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Fuzzy Connectedness and Object Definition: Theory, Algorithms, and Applications in Image Segmentation

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Images are by nature fuzzy. Approaches to object information extraction from images should attempt to use this fact and retain fuzziness as realistically as possible. In past image segmentation research, the notion of "hanging togetherness" of image elements specified by their fuzzy connectedness has been lacking. We present a theory of fuzzy objects for *n*-dimensional digital spaces based on a notion of fuzzy connectedness of image elements. Although our definitions lead to problems of enormous combinatorial complexity, the theoretical results allow us to reduce this dramatically, leading us to practical algorithms for fuzzy object extraction. We present algorithms for extracting a specified fuzzy object and for identifying all fuzzy objects present in the image data. We demonstrate the utility of the theory and algorithms in image segmentation based on several practical examples all drawn from medical imaging. © 1996 Academic Press, Inc.

1. INTRODUCTION

Image data captured by devices such as biomedical scanners have inherent inaccuracies. The degree of this inaccuracy depends on a number of factors including limitations in spatial, temporal, and parametric resolutions and other physical limitations of the device. The main 3D imaging operations of visualization, manipulation, and analysis are usually aimed toward certain "objects" which are represented in the image data as characteristic intensity patterns. When the object of interest has intensity patterns distinctly different from those of other objects, it is often possible to segment the object in a hard sense into a binary image. Although such a strategy does not account for most of the inaccuracies in acquired data, significant progress has been made over the past 15 years in effectively visualizing, manipulating, and analyzing multidimensional, multimodality object information [1–3].

Attempts to retain data inaccuracies and pass them on to the human observer were made in the past in the context of volume rendering developments in visualization [4, 5]. Although philosophically this was a departure from the principle of hard structure definition, just a provision to retain inaccuracies by itself does not guarantee the accuracy of their retention. From this consideration, the relative accuracy of these strategies compared to those using hard (binary) segmentation principles naturally becomes an important issue, which remains largely unexplored in medical 3D imaging [6, 7]. The principle of retention of data inaccuracies as realistically as possible in object representations and subsequently in object renditions and analysis is undoubtedly the right stand. However, no formal framework has yet been developed to handle object-related issues. Traditionally, in volume rendering, the given image data are considered to represent an amorphous volume, the emphasis being mainly on creating a rendition that depicts object structures represented in the image data. By bringing explicitly the notion of a structure, we have shown [8] that volume rendering operations become significantly more efficient computationally. We argue that the notion of object should be defined formally in the fuzzy setting in order to take operations that can handle data inaccuracies beyond mere visualization to object segmentation, manipulation, and analysis.

The basic mathematical framework toward this goal should be addressing issues of the following form: How are objects to be defined in a fuzzy setting? How are topological concepts such as connectivity and boundary to be handled in fuzzy situations? What are the algorithms to efficiently extract fuzzy connected components and fuzzy boundaries? Although the theory of fuzzy subsets is an appropriate mathematical vehicle for addressing these issues, the published literature on dealing with fuzzy topological notions is limited [9–12]. These publications deal with some of these notions for 2D digital pictures.

The fuzzy connectivity and object notions have significant implications in image segmentation. The main hardships encountered in the design of effective segmentation algorithms are often attributable to the inflexibility of the rigid, often contradicting, requirements that attempt to distinguish between object and nonobject regions. The

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flexibility afforded by fuzzy connectivity eases these requirements, making "fuzzy connected component" a computable alternative to the notion of an object. We argue that "fuzzy connectedness" is a concept that effectively captures fuzzy "hanging togetherness" of image elements—a notion that has been missing in past segmentation research. We demonstrate in this paper that finding fuzzy connected components is often a powerful solution to the difficult segmentation problem.

In this paper, we first present a theory of fuzzy connected objects in Section 2 for digital spaces of finite dimensionality. In Section 3, we describe efficient algorithms for extracting fuzzy connected components from membership images. In Section 4, we demonstrate the utility of these algorithms in image segmentation based on examples drawn from several clinical and medical imaging areas. We state our conclusions and describe future directions in Section 5.

2. THEORY

2.1. *Fuzzy Subsets, Membership Function, Fuzzy Relation*

We start by stating some known definitions from the theory of fuzzy subsets [13].

Let X be any reference set. A *fuzzy subset* $\mathscr{A}$ of X is a set of ordered pairs

$$\mathscr{A} = \{(x, \mu_{\mathscr{A}}(x)) \mid x \in X\}, \tag{2.1}$$

where

$$\mu_{\mathscr{A}}: X \to [0, 1] \tag{2.2}$$

is the *membership function* of $\mathscr{A}$ in X. We say $\mathscr{A}$ is *nonempty* if there exists $x \in X$ such that $\mu_{\mathscr{A}}(x) \neq 0$. The *empty* fuzzy subset of X, denoted Φ, satisfies $\mu_{\Phi}(x) = 0$ for all $x \in X$. We use Φ to denote the empty fuzzy subset of any reference set and ϕ to denote the empty hard set. The fuzzy union and intersection operations between fuzzy subsets $\mathscr{A}$ and $\mathscr{B}$ of X are defined as follows: $\mathscr{A} \cup \mathscr{B} = \{(x, \mu_{\mathscr{A}\cup\mathscr{B}}(x)) \mid x \in X\}$, where, for all $x \in X$, $\mu_{\mathscr{A}\cup\mathscr{B}}(x) = \max[\mu_{\mathscr{A}}(x), \mu\beta_{\mathscr{B}}(x)]$; $\mathscr{A} \cap \mathscr{B} = \{(x, \mu_{\mathscr{A}\cap\mathscr{B}}(x)) \mid x \in X\}$, where, for all $x \in X$, $\mu_{\mathscr{A}\cap\mathscr{B}}(x) = \min[\mu_{\mathscr{A}}(x), \mu_{\mathscr{B}}(x)]$.

A *fuzzy relation* ρ *in* X is a fuzzy subset of $X \times X$

$$\rho = \{((x, y), \mu_\rho(x, y)) \mid (x, y) \in X \times X\}, \tag{2.3}$$

where

$$\mu_\rho : X \times X \to [0, 1]. \tag{2.4}$$

Strictly speaking, what we have defined above is a fuzzy 2-ary relation. Since we are not interested in fuzzy m-ary relations for $m > 2$, we drop the qualifier "2-ary" for simplicity. We shall always use μ subscripted by the fuzzy subset under consideration to denote the membership function of the fuzzy subset. For hard subsets, μ will denote their characteristic function.

Let ρ be any fuzzy relation in x. ρ is said to be

reflexive if, for all $(x, x) \in X \times X$, $\mu_\rho(x, x) = 1$, (2.5)

symmetric if, for all $(x, y) \in X \times X$, $\mu_\rho(x, y) = \mu_\rho(y, x)$, (2.6)

transitive if, for all $(x, y), (y, z), (x, z) \in X \times X$,

$$\mu_\rho(x, z) = \max_y [\min(\mu_\rho(x, y), \mu_\rho(y, z))]. \tag{2.7}$$

ρ is called a *similitude relation* in X if it is reflexive, symmetric, and transitive. The analogous concept for hard binary relations is an *equivalence relation* [14, Chap. 1].

Before presenting the theory in the rest of Section 2, we first outline the main underlying ideas. This will hopefully help in understanding the motivation for the mathematics better.

There seem to be two important characteristics which need to be considered in describing objects in images. First, objects have a graded composition. In the CT slice of a patient knee shown in Fig. 1a, for example, the object called "bone" consists of both the hard cortical tissues (the brightest regions) as well as the softer cancellous tissues (regions of intermediate brightness surrounded by the bright region). Second, the image elements that constitute an object hang together in a certain way. Again, consider the CT slice of Fig. 1a which consists of two bones. The pixels that constitute the bigger bone, including those representing the cortical and the cancellous tissues, hang together to form an object called bone much more strongly than the pixels that represent cancellous tissues in the two bones (although they have very similar image intensity properties), and than the pixels that represent cortical tissues in the two bones (although they have very similar image intensity properties). Both these—graded composition and hanging togetherness—are fuzzy properties. Our aim is to capture these properties within the notion of a "fuzzy object."

Independent of any image data, we think of the digital grid points (image elements) as having a fuzzy adjacency relation—the closer the points are, the more adjacent they are to each other. This is intended to be a "local" phenomenon. How local it ought to be should perhaps depend on the point spread function of the imaging device. Now consider the grid points as having image intensity values. Our aim is to capture the global phenomenon of hanging togetherness in a fuzzy relation between grid points, called "connectedness." We do this first through a local fuzzy relation called "affinity." Affinity takes into account the degree of adjacency of the grid points as well as the similarity of their intensity values. The closer the grid points are and more similar their intensities are, the greater is the affinity between them. To see how affinity is used to assign a "strength of connectedness" to any pair of grid points (c, d), consider all possible connecting "paths" of grid points between c and d. We think of each such path as being formed from a sequence of links between successive grid points in the path. Each link has a "strength" which is simply the affinity between the corresponding two grid points. The strength of a path is simply the strength of the weakest link in it. Finally, the strength of connectedness between c and d is the strength of the strongest of all paths. In defining a fuzzy object, the strength of connectedness between all possible pairs of grid points must be taken into account, as described in the following sections.

2.2. R^n, Z^n, *Spels*, *Fuzzy Spel Adjacency*, *Fuzzy Digital Space*

Let n-dimensional Euclidean space R^n be subdivided into hypercuboids by n mutually orthogonal families of parallel hyperplanes. We shall assume, with no loss of generality for our purposes, that the hyperplanes in each family have equal unit spacing so that the hypercuboids are unit hypercubes, and we shall choose coordinates so that the center of each hypercube has integer coordinates. The hypercubes will be called *spels* (an abbreviation for "space elements"). When $n = 2$, spels are called *pixels*, and when $n = 3$ they are called *voxels*. The coordinates of a center of a spel are an n-tuple of integers, defining a point in Z^n. For any spel c and for $1 \le j \le n$, we denote by c_j the jth coordinate of the center of c. We shall think of Z^n itself as the set of all spels in R^n with the above interpretation of spels and use the concepts of spels and points in Z^n interchangeably.

A fuzzy relation α in Z^n is said to be a *fuzzy spel adjacency* if it is reflexive and symmetric. It is desirable that α be such that $\mu_\alpha(c, d)$ is a nonincreasing function of the distance $\|c - d\|$ between c and d, where $\|\cdot\|$ represents any $L2$ norm in R^n.

An example of fuzzy spel adjacency is the fuzzy relation ω defined by

$$\mu_\omega(c, d) = \begin{cases} \dfrac{1}{1 + k_1\left(\sqrt{\sum_{i=1}^{n} (c_i - d_i)^2}\right)}, & \text{if } \sum_{i=1}^{n} |c_i - d_i| \le n \\ 0, & \text{otherwise,} \end{cases} \tag{2.8}$$

k_1 being a nonnegative constant. It is easily verified that the hard adjacency relations commonly used in digital topology [15–17] are special cases of fuzzy spel adjacencies.

We call the pair (Z^n, α), where α is a fuzzy spel adjacency, a *fuzzy digital space*. Fuzzy digital space is a concept that characterizes the underlying digital grid system independent of any image-related concepts. We shall eventually tie this with image-related concepts to arrive at fuzzy object-related notions.

2.3. *Scenes*, *Membership Scenes*, *Binary Scenes*, *Slices of* $\mathscr{C}$, *Classification*, *Segmentation*

A *scene over* a fuzzy digital space (Z^n, α) is a pair $\mathscr{C} = (C, f)$, where $C = \{c \mid -b_j \le c_j \le b_j$ for some $b \in Z^n_+\}$, Z^n_+ is the set of n-tuples of positive integers, and f is a function whose domain is C, called the *scene domain*, and whose range is a set of numbers. Any scene $\mathscr{C}$ over (Z^n, α) in which the range of f is a subset of the closed unit interval [0, 1] is called a *membership scene over* (Z^n, α). We call $\mathscr{C}$ a *binary scene over* (Z^n, α) if $\mathscr{C}$ is a membership scene over Z^n in which the range of f is {0, 1}. $\mathscr{C}$ is said to be nonemplty if there exists $c \in C$ such that $f(c) \neq 0$.

A set of all spels $c \in C$, all but distinct two of whose coordinates c_i, c_j are fixed, together with the restriction of f to that set, will be called a c_ic_j-*slice of* $\mathscr{C}$.

Scenes contain information about objects that have been imaged. Spel values in a membership scene constitute the membership of spels in a particular object of interest. The notion of a membership scene is for developing fuzzy object concepts. The purpose of an imaging operation, such as CT scanning, is indeed to get membership scenes. If the scene representing acquired image data already portrays object membership adequately, there is no need for the secondary concept of a membership scene, and the scene itself, but for a linear scaling of its values, can be treated as a membership scene. (That the range of f in a membership scene (C, f) over (Z^n, α) be [0, 1] is a theoretical requirement stemming from our need to handle fuzzy concepts. For implementations, the range of f can be taken to be the range of the original scene values themselves.) In general, however, spel values in scenes do not represent directly their degree of membership in objects. For example, a spel may have low value, yet it may have higher membership in a certain object. An ideal membership scene should contain only the object of interest with the spel values indicating as closely as possible the degree of membership of spels in the object. We call any process that converts a scene to a membership scene *n-classification*. *n-segmentation* is any process that converts a scene over (Z^n, α) to a binary scene over (Z^n, α). The purpose of n-segmentation may be considered as to identify the object of interest as a hard subset of the scene domain. The purpose of n-classification may be thought of as to identify the object of interest as a fuzzy subset of the scene domain.

For now we assume that we are given a membership scene over (Z^n, α). We shall come back to the n-classification and n-segmentation problems in Section 4.

2.4. *Fuzzy Spel Affinities*

Let $\mathscr{C} = (C, f)$ be a membership scene over (Z^n, α). Any fuzzy relation κ in C is said to be a fuzzy spel affinity in $\mathscr{C}$ if it is reflexive and symmetric. In practice, we would want κ to be such that $\mu_\kappa(c, d)$ is a function of $\mu_\alpha(c, d)$ and of $f(c)$ and $f(d)$ and perhaps even of c and d themselves.

An example of fuzzy spel affinity is the fuzzy relation ζ defined as follows. Let $\mathscr{C} = (C, f)$ be a membership scene over (Z^n, ω), where ω is as defined in (2.8). For all $(c, d) \in C \times C$, define

$$\mu_\zeta(c, d) = \frac{\mu_\omega(c, d)}{1 + k_2|f(c) - f(d)|}, \tag{2.9}$$

where k_2 is a nonnegative constant. It is easy to verify that ζ is a fuzzy spel affinity in $\mathscr{C}$. Clearly, the closer c and d are to each other in location and in their membership values in $\mathscr{C}$, the greater is their affinity.

The functional form used for fuzzy spel affinity can be much more sophisticated than the example in (2.9). Note that there is no requirement of "localness" for this relation. In fact it may even be "shift variant" in the sense that spel affinity may depend on where c and d are in C. For computational reasons, however, we may have to bring in some of these restrictions. We shall come back to this issue in Sections 3 and 4.

2.5. *Path, Fuzzy κ-Net*

Let $\mathscr{C} = (C, f)$ be a membership scene over a fuzzy digital space (Z^n, α) and let κ be a fuzzy spel affinity in $\mathscr{C}$.

A *nonempty path* p_{cd} in $\mathscr{C}$ *from* a spel $c \in C$ *to* a spel $d \in C$ is a sequence $\langle c^{(1)}, c^{(2)}, \ldots, c^{(m)}\rangle$ of $m \geq 2$ spels, all in C, such that $c^{(1)} = c$ and $c^{(m)} = d$. Note that the successive spels in the sequence may be any, not necessarily distinct, elements of C. *An empty path in* $\mathscr{C}$ from c to d, denoted $\langle\,\rangle$, is a sequence of no elements. The set of all (empty and nonempty) paths in $\mathscr{C}$ from c to d is denoted by P_{cd}. We use $P_\mathscr{C}$ to denote the set of all paths in $\mathscr{C}$, defined as $P_\mathscr{C} = \bigcup_{(c,d)\in C\times C} P_{cd}$. The *fuzzy κ-net* $\mathscr{N}$ of $\mathscr{C}$ is a fuzzy subset of $P_\mathscr{C}$ with its membership function defined as follows: for all $p = \langle c^{(1)}, c^{(2)}, \ldots, c^{(m)}\rangle \in P_\mathscr{C}$,

$$\mu_\mathscr{N}(p) = \min[\mu_\kappa(c^{(1)}, c^{(2)}), \mu_\kappa(c^{(2)}, c^{(3)}), \ldots, \mu_\kappa(c^{(m-1)}, c^{(m)})], \tag{2.10a}$$

and

$$\mu_\mathscr{N}(\langle\,\rangle) = 0. \tag{2.10b}$$

The fuzzy κ-net concept captures the idea of assigning a strength to every path that connects any pair of spels in $\mathscr{C}$.

We define a binary *join to* operation on $P_\mathscr{C}$, denoted "+" as follows. For any two nonempty paths $p_{cd} = \langle c^{(1)}, c^{(2)}, \ldots, c^{(m)}\rangle \in P_\mathscr{C}$ and $p_{de} = \langle d^{(1)}, d^{(2)}, \ldots, d^{(l)}\rangle \in P_\mathscr{C}$,

$$p_{cd} + p_{de} = \langle c^{(1)}, c^{(2)}, \ldots, c^{(m)}, d^{(2)}, d^{(3)}, \ldots, d^{(l)}\rangle, \tag{2.11a}$$

and

$$p_{cd} + \langle\,\rangle = p_{cd}, \tag{2.11b}$$

and

$$\langle\,\rangle + p_{de} = p_{de}, \tag{2.11c}$$

and

$$\langle\,\rangle + \langle\,\rangle = \langle\,\rangle. \tag{2.11d}$$

Note that the join of p_{de} to p_{cd} is not defined if $e \neq c$.

PROPOSITION 2.1. *For any membership scene $\mathscr{C} = (C, f)$ over any fuzzy digital space (Z^n, α) and for any spels $c, e \in C$,*

$$P_{ce} = \{p_{cd} + p_{de} \mid d \in C \text{ and } p_{cd} \in P_{cd} \text{ and } p_{de} \in P_{de}\}. \tag{2.12}$$

Proof. Evidently, if $p \in P_{cd}$ and $p' \in P_{de}$ for any $d \in C$, then $p + p' \in P_{ce}$.

Conversely, let $p_{ce} = \langle c^{(1)}, c^{(2)}, \ldots, c^{(m)}\rangle \in P_{ce}$. If $m > 2$, then there exists j, $1 < j < m$, such that $c^j \in C$, $p_{cc^{(j)}} = \langle c^{(1)}, c^{(2)}, \ldots, c^{(j)}\rangle \in P_{cc^{(j)}}$, $p_{c^{(j)}e} = \langle c^{(j)}, c^{(j+1)}, \ldots, c^{(m)}\rangle \in P_{c^{(j)}e}$, and $p_{ce} = p_{cc^{(j)}} + p_{c^{(j)}e}$. If $m = 2$, then $p_{ce} = \langle c, e\rangle + \langle\,\rangle$, and if p_{ce} is the empty path, then $p_{ce} = \langle\,\rangle + \langle\,\rangle$. In all cases, p_{ce} is an element of the set on the right side of (2.12). ∎

2.6. *Fuzzy κ-Connectedness K, Binary Relation K_θ*

Let $\mathscr{C} = (C, f)$ be a membership scene over (Z^n, α), let κ be a fuzzy spel affinity in $\mathscr{C}$, and let $\mathscr{N}$ be the fuzzy κ-net of $\mathscr{C}$. Fuzzy *κ-connectedness in* $\mathscr{C}$, denoted K, is a fuzzy relation in C, defined as follows. For all c, d in C,

$$\mu_K(c, d) = \max_{p\in P_{cd}}[\mu_\mathscr{N}(p)]. \tag{2.13}$$

(For fuzzy connectedness, we shall always use the upper case form of the symbol used to represent the corresponding fuzzy spel affinity.)

The intuitive idea underlying the principle of fuzzy connectedness, as described at the end of Section 2.1, is to assign to every pair of spels (c, d) in C a "strength of connectivity" between them. This strength is determined as follows. There are numerous possible paths between c and d (expressed by the set P_{cd}). Along each path p, there is a "weakest link" (in the sense of the smallest affinity between spels along p) that determines the strength of connectivity along p. The actual strength of connectivity from c to d is the maximum of the strength of all paths. Note that in the definition of strength, α, $\mathscr{C}$, and κ all play important roles.

The following result is vital to the development of the notion of fuzzy objects.

PROPOSITION 2.2. For any fuzzy spel affinity κ in any membership scene $\mathscr{C}$ over any fuzzy digital space (Z^n, α), fuzzy κ-connectedness K in $\mathscr{C}$ is a similitude relation in C.

Proof. Let $\mathscr{N}$ be the fuzzy κ-net of $\mathscr{C}$. For any spel $c \in C$, $\mu_K(c, c) = \max_{p \in P_{cc}} [\mu_{\mathscr{N}}(p)] = \mu_K(c, c) = 1$. So K is reflexive.

For any spels, $c\ d \in C$, we first observe that there is a one-to-one correspondence between P_{cd} and P_{dc} as follows: If $p_{cd} = \langle c^{(1)}, c^{(2)}, \ldots, c^{(m)} \rangle \in P_{cd}$, then the corresponding element of P_{dc} is $p_{dc} = \langle c^{(m)}, c^{(m-1)}, \ldots, c^{(1)} \rangle$. Clearly, $\mu_{\mathscr{N}}(p_{cd}) = \mu_{\mathscr{N}}(p_{dc})$. Therefore, $\mu_K(c, d) = \max_{p_{cd} \in P_{cd}} [\mu_{\mathscr{N}}(p_{cd})] = \max_{p_{dc} \in P_{dc}} [\mu_{\mathscr{N}}(p_{dc})] = \mu_K(d, c)$, establishing the symmetry of K.

For any $c, e \in C$, by (2.13),

$$\begin{aligned}
\mu_K(c,e) &= \max_{p_{ce} \in P_{ce}} [\mu_{\mathscr{N}}(p_{ce})] \\
&= \max_{\substack{d \in C \\ p_{cd} \in P_{cd} \\ p_{de} \in P_{de}}} [\mu_{\mathscr{N}}(p_{cd} + p_{de})], \text{by (2.12)} \\
&= \max_{d \in C} \left[\max_{p_{cd} \in P_{cd}} \left[\max_{p_{de} \in P_{de}} [\min(\mu_{\mathscr{N}}(p_{cd}), \mu_{\mathscr{N}}(p_{de}))] \right] \right] \\
&= \max_{d \in C} \left[\max_{p_{cd} \in P_{cd}} \left[\min \left[\max_{p_{de} \in P_{de}} (\mu_{\mathscr{N}}(p_{de})), \mu_{\mathscr{N}}(p_{cd}) \right] \right] \right] \\
&= \max_{d \in C} \left[\min \left[\max_{p_{cd} \in P_{cd}} (\mu_{\mathscr{N}}(p_{cd})), \max_{p_{de} \in P_{de}} (\mu_{\mathscr{N}}(p_{de})) \right] \right] \\
&= \max_{d \in C} [\min(\mu_K(c,d), \mu_K(d,e))], \text{by (2.13)},
\end{aligned}$$

establishing the transitivity of K. ■

The reason for utilizing the max–min composition in the definition of κ came naturally from the physical analogy of links and strengths of paths described at the end of Section 2.1 and was not designed by any theoretical argument. There are perhaps other compositions (such as max-t [13]) that are appropriate for this purpose.

To define the notion of a fuzzy connected component, we need the following hard binary relation K_θ based on the fuzzy relation K. We use θ to denote any subset of $[0, 1]$ and, for $0 \leq x \leq 1$, define $\theta_x = [x, 1]$.

Let $\mathscr{C} = (C, f)$ be a membership scene over a fuzzy digital space (Z^n, α), and let κ be a fuzzy spel affinity in $\mathscr{C}$. For all $c, d \in C$ and for any $\theta \subset [0, 1]$, we define a (hard) binary relation K_θ in C as

$$\mu_{K_\theta}(c,d) = \begin{cases} 1, & \text{iff } \mu_K(c,d) \in \theta, \\ 0, & \text{otherwise.} \end{cases} \tag{2.14}$$

PROPOSITION 2.3. *For any membership scene $\mathscr{C} = (C, f)$ over any fuzzy digital space (Z^n, α), for any $x \in [0, 1]$, and for any fuzzy κ-connectedness K in $\mathscr{C}$, K_{θ_x} is an equivalence relation in C.*

Proof. K_{θ_x} is reflexive since K is reflexive.

Since $\mu_K(c, d) = \mu_K(d, c)$ for all $c, d \in C$ (by Proposition 2.2), either $\mu_{K_{\theta x}}(c, d) = \mu_{K_{\theta x}}(d, c) = 1$ (when $\mu_{K_{\theta x}}(c, d) \in \theta_x$) or $\mu_{K_{\theta x}}(c, d) = \mu_{K_{\theta x}}(d, c) = 0$ (when $\mu_{K_{\theta x}}(c, d) \notin \theta_x$). So $K_{\theta x}$ is symmetric.

Since K is transitive (by Proposition 2.2), for all (c, d), (d, e), (c, e) in $C \times C$,

$$\mu_K(c,e) = \max_{d \in C} [\min[\mu_K(c,d), \mu_K(d,e)]]. \tag{2.15}$$

Suppose $\mu_{K_{\theta x}}(c, g) = 1 = \mu_{K_{\theta x}}(g, e)$ for some $c, g, e \in C$. Then by (2.15),

$$\mu_K(c,e) \geq \min[\mu_K(c,g), \mu_K(g,e)]. \tag{2.16}$$

Since $\mu_K(c, g) \geq x$ and $\mu_K(g, e) \geq x$, by (2.16) $\mu_K(c, e) \geq x$. Hence, $\mu_{K_{\theta x}}(c, e) = 1$, establishing the transitivity of $K_{\theta x}$. $K_{\theta x}$ is therefore an equivalence relation in C. ■

2.7. *Fuzzy κ-Components, Fuzzy κθ-Objects, Fuzzy Object Extraction, Fuzzy Object Labeling*

Let $\mathscr{C} = (C, f)$ be a membership scene over a fuzzy digital space (Z^n, α), let κ be a fuzzy spel affinity in $\mathscr{C}$, let $x \in [0, 1]$, and let $O_{\theta x}$ be an equivalence class ([13, Chap. 10]) of the relation $K_{\theta x}$ in C. A *fuzzy κ-component $\mathscr{O}_{\theta x}$ of C of strength θ_x* is a fuzzy subset of C defined by the membership function

$$\mu_{\mathscr{O}_{\theta x}}(c) = \begin{cases} f(c), & \text{if } c \in O_{\theta x} \\ 0, & \text{otherwise.} \end{cases} \tag{2.17}$$

We use the notation $[o]_{\theta x}$ to denote the equivalence class of $K_{\theta x}$ that contains o for any $o \in C$.

The *fuzzy κ-component of C of strength θ_x that contains*

o, denoted $\mathscr{O}_{\theta x}(o)$, is a fuzzy subset of C whose membership function is

$$\mu_{\mathscr{O}_{\theta x}(o)}(c) = \begin{cases} f(c), & \text{if } c \in [o]_{\theta x} \\ 0, & \text{otherwise.} \end{cases} \tag{2.18}$$

A *fuzzy $\kappa\theta_x$-object of $\mathscr{C}$* is a fuzzy κ-component of $\mathscr{C}$ of strength θ_x. For any spel $o \in C$, a *fuzzy $\kappa\theta_x$-object of $\mathscr{C}$ that contains o* is a fuzzy κ-component of $\mathscr{C}$ of strength θ_x that contains o.

Given $\mathscr{C}$, κ, α, and $x \in [0, 1]$, and any spel $o \in C$, we refer to the process of finding the fuzzy $\kappa\theta_x$-object that contains o as *n-fuzzy object extraction*. We refer to the process of finding all fuzzy $\kappa\theta_x$-objects of $\mathscr{C}$, given $\mathscr{C}$, κ, α, and $x \in [0, 1]$, as *n-fuzzy object labeling*. In practice, we may do n-fuzzy object extraction and labeling directly on scenes (ignoring the theoretical requirement of the spel value range to be $[0, 1]$ for membership scenes and treating the given scene itself as a membership scene). Therefore, these processes can also be considered as solutions to the n-classification problem.

The idea of specifying a spel o in C for n-fuzzy object extraction comes from the practical consideration of indicating a particular object in $\mathscr{C}$ that the user is interested in detecting. In the CT slice of Fig. 1a, for example, the user may be interested in detecting one of the two bones. The specific bone is indicated by pointing to a spel o in this bone using the cursor of a pointing device on a display of an appropriate $c_i c_j$-slice of $\mathscr{C}$. When the number of objects to be detected is large (such as in the case of multiple sclerosis lesions of the brain), it may not be practical to specify a spel for each object. In this case, the approach of n-fuzzy object labeling is more appropriate.

The assignment of values $f(c)$ to spels c of a fuzzy $\kappa\theta_x$-object of $\mathscr{C}$ requires some explanation. Recall that we started with the assumption that $\mathscr{C}$ is a membership scene, wherein scene density is supposed to indicate "objectness." If $\mathscr{C}$ is a scene (rather than a membership scene), such as the MR slices of a brain, then scene density values may not directly indicate objectness and it may not be appropriate to assign $f(c)$ as the spel membership value. We shall come back to this issue in Section 4.2.

Both n-fuzzy object extraction and labeling are computationally formidable processes even for the case $n = 2$. Any method for these processes that proceeds directly from the definitions will be computationally impractical. However, there are certain properties of fuzzy $\kappa\theta_x$-objects which when exploited can lead to computationally practical algorithms for these processes. We now proceed to study these properties.

2.8. *Properties of Fuzzy $\kappa\theta_x$-Objects*

A property analogous to that of hard connected components follows for fuzzy κ-components directly from Proposition 2.3.

PROPOSITION 2.4. *For any membership scene $\mathscr{C} = (C, f)$ over any fuzzy digital space (Z^n, α), for any fuzzy spel affinity κ in $\mathscr{C}$ and for any $x \in [0, 1]$, the set $\{\mathscr{O}^{(1)}_{\theta x}, \mathscr{O}^{(2)}_{\theta x}, \ldots, \mathscr{O}^{(l)}_{\theta x}\}$ of all distinct fuzzy $\kappa\theta_x$-objects of $\mathscr{C}$ satisfies the following:*

(i) *For $1 \le i, j \le l$, and $i \ne j$, $\mathscr{O}^{(i)}_{\theta x} \cap \mathscr{O}^{(j)}_{\theta x} = \Phi$.*

(ii) $\bigcup_{1 \le i \le l} \mathscr{O}^{(i)}_{\theta x} = \mathscr{C}$.

Proof. (i) Follows from the disjointness of distinct equivalence classes of $K_{\theta x}$ and from the definition of fuzzy $\kappa_{\theta x}$-objects. (ii) Follows from the fact that the equivalence classes of $K_{\theta x}$ partition C and from the definition of fuzzy κ-components of $\mathscr{C}$ of strength θ_x. ■

The following lemma leads to one of our main results that has significant computational consequences for n-fuzzy object extraction.

LEMMA 2.5. *Let $\mathscr{C} = (C, f)$ be any membership scene over any fuzzy digital space (Z^n, α), let κ be any fuzzy spel affinity in $\mathscr{C}$, let $x \in [0, 1]$, and let o be any element of C. Define a special subset $\Omega_{\theta x}(o)$ of C as*

$$\Omega_{\theta x}(o) = \{c \in C \mid \mu_K(o, c) \in \theta_x\}. \tag{2.19}$$

Then, $\Omega_{\theta x}(o) = [o]_{\theta x}$.

Proof. For any spel $c \in [o]_{\theta x}$, by (2.14), $\mu_{K_{\theta x}}(o, c) = 1$, implying that $\mu_K(o, c) \in \theta_x$. Hence by (2.19), $c \in \Omega_{\theta x}(o)$, and thus $[o]_{\theta x} \subset \Omega_{\theta x}(o)$.

For any spel $c \in \Omega_{\theta x}(o)$, by (2.19) and (2.14), $\mu_{K_{\theta x}}(o, c) = 1$, and by Proposition 2.3, $c \in [o]_{\theta x}$. Therefore $\Omega_{\theta x}(o) \subset [o]_{\theta x}$. ■

The following theorem gives us practical methods for n-fuzzy object extraction.

THEOREM 2.6. *For any membership scene $\mathscr{C} = (C, f)$ over any fuzzy digital space (Z^n, α), for any fuzzy spel affinity κ in $\mathscr{C}$, for any $x \in [0, 1]$, and for any spel $o \in C$, the fuzzy $\kappa\theta_x$-object $\mathscr{O}_{\theta x}(o)$ containing o is given by the membership function*

$$\mu_{\mathscr{O}_{\theta x}(o)}(c) = \begin{cases} f(c), & \text{if } c \in \Omega_{\theta x}(o) \\ 0, & \text{otherwise.} \end{cases} \tag{2.20}$$

Proof. Combine Lemma 2.5 and (2.18). ■

This result is quite remarkable especially considering the minimal restrictions that are put on the functional forms of μ_α and μ_κ. The theorem implies that for finding the fuzzy $\kappa\theta_x$-object containing o, it is not necessary to compute $\mu_K(c, d)$ for each possible pair (c, d) of spels in C. Rather it is sufficient to compute $\mu_K(o, c)$ for each spel $c \in C$. This is a vast reduction in combinatorial complexity.

Despite this simplification, we still need to determine all possible paths from o to each $c \in C$, and for each such path, evaluate (2.10). In the next section, we shall describe efficient algorithms to carry out this computation.

The following result suggests a solution to the n-fuzzy object labeling problem, given that a method for n-fuzzy object extraction is available.

COROLLARY 2.7. *Let $\mathscr{C} = (C, f)$ be any membership scene over any fuzzy digital space (Z^n, α), let κ be any fuzzy spel affinity in $\mathscr{C}$, let $x \in [0, 1]$, let $\{\mathscr{O}^{(1)}_{\theta x}, \mathscr{O}^{(2)}_{\theta x}, \ldots, \mathscr{O}^{(l)}_{\theta x}\}$ be the set of all distinct $\kappa\theta_x$-objects of $\mathscr{C}$, and let $o^{(1)}, o^{(2)}, \ldots, o^{(l)}$ be spels that are contained in $\mathscr{O}^{(1)}_{\theta x}, \mathscr{O}^{(2)}_{\theta x}, \ldots, \mathscr{O}^{(l)}_{\theta x}$, respectively. Then*

(i) *For $1 \leq i, j \leq l$, and $i \neq j$, $\Omega_{\theta x}(o^{(i)}) \cap \Omega_{\theta x}(o^{(j)}) = \phi$, and*

(ii) $\bigcup_{1 \leq i \leq l} \Omega_{\theta x}(o^{(i)}) = C$.

Proof. Combine Proposition 2.4 and Theorem 2.6. ■

The following result specifies the necessary (but not sufficient) condition for the inclusion relationship among fuzzy $\kappa\theta_x$-objects.

PROPOSITION 2.8. *For any membership scene $\mathscr{C} = (C, f)$ over any fuzzy digital space (Z^n, α), for any fuzzy spel affinity κ in $\mathscr{C}$, for any spel $o \in C$, and for any $t, y \in [0, 1]$, $\mathscr{O}_{\theta y}(o) \subset \mathscr{O}_{\theta t}(o)$ if $t \leq y$.*

Proof. For any spel $c \in [o]_{\theta y}$, $\mu_K(o, c) \in \theta y$, which implies that $\mu_K(o, c) \in \theta_t$ since $\theta_y \subset \theta_t$. Hence $\mu_{K_{\theta_t}}(o, c) = 1$, and since K_{θ_t} is an equivalence relation, $c \in [o]_{\theta_t}$. Thus, whenever $c \in [o]_{\theta_y}$, $\mu_{\mathscr{O}_{\theta_y}(o)}(c) = \mu_{\mathscr{O}_{\theta_t}(o)}(c)$. For $c \in C - [o]_{\theta_y}$, $\mu_{\mathscr{O}_{\theta_y}(o)}(c) = 0$ by (2.18). Thus $\mathscr{O}_{\theta y}(o) \subset \mathscr{O}_{\theta t}(o)$. ■

We have been studying the notion of $\kappa\theta_x$-objects and their properties for certain special subsets θ_x of [0, 1]. This notion can be generalized to more general subsets of [0, 1] with similar attendant properties. We will not, however, pursue this here.

3. ALGORITHMS

In this section, we present algorithms, first for n-fuzzy object extraction and then for n-fuzzy object labeling.

3.1. *Fuzzy Object Extraction*

We present two algorithms for n-fuzzy object extraction, both based on dynamic programming [18, Chap. 25]. In the first algorithm named $\kappa\theta_xFOE$, the value of θ_x is specified beforehand, and the algorithm makes essential use of this predetermination. In the second algorithm, named κFOE, we do not assume that θ_x is known beforehand. $\kappa\theta_xFOE$ terminates faster than κFOE for two reasons. First, when θ_x is known in advance, there is no need to find *the* best path p_{oc} from o (the spel contained in the fuzzy $\kappa\theta_x$-object of $\mathscr{C}$ to be extracted) to $c \in C$ such that $\mu_K(o, c) = \mu_{\mathscr{N}}(p_{oc})$ (see (2.13)). Rather, it is enough to find *a* path p'_{oc} such that $\mu_{\mathscr{N}}(p'_{oc}) \geq x$. When the first p'_{oc} satisfying this condition is found, the search for the best path from o to c can be stopped. Second, certain computations can be avoided for those spels $d \in C$ for which $\mu_K(o, d) < x$.

Although κFOE terminates slower, it has the practical advantage that x can be specified interactively after the algorithm terminates and thereby it becomes possible to choose the appropriate strength of connectivity to define the fuzzy $\kappa\theta_x$-object properly. The algorithm essentially outputs a scene expressing strength of connectivity between o and all $c \in C$. We call this scene the K_o-*secne* $\mathscr{C}_o$ *of* $\mathscr{C}$ defined by $\mathscr{C}_o = (C_o, f_o)$, where $C_o = C$ and for all $c \in C_o$, $f_o(c) = \mu_K(o, c)$. By thresholding this scene at various values x, we can examine the various fuzzy $\kappa\theta_x$-objects that result. This scene has interesting properties that are relevant to n-classification and n-segmentation, and in shell rendering and manipulation [8] of $\kappa\theta_x$-objects. We will not pursue these directions in this paper. In closing this discussion, we wish to point out that, whenever $f_o(c) \geq x$ and $f_o(d) \geq x$ for any spels $c, d \in C$, the transitivity of K guarantees that $\mu_K(c, d) \geq x$.

As a compromise between speed and practical utility, it is better to run Algorithm $\kappa\theta_xFOE$ for $x > 0$ but for a sufficiently smaller value than the strength of connectivity y we expect for the object of interest. This will ensure that the algorithm will terminate substantially faster than κFOE, and, because of Proposition 2.8, the convenience of deciding and choosing the value of y after the termination of the algorithm is retained. This is the mode in which we use algorithm $\kappa\theta_xFOE$ in all of our current applications involving massive data.

We now present the two algorithms. A knowledge of dynamic programming is not essential to understand or to implement these algorithms. But this knowledge is helpful in appreciating and understanding the performance of these algorithms.

ALGORITHM $\kappa\theta_xFOE$

Input: $\mathscr{C}$, o, κ and θ_x as defined in Section 2.
Output: $\mathscr{O}_{\theta x}(o)$ as defined in Section 2.
Auxiliary Data Structures: An nD array representing the K_o-scene $\mathscr{C}_o = (C_o, f_o)$ of $\mathscr{C}$ and a queue Q of spels. We refer to the array itself by $\mathscr{C}_o$ for the purpose of the algorithm.

begin

0. set all elements of $\mathscr{C}_o$ to 0 except o which is set to 1;

1. push all spels $c \in C_o$ such that $\mu_\kappa(o, c) > 0$ to Q;
   while Q is not empty *do*
2. remove a spel c from Q;
3. *if* $f_o(c) < x$ *then*
4. find $f_{\max} = \max_{d \in C_o} [\min(f_o(d), \mu_\kappa(c, d))]$;
5. *if* $f_{\max} > f_o(c)$ and $f_{\max} \geq x$ *then*
6. set $f_o(c) = f_{\max}$;
7. push all spels e such that $\mu_\kappa(c, e) > 0$ to Q;
   endif;
   endif;
   endwhile;
8. Create and output $\mathscr{O}_{\theta_x}(o)$ by assigning to those spels c in C_o for which $f_o(c) \neq 0$ the value $f(c)$, and to the rest the value 0;

end

ALGORITHM κFOE

Input: $\mathscr{C}$, o, κ as defined in Section 2.
Output: K_o-scene $\mathscr{C}_o = (C_o, f_o)$ of $\mathscr{C}$.
Auxiliary Data Structures: An nD array representing the K_o-scene $\mathscr{C}_o = (C_o, f_o)$ of $\mathscr{C}$ and a queue Q of spels. For the purpose of the algorithm, we refer to the array itself by $\mathscr{C}_o$.

begin

0. set all elements of $\mathscr{C}_o$ to 0 except o which is set to 1;
1. push all spels c of C_o such that $\mu_\kappa(o, c) > 0$ to Q;
   while Q is not empty *do*
2. remove a spel c from Q;
4. find $f_{\max} = \max_{d \in C_o} [\min(f_o(d), \mu_\kappa(c, d))]$;
5. *if* $f_{\max} > f_o(c)$ *then*
6. set $f_o(c) = f_{\max}$;
7. push all spels e such that $\mu_\kappa(c, e) > 0$ to Q;
   endif;
   endwhile;

end

To generate the fuzzy $\kappa\theta_x$-object containing o, $\mathscr{C}_o$ should be thresholded at x and Step 8 of Algorithm $\kappa\theta_x FOE$ should be applied.

The algorithms are both iterative and they work as follows. Within the iterative loop, a spel c of C_o is examined to see if the paths from o coming up to each spel d can be expanded unto c itself profitably (Steps 4, 5). Which of the spels d actually matter depends on α. If we take fuzzy spel adjacency α to be any of the commonly used hard adjacency relations, then the spels that matter are just the immediate neighbors. The array $\mathscr{C}_o$, which will eventually contain the K_o-scene, contains the strength of connectivity $\mu_K(o, d)$ for the individual elements d of $\mathscr{C}_o$. To determine the profitable expandability of paths unto c, a min–max test is done in Step 4 and if a "stronger" path is found (in Step 5) the higher strength is assigned to c (in Step 6).

PROPOSITION 3.1. *Algorithms $\kappa\theta_x FOE$ and κFOE terminate. When they do so, they output respectively the fuzzy $\kappa\theta_x$-object containing o and the K_o-scene of $\mathscr{C}$.*

Sketch of Proof. That the standard dynamic programming algorithm terminates and produces the correct output is an established result [18, Section 25.2]. Our modifications are essentially in Steps 1 and 7 in Algorithm κFOE and in Steps 1, 3, and 7 in Algorithm $\kappa\theta_x FOE$.

We claim that Steps 1 and 7 do not affect the termination of κFOE and that they do not make the output κFOE different from that of the standard algorithm. To prove the claim, suppose a spel e such that $\mu_\kappa(c, e) = 0$ is pushed into Q in Step 1 or 7. Upon the removal of e in Step 2, $f_{\max}$ will be 0 in Step 4, and so Steps 6 and 7 will not be executed.

To prove the correctness of $\kappa\theta_x FOE$, we first observe that the value assigned to every spel in $\mathscr{C}_o$ never decreases with the increasing iteration number of the *do–while* loop. We claim that the outputs of $\kappa\theta_x FOE$ with and without Step 3 are identical. To prove the claim, let c be the first spel for which $f_o(c)(= x'$, say$) \geq x$ in Step 3. (At least one such spel must exist since $f_o(o) = 1$.) Suppose we ignore Step 3, carry out Steps 4 onward, and say $f_o(c)$ was updated in Step 6 to x'' (if this did not happen, there is nothing to prove). Obviously $x'' > x'$. Let a spel e for which $\mu_\kappa(c, e) > 0$ be put in Q in Step 7. (If no such spel exists, there is nothing to prove.) When this spel is removed from Q in Step 2 at the same later time, there are two cases to consider for e in Step 3: $f_o(e) \geq x$ and $f_o(e) < x$. In the first case, obviously, the change in value of e from x' to x'' did not cause e to attain $f_o(e) \geq x$. In the second case, it is easy to check that e's value will not change from $f_o(e) < x$ to $f_o(e) \geq x$ because of just the change in value of e from x' to x''. Therefore, the output in Step 8 will be identical with or without Step 3. ∎

Since we find Algorithm κFOE to be practically more useful, although less efficient, than Algorithm $\kappa\theta_x FOE$, the rest of our discussion in this section will refer to κFOE. Nonetheless many of these remarks apply to $\kappa\theta_x FOE$ as well.

The computational cost of κFOE is determined mainly by α and κ. Obviously, the larger the "neighborhood" of α (i.e., the number of spels d for which $\mu_\alpha(c, d) > 0$), the more expensive Steps 4 and 7 are likely to be. The functional form of μ_κ actually determines the cost of Steps 4 and 7. Since for many spels c, $\mu_\kappa(c, d)$ is evaluated more than once, it is advisable to store these values when they are first computed if adequate fast storage space is available. (Note that it is enough to store one of $\mu_\kappa(c, d)$ and $\mu_\kappa(d, c)$ since κ is symmetric.) It is possible that in Step 7 a spel e that is to be pushed into Q is already in Q. In some cases, it may be possible to search for e in Q less expensively than the cost incurred in the repeated subsequent computation for multiple copies of e that are otherwise needed. In our current implementation, we have not incorporated any of these computational optimizations.

3.2. *Fuzzy Object Labeling*

The algorithms presented above provide a solution to the n-fuzzy object extraction problem. The algorithm that we now describe, called $\kappa\theta_x FOL$, uses Algorithm $\kappa\theta_x FOE$ to solve the n-fuzzy object labeling problem.

ALGORITHM $\kappa_{\theta x} FOL$

Input: $\mathscr{C}$, κ, and θ_x as described in Section 2.
Output: The set of all $\kappa\theta_x$-objects of $\mathscr{C}$.
Auxiliary Data Structures: A list C' of spels which initially contains one copy of every spel in C.

begin
0. put a copy of every spel C into C';
repeat
1. remove a spel o from C';
find the fuzzy $\kappa\theta_x$-object $\mathscr{O}_{\theta x}(o)$ of $\mathscr{C}$ that contains o using Algorithm $\kappa\theta_x FOE$;
3. output $\mathscr{O}_{\theta x}(o)$;
4. remove spels of $[o]_{\theta x}$ that are in C' from C';
until C' is empty;
end

PROPOSITION 3.2. *Algorithm $\kappa\theta_x FOL$ terminates. When it does so, it outputs one copy of every $\kappa\theta_x$-object of $\mathscr{C}$.*

Proof. Combine Proposition 3.1 and Corollary 2.7. ■

The above algorithm is rather straightforward. It essentially finds all equivalence classes in C of the binary relation K_{θ_x} defined in (2.14). Obviously, it will be interesting to combine the ideas underlying (hard) connected component labeling algorithms (e.g., [19, 20]) and Algorithm $\kappa\theta_x FOE$. We will not pursue this direction here.

We wish to point out that, in many situations, the number of spels in the set $\Omega_{\theta x}(o)$ (defined in (2.19)) for $x > 0$ may be just 1; that is, the set contains only o. For example, if $\mathscr{C}$ represents the membership scene of the bones of a joint, there are many spels in $\mathscr{C}$ that do not contain any bone. For each spel o, $\Omega_{\theta x}(o)$ will be a singleton set (this is because fuzzy κ-connectedness K is reflexive). The problem of finding all bones of the joint, each expressed as a fuzzy $\kappa\theta_x$-object, however, is certainly legitimate. Algorithm $\kappa\theta_x FOL$ is easily modified to skip all singleton-set components so that only the real bone components are extracted.

4. APPLICATIONS IN IMAGE SEGMENTATION

In this section, we demonstrate the use of the theory and of the algorithms presented in the previous sections in image segmentation based on several examples drawn from medical imaging. First we take up some of the issues left open in the previous sections.

4.1. *Selection of Fuzzy Spel Adjacencies and Fuzzy Spel Affinities*

In all results presented in this section, α was chosen to be a hard adjacency relation, 4-adjacency for $n = 2$ and 6-adjacency for $n = 3$. That is, for all c, d in C,

$$\mu_\alpha(c,d) = \begin{cases} 1, & \text{if } \sqrt{\sum_i (c_i - d_i)^2} \le 1 \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

The general form of μ_κ can be written as follows. For all $c, d \in C$,

$$\mu_\kappa(c,d) = h(\mu_\alpha(c,d), f(c), f(d), c, d), \tag{4.2}$$

where h is a scalar-valued function with range [0, 1]. The dependence of h on the location of spels indicates that it may be shift variant. For the results presented in this section, $\mu_\kappa(c,d)$ was independent of c and d with the following form. For all $c, d \in C$,

$$\mu_\kappa(c,d) = \mu_\alpha(c,d)[\omega_1 h_1(f(c), f(d)) + \omega_2 h_2(f(c), f(d))], \quad \text{if } c \ne d, \tag{4.3a}$$

$$\mu_\kappa(c,c) = 1, \tag{4.3b}$$

where ω_1 and ω_2 are free parameters satisfying

$$\omega_1 + \omega_2 = 1. \tag{4.3c}$$

The functional forms for h_1 and h_2 are chosen from one of the following:

$$g_1(f(c), f(d)) = e^{-(1/2)[((1/2)(f(c)+f(d))-m_1)/s_1]^2}, \tag{4.4a}$$

$$g_2(f(c), f(d)) = e^{-(1/2)[(|f(c)-f(d)|-m_2)/s_2]^2}, \tag{4.4b}$$

$$g_3(f(c), f(d)) = 1 - g_1(f(c), f(d)), \tag{4.4c}$$

$$g_4(f(c), f(d)) = 1 - g_2(f(c), f(d)). \tag{4.4d}$$

In these expressions, m_1, m_2 and s_1, s_2 represent the mean and standard deviation of spel values and their differences (gradient magnitudes) in the membership scene for spels that are in the object of interest. For illustration, by choosing $h_1(f(c), f(d)) = g_1(f(c), f(d))$, $\omega_1 = 1$, and $\omega_2 = 0$ in (4.3a), we specify a fuzzy spel affinity in which affinity between c and d is greater when their spel values are closer to a mean (expected) spel value m_1. By choosing $h_1(f(c), f(d)) = g_1(f(c), f(d))$, $h_2(f(c), f(d)) = g_4(f(c), f(d))$ and appropriate values for ω_1 and ω_2 (say, $\omega_1 = \omega_2 = 0.5$), we introduce an additional boundary constraint which makes the affinity between c and d lower when the gradient (difference) between their values is closer to a mean gradient value m_2. This component may be thought of as representing enmity (reverse affinity) between c and d. The forms of μ_κ specified by these two examples are the only forms used in the results presented in this section. Clearly, a variety of other more sophisticated forms can also be employed as long as they make κ reflexive and symmetric. Obviously, in place of $f(c)$ and $f(d)$ any features derived from scene intensities evaluated at c and d may also be used and $f(c)$ and $f(d)$ may even be vector-valued. For vector-valued features, we use multivariate versions of (4.4a) and (4.4b)

$$g_1(\mathbf{f}(c), \mathbf{f}(d)) = \frac{1}{(2\pi)^{r/2}|S_1|^{1/2}} e^{-(1/2)[(1/2)(\mathbf{f}(c)+\mathbf{f}(d))-\mathbf{m}_1]^t S_1^{-1}[(1/2)(\mathbf{f}(c)+\mathbf{f}(d))-\mathbf{m}_1]}, \tag{4.5a}$$

$$g_2(\mathbf{f}(c), \mathbf{f}(d)) = \frac{1}{(2\pi)^{r/2}|S_2|^{1/2}} e^{-(1/2)[|\mathbf{f}(c)-\mathbf{f}(d)|-\mathbf{m}_2]^t S_2^{-1}[|\mathbf{f}(c)-\mathbf{f}(d)|-\mathbf{m}_2]}, \tag{4.5b}$$

where $\mathbf{f}(c)$ and $\mathbf{f}(d)$ are r-component column vectors, $\mathbf{m}_1$ and $\mathbf{m}_2$ are r-component mean vectors, S_1 and S_2 are $r \times r$ covariance matrices, S_1^{-1} and S_2^{-1} are the inverses of S_1 and S_2, $|S_1|$ and $|S_2|$ are the determinants of S_1 and S_2, and $|\mathbf{f}(c) - \mathbf{f}(d)|$ denotes componentwise absolute difference between $\mathbf{f}(c)$ and $\mathbf{f}(d)$.

The mean and the standard deviation values in the above fuzzy spel affinities can be determined via any parameter estimation method. Any rough segmentation method such as thresholding, clustering, or user painting of regions on c_ic_j-slices of the given scene can be used to specify spels that are very likely to belong to the object of interest. In applications involving the processing of a large number of scenes of a particular kind (such as the MR images of the brain), this estimation needs to be done only once.

4.2. *Segmentation and Classification*

To do n-segmentation, we may simply threshold the K_o-scene at an appropriate strength of connectedness. Note that this is a thresholding of the hanging togetherness or objectness of the spels and is vastly different from thresholding of the original scene.

Since images are by nature fuzzy, it is more appropriate, and often more accurate, to do n-classification than n-segmentation. One possible approach to n-classification is to express the fuzzy $\kappa\theta_x$-object extracted from the given scene (strictly speaking, a scaled version of it to make it a membership scene) as a membership scene. Another attractive alternative is to use the K_o-scene $\mathscr{C}_o = (C_o, f_o)$ with the following modification as the output membership scene: set the values of those spels c such that $f_o(c) < x$ to 0 and the values of other spels to $f_o(c)$. This is sensible since $f_o(c)$ seems to be a better indicator of objectness than the spel value $f(c)$ in the original scene.

4.3. *Results*

We will present several examples, all based on scenes derived from medical CT and MR imaging. We have conducted extensive evaluation studies in one application (the detection of tissues and multiple sclerosis lesions of the brain via MR imaging) to determine the effectiveness of object identification via n-fuzzy object extraction, n-fuzzy object labeling, and n-classification.

Our first example, shown in Fig. 1, is for illustrating the concepts of K, K_o-scenes, and fuzzy $\kappa\theta_x$-objects. The scene data are obtained via CT of a patient's knee. A c_1c_2-slice of this scene is shown in Fig. 1a. Figure 1b shows the K_o-scene for the 2D scene in Fig. 1a for a spel o selected in the dense part of the bone. Here, algorithm κFOE was run with $n = 2$ and κ described by $\mu_\kappa(c, d) = g_2(f(c), f(d))$ ((4.4b)). The K_o-scene exhibits the following interesting phenomena: (i) The dense parts of the bone within the same bone in which o was specified are strongly connected to o and strongly connected among themselves (by Theorem 2.6), whereas even the dense parts in the other bones are weakly connected to o and to other spels that strongly hang together with o. (ii) The less dense parts of the bone are connected to o and among themselves with moderate strength, whereas they are connected to other aspects in the scene including spels in the other bone very weakly. Figures 1c to 1g show the fuzzy $\kappa\theta_x$-object of this scene containing o for increasing values of x. Figures 1c and 1g represent somewhat the two extremes at and beyond which the object definition is clearly unacceptable. Note that the smaller bone is very weakly connected to the bone of interest and is therefore picked up as part of the $\kappa\theta_x$-object for low values of x (Fig. 1c and 1d). Figure 1h is a shell rendition [8] of all bones in the scene created using a trapezoidal opacity function [5]. Figure 1i shows a shell rendition of one of the bones identified automatically as a fuzzy $\kappa\theta_x$-object. In this case Algorithm κFOE was run with $n = 3$. The two bones come very close to each other in three dimensions (although not apparent in Fig. 1a) and are very difficult to segment in their entirety using hard segmentation and/or connectivity strategies.

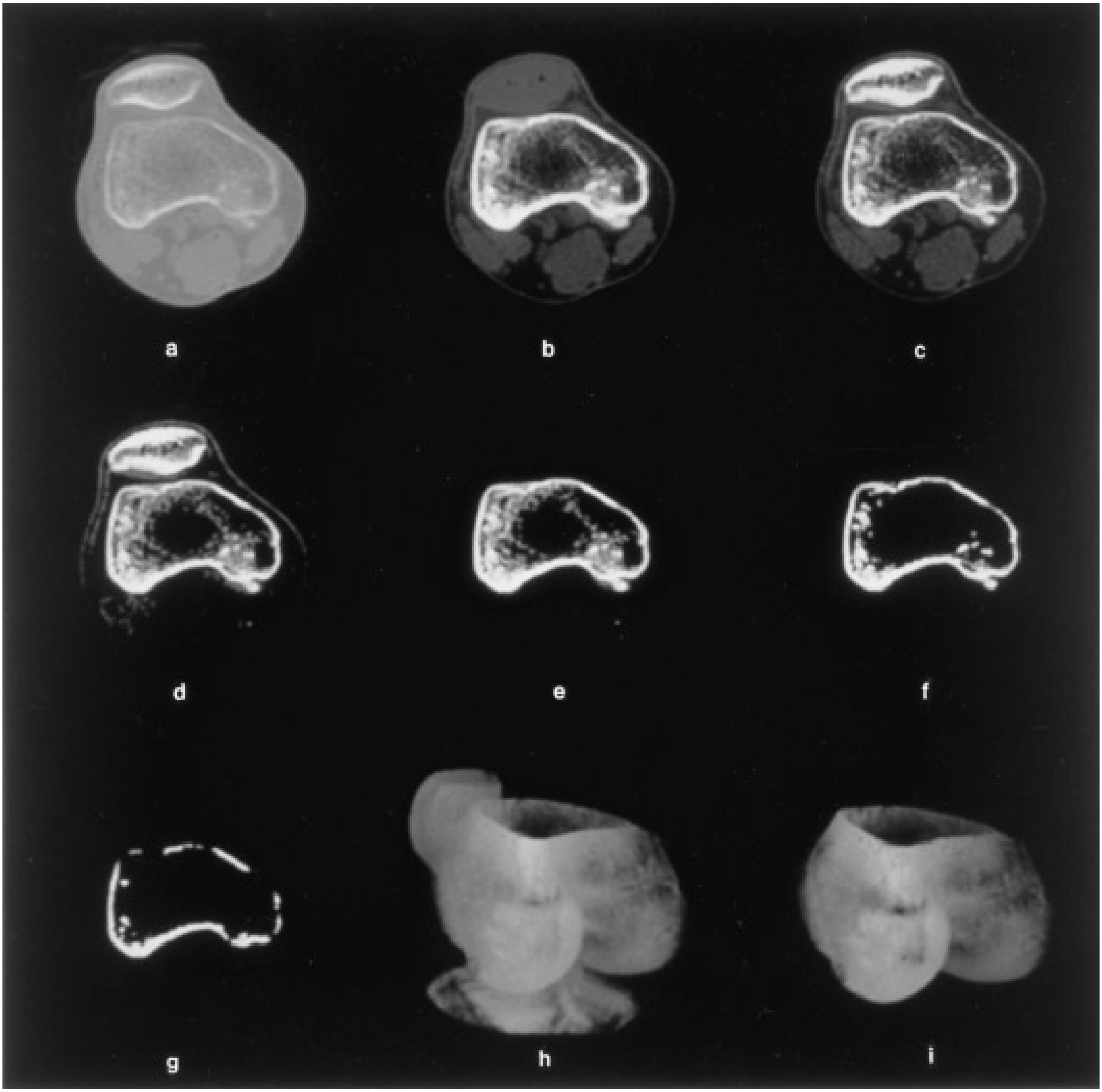

FIG. 1. (a) A c_1c_2-slice of a 3D CT scene of a patient's knee. (b) The K_o-scene for the 2D scene in (a). The spel o was specified in the dense part of the bone. (c) The fuzzy $\kappa\theta_x$-object of the scene in (a) containing o for a "very low" value of x. (d) As in (c) for a "low" value of x. (e) As in (c) for a "medium" value of x. (f) As in (c) a "high" value of x. (g) As in (c) for a "very high" value of x. (h) A shell rendition [8] of the bones in the scene of (a) using a trapezoidal opacity function [5]. (i) A shell rendition of the fuzzy $\kappa\theta_x$-object shown in (e).

In Fig. 2, we demonstrate how some of the soft-tissue regions in the scene of Fig. 1 can be identified as fuzzy $\kappa\theta_x$-objects. To make matters worse, we added to the scene of Fig. 1 a ramp function that increases from left to right but remains constant in the vertical direction. The resulting scene is shown in Fig. 2a. Figure 2b shows the K_o-scene for a spel o chosen in the soft-tissue blob in the lower center in Fig. 2a. It is clear how the bony regions are strongly dissimilar to and disconnected from the specified soft-tissue blob. It is also clear that spels in other soft-tissue blobs which hang together and which are loosely connected with the specified blob have moderate strength

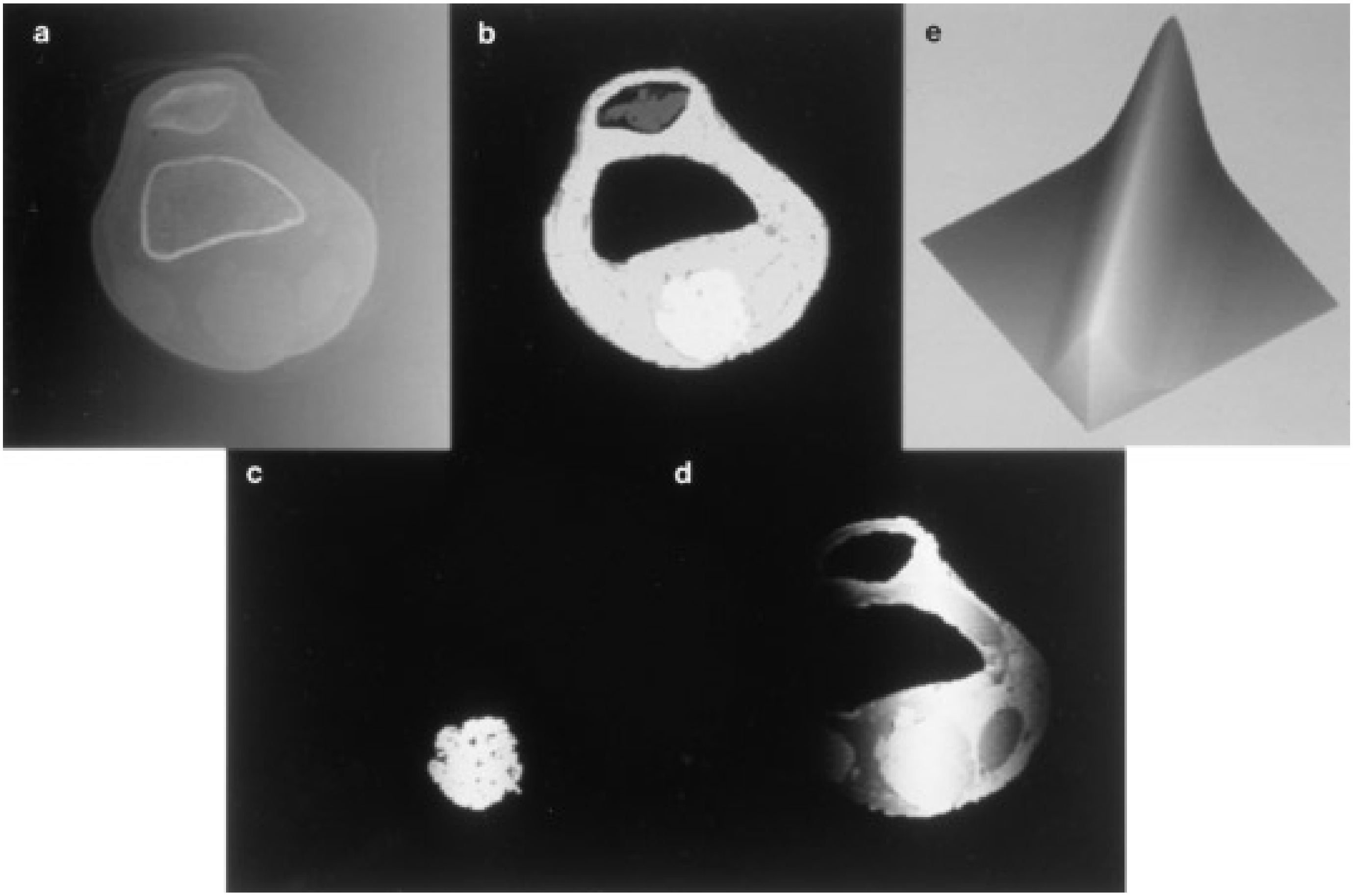

FIG. 2. (a) The c_1c_2-slice in Fig. 1a with an added ramp going from left to right. (b) The K_o-scene for the scene in (a) for a spel o specified in the soft-tissue blob in the lower center. (c) A fuzzy $\kappa\theta_x$-object of the scene in (a) containing o for a "high" value of x. (d) As in (c) for a "medium" value of x. (e) The functional form of $\mu_\kappa(c, d)$ used to detect the $\kappa\theta_x$-objects.

of connectivity. Figures 2c and 2d show two fuzzy $\kappa\theta_x$-objects obtained for a high and a medium value of x. We have displayed in Fig. 2e the functional form of $\mu_\kappa(c, d)$ used in this example which depended only on $f(c)$ and $f(d)$.

Our third example, illustrated in Fig. 3, pertains to MR angiography (MRA). In this application, the MR imaging protocols are such that higher values representing blood flow are assigned to spels inside vessels. The clinical aim of imaging here is to identify regions of the vessels with constriction, narrowing, or stenosis. A popular method of visualizing the vessels in this application is via 3D renditions created by maximum intensity projection (MIP) [21]. The value assigned to a pixel in a MIP rendition is the maximum of all values encountered in the scene along the line of sight associated with the pixel. Such an approach, which does not require segmentation or object model construction, is taken because the latter are very difficult in these scenes due to a variety of image artifacts. A problem with MIP is that it is accompanied by considerable clutter, and since there is no model of reflection, aspects of the vessels at different distances with respect to the viewpoint are not distinguished easily. This leads to some confusion in stationary views. From over 10 patient studies we have done so far, 3-fuzzy object extraction using Algorithm κFOE seems to be an effective solution to extract vessels in MRA. We are currently in the process of conducting studies to compare among MIP, shell rendering based on fuzzy $\kappa\theta_x$-objects, MIP based on fuzzy $\kappa\theta_x$-objects, and a host of other methods.

Our final example is illustrated in Fig. 4. One of the main aims of this application is to identify and compute the volume of the various component tissue regions and multiple sclerosis (MS) lesions in human brains. The imaging modality used is MR. Often identification and volume computation of the tissue regions is done for image data acquired on a longitudinal basis, usually for assessing the progression of the disease or of the effect of a drug on the disease. In a large study, Dr. Robert Grossman of our department has acquired over 1000 3D scenes, some of which are vector-valued, including several longitudinal

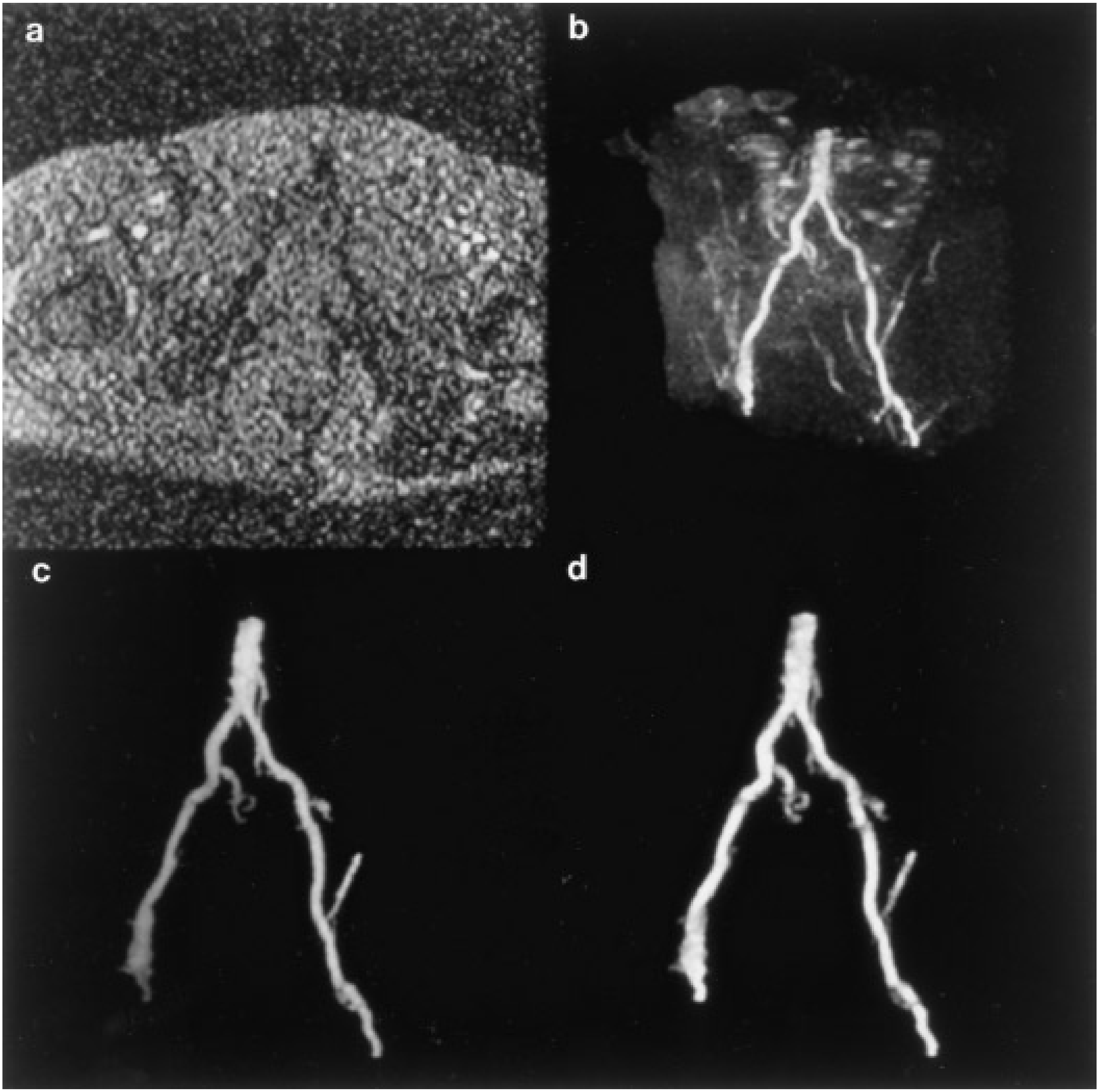

FIG. 3. (a) A c_1c_2-slice of a 3D MR angiographic scene. Bright values correspond to flow in vessels. (b) A maximum intensity projection [21] (MIP) of the scene in (a). (c) A shell rendition of a $\kappa\theta_x$-object of the scene in (a) detected for a ''medium'' value of x. (d) A MIP rendition of the $\kappa\theta_x$-object shown in (c).

acquisitions for MS patients. We have so far processed over 600 3D scenes with excellent results, each of which was verified by a neuroradiologist for accuracy. The methodology of object identification and volume computation is quite involved; Algorithm κFOE forming its core. We have conducted extensive experiments to determine the repeatability and accuracy of the methodology. In its current set up, it requires the operator to specify a few spels contained in the white matter, gray matter, and the ventricle (but not in the lesions) in one c_1c_2-slice, which requires about 30 s per 3D study. From this point, all fuzzy $\kappa\theta_x$-objects are identified automatically using such application-specific knowledge as that MS is mainly a disease of the white matter but the lesions may also occur in the gray

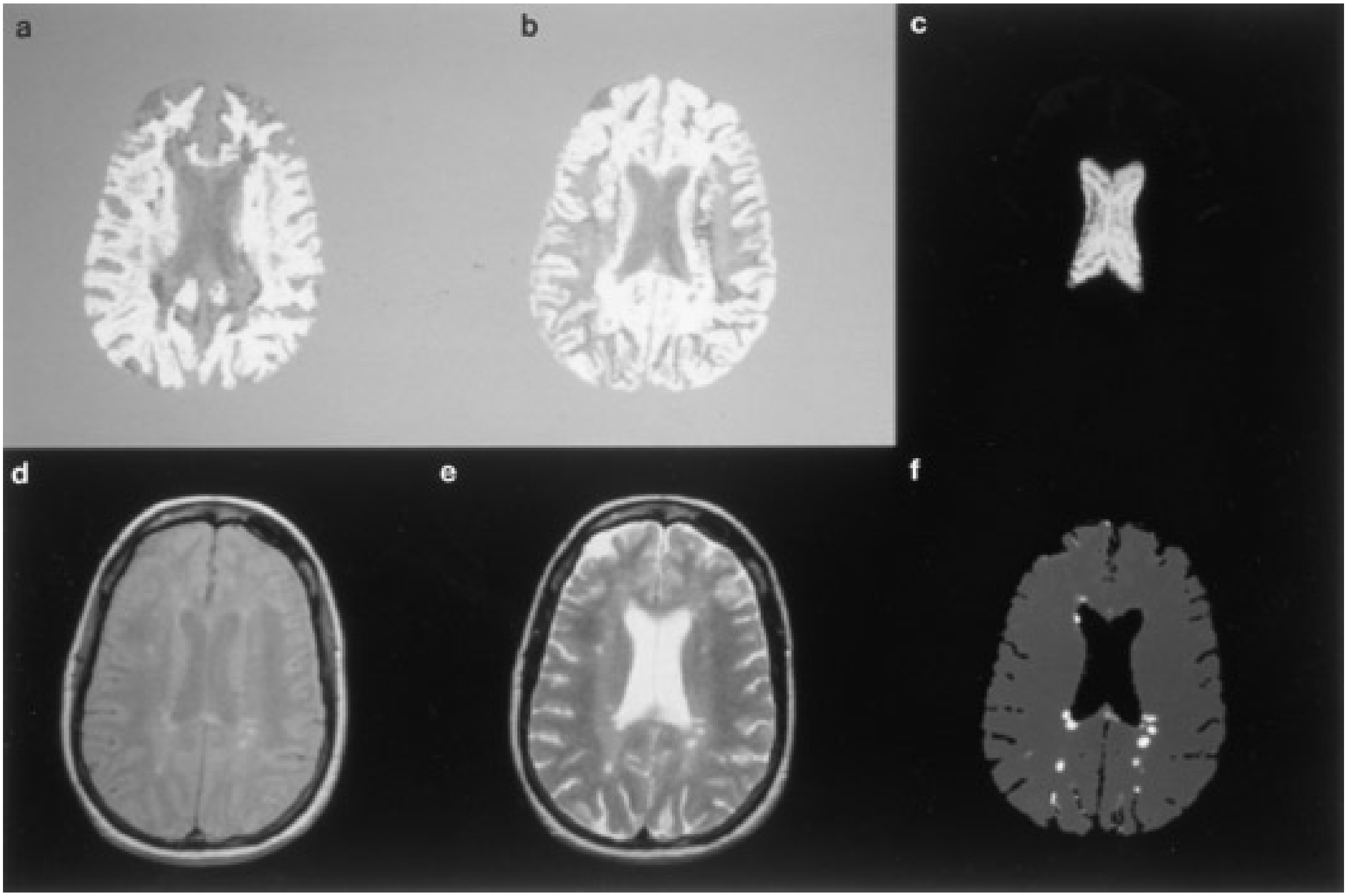

FIG. 4. (a) A c_1c_2-slice of the K_o-scene of the scene in (d,e) for the white matter. (b) As in (a) but for the gray matter. (c) As in (a) but for the ventricles. (d) A c_1c_2-slice of a vector-valued 3D MR scene. The picture shows the proton density values. (e) As in (d) showing the T2 values. (f) A c_1c_2-slice of the K_o-scene of the scene in (d,e) for the lesions.

matter and the periventricular region but never inside the ventricle or outside the brain. These criteria are easily incorporated and they aid in effectively detecting the component fuzzy objects. Because of its complexity, importance, and extent, this topic will be covered in a separate paper. Here we will give one example.

Figures 4d and 4e show a c_1c_2-slice of a vector-valued (T2 and proton density) MR 3D scene. Figures 4a to 4c show c_1c_2-slices of the K_o-scenes corresponding to the white matter, the gray matter, and the ventricle fuzzy objects for this input scene. Note how these displays depict "white-matterness," "gray-matterness," and "ventricleness" of spels that hang together to form the respective objects. Figure 4f shows a c_1c_2-slice of the K_o-scene corresponding to the lesions. All fuzzy $\kappa\theta_x$-objects in this example are detected in three dimensions. The lesions are inherently fuzzy and manual delineation, therefore, even by experts, is an ill-defined task.

We have implemented Algorithms $\kappa\theta_x FOE$ and κFOE for $n = 2$ and $n = 3$ within an internal version of the 3DVIEWNIX software system [22]. On a Sparc10/51 workstation, the run time for the 2D version of Algorithm κFOE is about 20 s and about 5 s for Algorithm $\kappa\theta_x FOE$ even for small values of x for a scene of domain 256×256. The 2D version facilitates experimentation since it operates at interactive speeds. However, to reap the full power of these algorithms, fuzzy object extraction should be done in the natural dimensionality of the scene. The run time on a Sparc 10/51 workstation for the 3D version of Algorithm κFOE is about 20 min for a scene of domain $256 \times 256 \times 64$. This figure reduces to about 2 min for Algorithm $\kappa\theta_x FOE$ for $x = 0.1$. As we pointed out earlier, running $\kappa\theta_x FOE$ with a small value of x provides an optimal tradeoff between speed and the convenience of selection of the strength of connectedness after the algorithm's termination.

5. CONCLUSIONS AND DISCUSSION

We have presented a new theory in this paper for fuzzy object definition in n-dimensional (fuzzy) digital spaces. A fuzzy object is defined to be a fuzzy connected component of spatial elements (spels). Fuzzy connectedness is a fuzzy relation in the set of all spels which combines together the notion of fuzzy adjacency of spels, which is independent of any image information, and fuzzy affinity between spels, which depends on image intensity values. Although the definition of a fuzzy object involves combinatorics of impractical proportion even for 2D digital spaces, with the help of some basic results relating to fuzzy connectedness and fuzzy objects, we have developed and presented practical algorithms for their extraction in given multidimensional image data. We have demonstrated the power of these algorithms in accurate object definition in digital imagery using several practical applications drawn from medical imaging which are currently run routinely in a clinical setting. We conclude that attempting to retain in object information extracted from images the inaccuracies inherent in image data is a right stand in image analysis and that the notion of fuzzy connectedness which has been missing in previous image segmentation research has much to offer in practical image analysis.

The research reported in this paper opens numerous new directions. We describe some of these below.

The fuzzy adjacency relation α needs further investigation. We used α mainly as a hard binary relation in all experiments we have done. More realistically, α should perhaps reflect the form of the point-spread function of the imaging device. Of course, taking more neighbors into account may increase the cost of the algorithms. The fuzzy affinity relation κ similarly requires further study. In all our experiments, we have used rather simple functional forms for κ_κ. More sophisticated and general forms as per (4.2) are worth investigating. It is also possible to design μ_κ based on features (such as texture measures) extracted from spel values rather than based directly on spel values only. More generally, fuzzy connectedness may be interpreted as a (fuzzy) spatial contiguity of object structural and intensity properties that can be measured locally. For the fuzzy κ-connectedness relation K, other functional forms more sophisticated than (2.10a) may exist that do not violate any of our results.

The K_o-scene of a given membership scene $\mathscr{C}$ has several interesting properties. Note that it does not specify K completely but it has enough information to define the fuzzy $\kappa\theta_x$-objects of $\mathscr{C}$ containing o for any $x \in [0, 1]$. It can be treated as a new membership scene containing (hopefully) refined information about the object of interest and define fuzzy objects in it (hopefully) for improved object definition.

Our theory and algorithms have been developed in such a way that for hard adjacency relations and certain special affinity relations, concepts and algorithms related to binary scenes are realized. It is readily seen that Algorithms $\kappa\theta_x FOE$ and κFOE can be used to extract hard connected components in nD binary scenes. We wish to point out that, even for binary scenes, there are more general concepts and operations possible in the fuzzy setting. For example, by choosing $\mu_\kappa(c, d)$ the expression on the right side of (2.8) for all c and d that are 1-spels and setting $\mu_\kappa(c, d) = 0$ if c or d is a 0-spel, we can distinguish among components of different strengths of connectedness. Results analogous to those obtained by erosion, (hard) connected component extraction, and dilation can be obtained by extracting fuzzy κ-components for proper choices of κ. Operations on binary as well as nonbinary scenes that require connectivity analysis such as growing, dilation, hole filling, thinning, erosion, skeletonization, and shrinking are perhaps better reexamined using fuzzy connectivity notions since scenes are by nature fuzzy.

We hypothesize that extending fuzzy analysis to quantitative object-related measures derived from images allows extracting the fuzzy object information inherent in images more accurately than if hard analysis techniques were used. We do not have a proof of this hypothesis at present. To clarify this statement, consider an example involving volume computation. Corresponding to each value of x between $x_{\min}$ and $x_{\max}$ such that $[x_{\min}, x_{\max}] \subset [0, 1]$, we determine the volume of the fuzzy $\kappa\theta_x$-object that contains a given fixed spel o by taking into account the fractional contribution of spels to the volume depending on their membership value in the object. $x_{\min}$ and $x_{\max}$ are chosen to represent the extreme strengths beyond which object definition is clearly unacceptable. Thus $V(x)$, the volume as a function of x, itself can be thought of (after proper scaling) as a fuzzy subset of the set of all fuzzy $\kappa\theta_x$-objects that contain o. In a longitudinal analysis of the changes in an object, such as the MS lesions, we have $V_t(x)$ for each longitudinal instance t. Now we can analyze more comprehensively (than if hard analysis techniques were used) as to what happens from one instance t_1 to another instance t_2 to the object components of various strengths by examining the distributions $V_{t_1}(x)$ and $V_{t_2}(x)$. In the context of the MS lesions, for example, from t_1 to t_2 the weaker components may have grown in size while the stronger ones may have diminished.

Our final comment relates to fuzzy object rendition. There are two key considerations in volume rendering—assignment of opacity values to spels and the estimation of surface normals at interfaces. Clearly, the utility of the K_o-scenes for these purposes is worth exploring. More importantly, our ability to extract different fuzzy objects (such as the gray matter, white matter, and the ventricles in the brain) from the same given scene, possibly using independent criteria, calls for formal models for volume rendering that can handle mixtures that result when fuzzy objects are put together. Current volume rendering concepts lack such models and cannot handle these situations.

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