|1/p-1/2|}. \tag{5.9}$$

Step 2: The lower estimate in (5.9). For any $\xi > 0$, let

$$\beta_{p,u}(\xi) = e^{-iu\log(\xi^2/u^2+4|\rho_p|^2)/2} e^{-uY_p(\xi/u)} \eta\Big(\frac{\xi}{u}\Big);$$

then

$$q_{p,u}(\xi) = e^{-iu\log\xi}\beta_{p,u}(u\xi). \tag{5.10}$$

Notice that $|\beta_{p,u}(\xi)| \geq c$ for all $\xi \in [0,4]$ (recall that u is a large positive number) and that there exists a number $c_3 > 0$ such that

$$\Big|\frac{\partial^j}{\partial\xi^j}\beta_{p,u}(\xi)\Big| \leq C_j e^{-c_3\xi} \tag{5.11}$$

for all integers $j = 0, 1, \dots$ and real numbers $\xi > 0$. Let γ be a smooth cutoff function supported in the interval $[1/2, 4]$ and equal to 1 in the interval $[1, 2]$, and let a_u be the function whose Euclidean Fourier transform is the function $\xi \to \gamma(u\xi)$. Then

$$\begin{aligned}|S_{p,u}a_u(r)| &= C\Big|\int_0^\infty e^{i\xi r} q_{p,u}(\xi)\gamma(u\xi)\,d\xi\Big| \\ &= \frac{C}{u}\Big|\int_0^\infty e^{i[\xi r/u - u\log\xi]}\beta_{p,u}(\xi)\gamma(\xi)\,d\xi\Big|\end{aligned}$$

and the method of stationary phase shows that $|S_{p,u}a_u(r)| \geq cu^{-3/2}$ for any $r \in [u^2/2, u^2]$. Therefore

$$||S_{p,u}f_u||_{L^p(\mathbb{R})} \geq cu^{2/p-3/2}.$$

On the other hand, $||a_u||_{L^p(\mathbb{R})} \approx u^{1/p-1}$, and the lower estimate in (5.9) follows.

Step 3: The upper estimate in (5.9). This is based on the following lemma.

LEMMA 10

If $q \in (1,\infty)$ and $u \in \mathbb{R}$, then the operator S_u defined by the Fourier multiplier $l_u : \mathbb{R} \to \mathbb{C}$,

$$l_u(\xi) = \begin{cases} 0 & \text{if } \xi \leq 0, \\ e^{-iu\log\xi} & \text{if } \xi > 0, \end{cases}$$

extends to a bounded operator on $L^q(\mathbb{R})$ and

$$|||S_u|||_{L^q(\mathbb{R})\to L^q(\mathbb{R})} \leq C_q\big(1+|u|\big)^{|1/q-1/2|}.$$

Assuming this lemma for a moment, notice that the operator $S_{p,u}$ is the composition of the operator S_u and the operator defined by the Fourier multiplier $\xi \to \beta_{p,u}(u\xi)$. By (5.11), this second operator is bounded on $L^q(\mathbb{R})$ for any $q \in [1,\infty]$ and its L^q-norm is dominated by an absolute constant. Thus the upper estimate in (5.9) follows from Lemma 10. □

Proof of Lemma 10

Variants of this lemma are certainly known; however, for lack of a precise reference, we give a short proof of Lemma 10. Notice that we can assume that $q \in (1, 2]$ and $u \geq 0$. By Plancherel theorem,

$$|||S_u|||_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \leq C. \tag{5.12}$$

Therefore it suffices to prove that the operator S_u is bounded from the Hardy space $H^1(\mathbb{R})$ to $L^1(\mathbb{R})$ and that

$$|||S_u|||_{H^1(\mathbb{R})\to L^1(\mathbb{R})} \leq C\big(1 + |u|\big)^{1/2}. \tag{5.13}$$

It is well known that singular integrals are bounded from $H^1(\mathbb{R})$ to $L^1(\mathbb{R})$ (see [10]); thus we only have to prove (5.13) when u is large. The operator S_u is translation and scale invariant. Therefore it suffices to prove that if $a : \mathbb{R} \to \mathbb{C}$ is a measurable function supported in the interval $[-1, 1]$ with the properties $|a(x)| \leq 1/2$ and

$$\int_{\mathbb{R}} a(x)\, dx = 0,$$

then

$$||S_u a||_{L^1(\mathbb{R})} \leq C\big(1 + |u|\big)^{1/2}.$$

In the terminology of [22, Chap. III], the function a is a standard atom on the real line. It follows from (5.12) that

$$\int_{-2u}^{2u} |S_u a(x)|\, dx \leq C u^{1/2} ||S_u a||_{L^2(\mathbb{R})} \leq C\big(1 + |u|\big)^{1/2}.$$

The kernel k of the operator S_u is a smooth function away from the origin of $\mathbb{R}$, and the method of stationary phase shows that

$$|k'(y)| \leq C u^{3/2} |y|^{-2}$$

for any $y \in \mathbb{R} \setminus \{0\}$. Therefore if $|x| \geq 2u \geq 2$, then

$$\begin{aligned} |S_u a(x)| &= \Big|\int_{-1}^{1} a(y) k(x - y)\, dy\Big| = \Big|\int_{-1}^{1} a(y)\big(k(x - y) - k(x)\big)\, dy\Big| \\ &\leq C u^{3/2} |x|^{-2}, \end{aligned}$$

which shows that

$$\int_{|x|\geq 2u} |S_u a(x)|\, dx \leq C\big(1 + |u|\big)^{1/2}.$$

Estimate (5.13) and the lemma follow. □

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School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540, USA