Chapter 1

Nonlinear Flight Control Techniques for Unmanned Aerial Vehicles

Girish Chowdhary, Emilio Frazzoli, Jonathan P. How, and Hugh Liu
Abstract

In order to meet increasing demands on performance and reliability of Unmanned Aerial Vehicles, nonlinear and adaptive control techniques are often utilized. These techniques are actively being studied to handle nonlinear aerodynamic and kinematic effects, actuator saturations and rate limitations, modeling uncertainty, and time varying dynamics. This chapter presents an overview of some tools and techniques used for designing nonlinear flight controllers for UAVs. A brief overview of Lyapunov stability theory is provided. Nonlinear control techniques covered include gain scheduling, model predictive control, backstepping, dynamic inversion based control, model reference adaptive control, and model based fault tolerant control.
1.1 Introduction

UAV flight control systems provide enabling technology for the aerial vehicles to fulfill their flight missions, especially when these missions are often planned to perform risky or tedious tasks under extreme flight conditions that are not suitable for piloted operation. Not surprisingly, UAV flight control systems are often considered safety/mission critical, as a flight control system failure could result in loss of the UAV or an unsuccessful mission. The purpose of this chapter, and its companion chapter on linear control (“Linear Flight Control Techniques for Unmanned Aerial Vehicles”), is to outline some well studied linear control methods and their applications on different types of UAVs as well as their customized missions.

The federal inventory of UAVs grew over 40 times in the last decade [35]. Most UAVs in operation today are used for surveillance and reconnaissance (S&R) purposes [123], and in very few cases for payload delivery. In these cases a significant portion of the UAVs in operation remain remotely piloted, with autonomous flight control restricted to attitude hold, non-agile way-point flight, or loiter maneuvers. Linear or gain-scheduled linear controllers are typically adequate for these maneuvers. But in many future scenarios collaborating UAVs will be expected to perform agile maneuvers in the presence of significant model and environmental uncertainties [35,99,100]. As seen in the companion chapter on linear flight control (“Linear Flight Control Techniques for Unmanned Aerial Vehicles”), UAV dynamics are inherently nonlinear. Thus any linear control approach can only be guaranteed to be locally stable and it may be difficult to extract the desired performance or even guarantee stability when agile maneuvers are performed or when operating in the presence of significant nonlinear effects. This is particularly true for rotorcraft and other non-traditional fixed-wing planform configurations that might be developed to improve sensor field-of-view, or for agile flight. In these cases, nonlinear and adaptive control techniques must be utilized to account for erroneous linear models (e.g., incorrect representation of the dynamics, or time-varying/state-varying dynamics), nonlinear aerodynamic and kinematic effects, and actuator saturations and rate limitations.

The chapter begins with an overview of Lyapunov stability theory in Section 1.2. An overview of Lyapunov based control techniques including gain scheduling, backstepping, and model predictive control is presented in Section 1.3. Dynamic inversion based techniques are introduced in Section 1.3.4. A brief overview of model reference adaptive control is presented in Section 1.3.5. The intent of this chapter is to provide an overview of some of the more common methods used for nonlinear UAV control. There are several method and active research direction beyond those discussed here for UAV nonlinear control; two examples are sliding mode control (see e.g. [63,74,116]), and reinforcement learning based methods [2].

1.2 An Overview of Nonlinear Stability Theory

This section presents a brief overview of Lyapunov theory based mathematical tools used in nonlinear stability analysis and control. A detailed treatment of Lyapunov based methods can be found
in [36,58,64,118]. A state space representation of UAV dynamics will be used. Let \( t \in \mathbb{R}^+ \) denote the time, then the state \( x(t) \in D \subset \mathbb{R}^n \) is defined as the minimal set of variable required to describe a system. Therefore, the choice of state variables is different for each problem. For a typical UAV control design problem, the state consists of the position of the UAV, its velocity, its angular rate, and attitude. The admissible control inputs are defined by \( u(t) \in U \subset \mathbb{R}^m \), and typically consist of actuator inputs provided by elevator, aileron, rudder, and throttle for a fixed wing UAV. Detailed derivation of UAV dynamics can be found in “Linear Flight Control Techniques for Unmanned Aerial Vehicles” in this book. For the purpose of this chapter, it is sufficient to represent the UAV dynamics in the following generic form

\[
\dot{x}(t) = f(x(t), u(t)).
\] (1.1)

The initial condition for the above system is \( x(0) = x_0 \), with \( t = 0 \) being the initial time. Note that if the dynamics are time-varying, then time itself can be considered as a state of the system (see e.g. Chapter 4 of [36]). In a state-feedback framework, the control input \( u \) is often a function of the states \( x \). Therefore, for stability analysis, it is sufficient to consider the following unforced dynamical system

\[
\dot{x}(t) = f(x(t)).
\] (1.2)

The set of all states \( x_e \in \mathbb{R}^n \) that satisfy the equation \( f(x_e) = 0 \) are termed as the set of equilibrium points. Equilibrium points are of great interest in study of nonlinear dynamical systems as they are a set of states in which the system can stay indefinitely if not disturbed. A general nonlinear system may have several equilibrium points, or may have none at all. The rigid body equations of motion of a UAV as studied in “Linear Flight Control Techniques for Unmanned Aerial Vehicles” has several equilibrium points. Since a simple linear transformation can move an equilibrium point to the origin in the state-space, it is assumed in the following that the nonlinear dynamical system of Equation (1.2) has an equilibrium at the origin, that is \( f(0) = 0 \). The solution \( x(t) \) to Equation (1.2) is

\[
x(t) = x(0) + \int_0^t f(x(t))dt, \quad x(0) = x_0.
\] (1.3)

A unique solution is guaranteed to exist over \( D \) if \( f(x) \) is Lipschitz continuous over \( D \), that is for all \( x \) and all \( y \) within a bounded distance of \( x \), \( ||f(x) - f(y)|| \leq c||x - y|| \) for some constant \( c \) [36,58]. Most UAVs also satisfy some condition on controllability [98] that guarantees the existence of an admissible control \( u \) that drives the state close to any point in \( D \) in finite time.

### 1.2.1 Stability of a Nonlinear Dynamical System

The stability of a dynamical system is closely related with the predictability or well-behavedness of its solution. Particularly, the study of stability of a dynamical system answers the question: How far the solution of a dynamical system would stray from the origin if it started away from the origin? The most widely studied concept of stability are those of Lyapunov stability [36,46,58,118].
**Definition** The origin of the dynamical system of Equation (1.2) is said to be Lyapunov stable if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\|x(0)\| \leq \delta$ then $\|x(t)\| \leq \epsilon$.

**Definition** The origin of the dynamical system of Equation (1.2) is said to be asymptotically stable if it is Lyapunov stable and $\|x(t)\| \leq \delta$ then $\lim_{t \to \infty} x(t) = 0$.

**Definition** The origin of the dynamical system of Equation (1.2) is said to be exponentially stable if there exists positive constants $\alpha$ and $\beta$ such that the solution $x(t)$ satisfies $\|x(t)\| \leq \alpha \|x(0)\| e^{-\beta t}$.

If the above definitions hold for all initial conditions $x(0) \in \mathbb{R}^n$, then the stability definitions are said to be global (assuming a unique solution exists everywhere on $\mathbb{R}^n$). If a dynamical system is not Lyapunov stable, then it is called unstable. It follows from the above definitions that the stronger notion of stability is that of exponential stability, which requires that the solution go to the origin at an exponential rate. This is different than the notion of asymptotic stability, which requires the solution to go to the origin eventually. The notion of exponential stability encompasses asymptotic stability, and the notion of asymptotic stability encompasses Lyapunov stability. A geometric depiction of Lyapunov stability is presented in Figure 1.1.

![Figure 1.1: A geometric depiction of Lyapunov stability concepts](image)

Lyapunov’s direct method, often referred to as Lyapunov’s second method, is a powerful technique that provides sufficient conditions to determine the stability of a nonlinear system.

**Theorem 1.2.1** [36, 58, 118] Consider the nonlinear dynamical system of Equation (1.2), and assume that there exists a continuously differentiable real valued positive definite function $V(x) : D \to \mathbb{R}^+$ such that for $x \in D$

$$
V(0) = 0,
$$

$$
V(x) > 0 \quad x \neq 0,
$$

$$
\frac{\partial V(x)}{\partial x} f(x) \leq 0,
$$

(1.4)
then the origin is Lyapunov stable. If in addition
\[ \frac{\partial V(x)}{\partial x} f(x) < 0, \] (1.5)
the origin is asymptotically stable. Furthermore, if there exist positive constants $\alpha, \beta, \epsilon$ such that
\[ \alpha \|x\|^2 \leq V(x) \leq \beta \|x\|^2, \] (1.6)
\[ \frac{\partial V(x)}{\partial x} f(x) \leq -\epsilon V(x), \]
then the origin is exponentially stable.

The function $V(x)$ is said to be a Lyapunov function of the dynamical system in Equation (1.2) if it satisfies these conditions. Therefore, the problem of establishing stability of a dynamical system can be reduced to that of finding a Lyapunov function for the system. If a system is stable, a Lyapunov function is guaranteed to exist [36,58,118], although finding one may not be always straightforward.

It should be noted that the inability to find a Lyapunov function for a system does not imply its instability. A Lyapunov function is said to be radially unbounded if as $x \to \infty$, $V(x) \to \infty$. If a radially unbounded Lyapunov function exists for a dynamical system whose solution exists globally, then the stability of that system can be established globally. In the following, $\dot{V}(x(t))$ and $\frac{\partial V(x(t))}{\partial x(t)} f(x(t))$ are used interchangeably.

It is possible to relax the strict negative definiteness condition on the Lyapunov derivative in Equation (1.5) if it can be shown that the Lyapunov function is non-increasing everywhere, and the only trajectories where $\dot{V}(x) = 0$ indefinitely is the origin of the system. This result is captured by the Barbashin-Krasovskii-LaSalle invariance principle (see e.g. [36,58,118]), which provides sufficient conditions for convergence of the solution of nonlinear dynamical system to its largest invariant set (described shortly). A point $x_p$ is said to be a positive limit point of the solution $x(t)$ to the nonlinear dynamical system of Equation (1.2) if there exists an infinite sequence $\{t_i\}$ such that $t_i \to +\infty$ and $x(t_i) \to x_p$, as $i \to \infty$. The positive limit set (also referred to as $\omega$-limit set) is the set of all positive limit points. A set $M$ is said to be positively invariant if $x(0) \in M$ then $x(t) \in M$ for all $t \geq 0$. That is, a positively invariant set is the set of all initial conditions for which the solution of Equation (1.2) does not leave the set. The set of all equilibriums of a nonlinear dynamical system for example is a positively invariant set. The Barbashin-Krasovskii-LaSalle theorem can be stated as follows:

**Theorem 1.2.2** [36,58,118] Consider the dynamical system of Equation (1.2), and assume that $\bar{D}$ is a compact positively invariant set. Furthermore, assume that there exists a continuously differentiable function $V(x) : D \to R^+$ such that $\frac{\partial V(x)}{\partial x} f(x) \leq 0$ in $\bar{D}$, and let $M$ be the largest invariant set contained in the set $S = \{x \in D | \dot{V}(x) = 0\}$. Then, if $x(0)$ in $\bar{D}$, the solution $x(t)$ of the dynamical system approaches $M$ as $t \to \infty$.

A corollary to this theorem states that if $V(x)$ is also positive definite over $\bar{D}$ and no solution except the solution $x(t) = 0$ can stay in the set $S$, then the origin is asymptotically stable [36,58].
In the following the quaternion attitude controller discussed in “Linear Flight Control Techniques for Unmanned Aerial Vehicles” is used to illustrate the application of Lyapunov stability analysis. A unit quaternion \( q = (\tilde{q}, \vec{q}) \), can be interpreted as a representation of a rotation of an angle \( \theta = 2 \arccos(\tilde{q}) \), around an axis parallel to \( \vec{q} \). The attitude dynamics of an airplane modeled as a rigid body with quaternion attitude representation can be given by

\[
\begin{align*}
J_B \dot{\omega}_B &= -\omega_B \times J_B \omega_B + u \\
\dot{q} &= \frac{1}{2}q \circ (0, \omega_B).
\end{align*}
\]

(1.7)

where \( J_B \) is the inertia tensor expressed in the body frame, \( \omega_B = [\omega_1, \omega_2, \omega_3]^T \in \mathbb{R}^3 \) is the angular velocity of the body frame with respect to the inertial frame, and \( u \) is the control input. The operator \( \circ \) denotes the quaternion composition operator, using which the quaternion kinematics equation 1.7 can be expanded as

\[
q_1 \circ (0, \omega_B) = (-\tilde{q}_1 \cdot \omega_B, \quad \tilde{q}_1 \omega_B + \tilde{q}_1 \tilde{q}_1^* + \tilde{q}_1 \times \omega_B).
\]

The goal is to design a control law \( u \) such that the system achieves the unique zero attitude represented by the unit quaternions \([1, 0, 0, 0]\) and \([-1, 0, 0, 0]\). It is shown that a control law of the form

\[
u = -\frac{\tilde{q}}{2}K_p \tilde{q} - K_d \omega_B
\]

(1.8)

where \( K_p \) and \( K_d \) are positive definite gain matrices, guarantees that the system achieves zero attitude using Lyapunov analysis. Consider the Lyapunov candidate

\[
V(q, \omega) = \frac{1}{2}q \cdot K_p \tilde{q} + \frac{1}{2} \omega \cdot J \omega.
\]

Note that \( V(q, \omega) \geq 0 \), and is zero only at \( q = (\pm 1, \tilde{0}), \omega = 0 \); both points correspond to the rigid body at rest at the identity rotation. The time derivative of the Lyapunov candidate along the trajectories of system Equation (1.7) under the feedback Equation (1.8) is computed as:

\[
\dot{V}(q, \omega) = \tilde{q} \cdot K_p \dot{q} + \omega \cdot J \dot{\omega}.
\]

Note that from the kinematics of unit quaternions, \( \dot{q} = 1/2(\tilde{q} \omega + \tilde{q} \times \omega) \). Then,

\[
\dot{V}(q, \omega) = \frac{1}{2}q \cdot K_p (\tilde{q} \omega + \tilde{q} \times \omega) - \omega \cdot \left( \omega \times J \omega + \frac{\tilde{q}}{2}K_p \tilde{q} + K_d \omega \right) = -\omega \cdot K_d \omega \leq 0.
\]

Therefore, \( V(q, \omega) \leq 0 \), furthermore the set \( S = \{ x \in D | \dot{V}(x) = 0 \} \) consists of only \( q = (\pm 1, \tilde{0}), \omega = 0 \). Therefore, Theorem 1.2.2 guarantees asymptotic convergence to this set.
1.3 Lyapunov Based Control

1.3.1 Gain Scheduling

In the companion chapter titled “Linear Flight Control Techniques for Unmanned Aerial Vehicles” it was shown that aircraft dynamics can be linearized around equilibrium points (or trim conditions). A commonly used approach in aircraft control leverages this fact by designing a finite number of linear controllers, each corresponding to a linear model of the aircraft dynamics near a design trim condition. The key motivation in this approach is to leverage well understood tools in linear systems design. Let $A_i, B_i, i \in \{1, \ldots, N\}$ denote the matrices containing the aerodynamic and control effectiveness derivatives around the $i$th trimmed condition $\bar{x}_i$. Let $X_1, \ldots, X_N$ be a partition of the state space, i.e., $\bigcup_{i=1}^N X_i = \mathbb{R}^n$, $X_i \cap X_j = \emptyset$ for $i \neq j$, into regions that are “near” the design trim conditions; in other words, whenever the state $x$ is in the region $X_i$, the aircraft dynamics are approximated by the linearization at $\bar{x}_i \in X_i$. Then, the dynamics of the aircraft can be approximated as a state-dependent switching linear system as follows

$$\dot{x} = A_i x + B_i u \quad \text{when} \quad x \in X_i. \quad (1.9)$$

The idea in gain scheduling based control is to create a set of gains $K_i$ corresponding to each of the switched model and apply the linear control $u = K_i x$. Contrary to intuition, however, simply ensuring that the $i$th system is rendered stable (that is, the real parts of the eigenvalues of $A_i - B_i K_i$ are negative) is not sufficient to guarantee the closed loop stability of Equation (1.9) [12,77,78]. A Lyapunov based approach can be used to guarantee the stability of the closed-loop when using gain scheduling controller.

Consider the following Lyapunov candidate

$$V(x(t)) = x(t)^T P x(t), \quad (1.10)$$

where $P$ is a positive definite matrix, that is, for all $x \neq 0$, $x^T P x > 0$. Therefore, $V(0) = 0$, and $V(x) > 0$ for all $x \neq 0$ making $V$ a valid Lyapunov candidate. The derivative of the Lyapunov candidate is

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x}. \quad (1.11)$$

For the $i$th system, Equation (1.11) can be written as

$$\dot{V}(x) = (A_i x - B_i K_i x)^T P x + x^T P (A_i x - B_i K_i x). \quad (1.12)$$

Let $\bar{A}_i = (A_i - B_i K_i)$, then from Lyapunov theory it follows that for a positive definite matrix $Q$ if for all $i$

$$\bar{A}_i^T P + P \bar{A}_i < -Q, \quad (1.13)$$

$$\frac{\partial V(x)}{\partial x} f(x) < -x^T Q x. \quad \text{In this case,} \quad V(x) \quad \text{is a common Lyapunov function for the switched closed}$$
loop system (see e.g. [77]) establishing the stability of the equilibrium at the origin. Therefore, the control design task is to select the gains $K_i$ such that Equation (1.13) is satisfied. One way to tackle this problem is through the framework of Linear Matrix Inequalities (LMI) [10,33]. It should be noted that the condition in Equation (1.13) allows switching between the linear models to occur infinitely fast, this can be a fairly conservative assumption for most UAV control applications. This condition can be relaxed to $\bar{A}_i^T P_i + P_i \bar{A}_i < -Q_i$ for $Q_i > 0$ to guarantee the stability of the system if the system does not switch arbitrarily fast. A rigorous condition for proving asymptotic stability of a system of the form 1.9 was introduced in [12]. Let $V_i, i \in \{1, \ldots, N\}$ be Lyapunov-like functions, i.e., positive definite functions such that $\dot{V}_i(x) < 0$ whenever $x \in X_i \setminus \{0\}$. Define $V_i[k]$ as the infimum of all the values taken by $V_i$ during the $k$-th time interval over which $x \in X_i$. Then, if the system satisfies the sequence non-increasing condition $V_i[k+1] < V_i[k]$, for all $k \in \mathbb{N}$, asymptotic stability is guaranteed [12,77].

The above example illustrates the use of Lyapunov techniques in synthesizing controllers. In general, given the system of equation 1.2, and a positive definite Lyapunov candidate, the Lyapunov synthesis approach to create robust exponentially stable controllers can be summarized as follows: Let $\dot{V}(x) = g(x,u)$, find a control function $u(x)$ such that $\dot{V}(x) < -\epsilon V(x)$ for some positive constant $\epsilon$. Robust methods for Lyapunov based control synthesis are discussed in [36]. Furthermore, [111] provides an output feedback control algorithms for system with switching dynamics.

The framework of Linear Parameter Varying (LPV) systems lends naturally to design and analysis of controllers based on a UAV dynamics representation via linearization across multiple equilibria. Gain scheduling based LPV control synthesis techniques have been studied for flight control and conditions for stability have been established (see e.g. [7,40,69,84,115]).
1.3.2 Backstepping Control

Backstepping is an example of a Lyapunov based technique, providing a powerful recursive approach for stabilizing systems that can be represented in nested loops [61,64].

1.3.2.1 Systems in strict-feedback form

As a basic example, consider a system of the (strict-feedback) form

\[
\begin{align*}
\dot{x} &= f_0(x) + g_0(x)z, \\
\dot{z} &= f_1(x,z) + g_1(x,z)u,
\end{align*}
\]

(1.14)

where \(x \in \mathbb{R}^n\), \(z\) is a scalar, and \(u\) is a scalar input. Assume that the “inner” system

\[
\dot{x} = f_0(x) + g_0(x)z,
\]

has an equilibrium point for \(x = 0, z = 0\), and admits a known stabilizing feedback control law \(x \mapsto u_0(x)\) with Lyapunov function \(V_0 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}, V_0(0) = 0\). In other words, if \(z = u_0(x)\), then

\[
\frac{d}{dt}V_0(x) = \frac{\partial V_0(x)}{\partial x} (f_0(x) + g_0(x)u_0(x)) = -W(x) < 0, \quad \forall x \neq 0.
\]

Now consider the Lyapunov function candidate

\[
V_1(x,e) = V_0(x) + \frac{1}{2}e^2;
\]

its time derivative along the system’s trajectories is

\[
\frac{d}{dt}V_1(x,e) = \frac{\partial V_0(x)}{\partial x} (f_0(x) + g_0(x)u_0(x) + g_0(x)e) + ev.
\]

If one picks \(v = -\frac{\partial V_0(x)}{\partial x} g_0(x) - k_1 e\), with \(k_1 > 0\), then

\[
\frac{d}{dt}V_1(x,e) = -W(x) - k_1 e^2 < 0, \quad \forall (x,e) \neq 0,
\]

thus proving stability of system Equation (1.15).

Finally, assuming that \(g_1(x,z) \neq 0, \forall (x,z) \in \mathbb{R}^{n+1}\), a stabilizing feedback for the original system
in Equation (1.14) can be recovered as

\[ u(x, z) = \frac{1}{g_1(x, z)} \left( \frac{\partial u_0(x)}{\partial x} (f_0(x) + g_0(x)z) - f_1(x, z) + v \right) \]

\[ = \frac{1}{g_1(x, z)} \left( \frac{\partial u_0(x)}{\partial x} (f_0(x) + g_0(x)z) - f_1(x, z) - \frac{\partial V_0(x)}{\partial x} g_0(x) - k_1(z - u_0(x)) \right). \] (1.16)

Since the control law in Equation (1.16) stabilizes system Equation (1.14), with Lyapunov function \( V_1 \), a similar argument can be used to recursively build a control law for a system in strict-feedback form of arbitrary order,

\[
\begin{align*}
\dot{x} &= f_0(x) + g_0(x)z_1, \\
\dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1)z_2, \\
&\vdots \\
\dot{z}_m &= f_m(x, z_1, \ldots, z_m) + g_1(x, z_m)u,
\end{align*}
\] (1.17)

The procedure is summarized as follows: start from the “inner” system, for which a stabilizing control law and a Lyapunov function are known. Define an error between the known control law and the actual input to the inner system. Augment the Lyapunov function with the square of the error. Design a new control law that ensures the stability of the augmented system. Repeat using the augmented system just defined as the new inner system. This recursion, “stepping back” from the inner system all the way to the control input gives the name to the method. A detailed description of control synthesis for small unmanned helicopters using backstepping techniques is provided in [106].

### 1.3.3 Model Predictive Control (MPC)

MPC has been successfully used for many years in industrial applications with relatively slow dynamics (e.g., chemical reactions [55,82,86]), it is only in the past decade that the computational power been available to enable online optimization for fast system dynamics typical in aerospace applications (early relevant demonstrations are in [27–29,83,108,110]). MPC attempts to solve the following problem: Design an admissible piecewise continuous control input \( u(t) \) that guarantees the system behaves like a reference model without violating the given state and input constraints. As such, a key benefit MPC is the ability to optimize the control input in the presence of state and input constraints. Furthermore, because MPC explicitly considers the operating constraints, it can operate closer to hard constraint boundaries than traditional control schemes.

MPC can be formulated using several types of cost function, however, due to the availability of robust quadratic solvers, the following formulation is popular. Find the optimal control \( u(t) \) such that for given positive (semi) definite matrices \( Q \), \( R \), and \( S \), the following quadratic cost is minimized

\[ J_\infty(e, u) = \int_{t_0}^\infty e^T(t)Qe(t) + u^T(t)Ru(t)dt, \quad x \in \Xi, \ u \in \Pi. \] (1.18)

In the presence of constraints, a closed form solution for an infinite horizon optimization problem
cannot be found in the general case [15,85,97,107]. Hence, the approach in MPC is to numerically solve a Receding Horizon Optimization (RHO) problem online over the interval \([t, t + N]\) to find the new control input at time \(t + 1\), this process is then repeated over every discrete update (see Figure 1.3). The idea is that if the horizon is sufficiently large, the solution to the receding horizon optimization problem can guarantee stability. Let \(h\) be a mapping between the states \(x\) of the nonlinear dynamical system in equation 1.2 and the output \(z\) such that \(z = h(x)\). If the function \(h(x)\) is observable this problem can be recast into a problem of minimizing a discrete output based cost. Observability for linear systems was discussed in “Linear Flight Control Techniques for Unmanned Aerial Vehicles” in this book. Local observability for nonlinear systems can be established in an analogous manner by considering the rank of the Jacobian matrix of \(n - 1\) Lie-derivatives of \(h(x)\) along the trajectories of equation 1.2 (see e.g. [62,98]). Typically, a quadratic cost function is preferred to leverage existing results in quadratic programming [15,26,107]. Let \(z_{rm_k}\) denote the sampled output of the reference model, and define the output error \(\bar{e}_k = z_k - z_{rm_k}\). Then the problem can be reformulated to finding the optimal sequence of inputs \(u\) such that the following quadratic cost function is minimized for given positive (semi) definite matrices \(\bar{Q}\) and \(\bar{R}\), subject to the reformulated output constraint \(y \in \bar{\Xi}\), where \(\bar{\Xi} = \{y : y = z_{\sigma}(x(t), u(t)), x \in \Xi\}\) and the input constraint \(u \in \Pi\)

\[
J_T(\bar{e}, u) = \sum_{k=t}^{t+N} \bar{e}_k^T \bar{Q} \bar{e}_k + u_k^T \bar{R} u_k + V_f(e_{t+N}), \quad y \in \bar{\Xi}, \; u \in \Pi, \quad (1.19)
\]

where the term \(V_f(e_{t+N})\) denotes a terminal penalty cost. Several computationally efficient nonlinear MPC algorithms have been proposed and their stability properties established [130,131].

To numerically solve the RHO problem in Equation (1.19) a prediction model \(\hat{x} = \hat{f}(\hat{x})\) is required to predict how the system states behave in the future, where \(\hat{x}\) is the estimated state, and \(\hat{f}\) is the prediction model. The prediction model is a central part of MPC, and in many cases (especially when the system dynamics are nonlinear or unstable), an inaccurate prediction model can result in instability [107]. In many MPC implementations, one of the most costly effort in control design is to develop a reliable prediction model [15,70,107]. Furthermore, approximations made in modeling, changes in the system dynamics due to wear and tear, reconfiguration of the system, or uncertainties introduced due to external effects can affect the accuracy of the predicted system response. Robustness of MPC methods to estimation errors [71,88,105], plant variability [17,25,73,110], and disturbances [8,56,57,72,109], remain active research areas. The resulting optimization problems are typically solved using linear matrix inequalities, linear programming, or quadratic programming. The key challenge here is to provide sufficient robustness guarantees while keeping the problem computationally tractable. The fact remains that without an accurate prediction model, the performance and stability guarantees remain very conservative [97,107].

To this effect, some authors have recently explored adaptive-MPC methods that estimate the modeling uncertainty [4,15,32,54]. Another active area of research in MPC is that of analytically guaranteeing stability. It should be noted that stability and robustness guarantees for MPC when
the system dynamics are linear are at least partially in place [15]. With an appropriate choice of $V_f(e_{t+N})$ and with terminal inequality state and input constraints (and in some cases without) stability guarantees for nonlinear MPC problems have been established [79,97,129]. However, due to the open loop nature of the optimization strategy, and dependence on the prediction model, guaranteeing stability and performance for a wide class of nonlinear systems still remains an active area of research. A challenge in implementing MPC methods is to ensure that the optimization problem can be solved in real-time. While several computationally efficient nonlinear MPC strategies have been devised, ensuring that a feasible solution is obtained in face of processing and memory constraints remains an active challenge for MPC application on UAVs, where computational resources are often constrained [121].

1.3.4 Model Inversion Based Control

1.3.4.1 Dynamic Model Inversion using Differentially Flat Representation of Aircraft Dynamics

A complete derivation of the nonlinear six degree of freedom UAV dynamics was presented in “Linear Flight Control Techniques for Unmanned Aerial Vehicles” in this book. For trajectory tracking control using feedback linearization, simpler representations of UAV dynamics are often useful. One such representation is the differentially flat representation (see e.g. [39]). The dynamical system in Equation (1.1) with output $y = h(x, u)$ is said to be differentially flat with flat output $z$ if there exists a function $g$ such that the state and input trajectories can be represented as a function of the flat output and a finite number of its derivatives:

$$(x, u) = g \left( y, \dot{y}, \ddot{y}, ..., \frac{d^n y}{dt^n} \right).$$  \hspace{1cm} (1.20)

In the following, assume that a smooth reference trajectory $p_d$ is given for a UAV in the inertial frame (see “Linear Flight Control Techniques for Unmanned Aerial Vehicles” in this book), and
that the reference velocity \( \dot{p}_d \) and reference acceleration \( \ddot{p}_d \) can be calculated using the reference trajectory. Consider the case of a conventional fixed wing UAV that must be commanded a forward speed to maintain lift, hence \( \dot{p}_d \neq 0 \). A right handed orthonormal frame of reference wind axes can now be defined by requiring that the desired velocity vector is aligned with the \( x_W \) axis of the wind axis (see “Linear Flight Control Techniques for Unmanned Aerial Vehicles” in this book for details), and the lift and drag forces are in the \( x_W-z_W \) plane of the wind axis the \( z_W \) axis such that there are no side forces. The acceleration can be written as

\[
\ddot{p}_d = \dot{g} + f_I/m, \tag{1.21}
\]

where \( f_I \) is the sum of propulsive and aerodynamic forces on the aircraft in the inertial frame. Let \( R_{IW} \) denote the rotation matrix that transports vectors from the defined wind reference frame to the inertial frame. Then

\[
\dot{p}_d = g + R_{IW}a_W, \tag{1.22}
\]

where \( a_W \) is the acceleration in the wind frame. Let \( \omega = [\omega_1, \omega_2, \omega_3]^T \) denote the angular velocity in the wind frame; then the above equation can be differentiated to obtain

\[
\frac{d^3 p_d}{dt^3} = R_{IW}(\omega \times a_W) + R_{IW}\dot{a}_W, \tag{1.23}
\]

by using the relationship \( \dot{R}_{IW}a_W = R_{IW}\dot{a}_W = R_{IW}(\omega \times a_W) \) (see Section 1.2.2 of “Linear Flight Control Techniques for Unmanned Aerial Vehicles” in this book). Let \( a_t \) denote the tangential acceleration along the \( x_W \) direction, \( a_n \) denote the normal acceleration along the \( z_W \) direction, and \( V \) denote the forward speed. Furthermore, let \( e_1 = [1, 0, 0]^T \). Then in coordinated flight \( \dot{p}_d = VR_{IW}e_1 \), hence

\[
\ddot{p}_d = \dot{V}R_{IW}e_1 + VR_{IW}(\omega \times e_1). \tag{1.24}
\]

Combining equation 1.22 and equation 1.24 the following relationship can be formed

\[
\ddot{p}_d = R_{IW}\begin{bmatrix}
\dot{V} \\
V\omega_3 \\
-V\omega_2
\end{bmatrix}. \tag{1.25}
\]

Equation 1.25 allows the desired \( \omega_2 \) and \( \omega_3 \) to be calculated from the desired acceleration \( \ddot{p}_d \). Furthermore, from equation 1.23

\[
\begin{bmatrix}
\dot{a}_t \\
\omega_1 \\
\dot{a}_n
\end{bmatrix} = \begin{bmatrix}
-\omega_2 a_n \\
\omega_3 a_t/n \\
\omega_2 a_t
\end{bmatrix} + \begin{bmatrix}
1 & 0 & 0 \\
0 & -1/a_n & 0 \\
0 & 0 & 1
\end{bmatrix} R_{IW}^T \frac{d^3 p_d}{dt^3}. \tag{1.26}
\]

The above equation defines a differentially flat system of aircraft dynamics with flat output \( p_d \) and inputs \([\dot{a}_t, \omega_1, \dot{a}_n] \), if \( V = \|\ddot{p}_d\| \neq 0 \) (nonzero forward speed) and \( a_n \neq 0 \) (nonzero normal
acceleration). The desired tangential and normal accelerations required to track the path \( p_d \) can be controlled through the thrust \( T(\delta_T) \) which is a function of the throttle input \( \delta_T \), the lift \( L(\alpha) \), and the drag \( D(\alpha) \) which are functions of the angle of attack \( \alpha \) by noting that in the wind axes

\[
a_t = T(\delta_T) \cos \alpha - D(\alpha),
\]

\[
a_n = -T(\delta_T) \sin \alpha - L(\alpha).
\]

To counter any external disturbances that cause trajectory deviation, a feedback term can be added. Let \( p \) be the actual position of the aircraft, \( u \) be the control input, and consider a system in which \( \frac{d^3 p_d}{dt^3} = u \):

\[
\frac{d}{dt} \begin{bmatrix} p \\ \dot{p} \\ \ddot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ \dot{p} \\ \ddot{p} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u.
\] (1.28)

Defining \( e = p - p_d \) the above equation can be written in terms of the error

\[
\frac{d}{dt} \begin{bmatrix} e \\ \dot{e} \\ \ddot{e} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \\ \ddot{e} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \left( u - \frac{d^3 p_d}{dt^3} \right).
\] (1.29)

Therefore, letting \( u = \frac{d^3 p_d}{dt^3} - K \begin{bmatrix} e \\ \dot{e} \\ \ddot{e} \end{bmatrix}^T \) where \( K \) is the stabilizing gain, and computing \( a_t, a_n, \omega_1 \) from \( (p, \dot{p}, \ddot{p}, u) \) guarantees asymptotic closed loop stability of the system (see [39] for further detail of the particular approach presented). This approach is essentially that of feedback linearization and dynamic model inversion, which is explored in the general setting in the next section.

### 1.3.4.2 Approximate Dynamic Model Inversion

The idea in approximate dynamic inversion based controllers is to use a (approximate) dynamic model of the UAV to assign control inputs based on desired angular rates and accelerations. Let \( x(t) = [x_1(t), x_2(t)]^T \in \mathbb{R}^n \) be the known state vector, with \( x_1(t) \in \mathbb{R}^{n_2} \) and \( x_2(t) \in \mathbb{R}^{n_2} \), let \( u(t) \in \mathbb{R}^{n_2} \) denote the control input, and consider the following multiple-input nonlinear uncertain dynamical system

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= f(x(t), u(t)),
\end{align*}
\] (1.30)

where the function \( f \) is assumed to be known and globally Lipschitz continuous, and control input \( u \) is assumed to be bounded and piecewise continuous. These conditions are required to ensure the existence and uniqueness of the solution to Equation (1.30). Furthermore, a condition on controllability of \( f \) with respect to \( u \) must also be assumed. Note also the requirement on as many control inputs as the number of states directly affected by the input \( x_2 \). For UAV control
problem this assumption can usually be met through the successive loop closure approach (see “Linear Flight Control Techniques for Unmanned Aerial Vehicles” in this book). For example, for fixed wing control aileron, elevator, rudder, and throttle control directly affect roll, pitch, yaw rate, and velocity. This assumption can also be met for rotorcraft UAV velocity control with the attitudes acting as virtual inputs for velocity dynamics, and the three velocities acting as virtual inputs for the position dynamics [49].

In dynamic model inversion based control the goal is to find the desired acceleration, referred to as the pseudo-control input $\nu(t) \in \mathbb{R}^{n_2}$, which can be used to find the control input $u$ such that the system states track the output of a reference model. Let $z = (x, u)$, if the exact system model $f(z)$ in Equation (1.30) is invertible, for a given $\nu(t)$, $u(t)$ can be found by inverting the system dynamics. However, since the exact system model is usually not invertible, let $\nu$ be the output of an approximate inversion model $\hat{f}$ such that $\nu = \hat{f}(x, u)$ is continuous and invertible with respect to $u$, that is, the operator $\hat{f}^{-1} : \mathbb{R}^{n+n_2} \rightarrow \mathbb{R}^l$ exists and assigns for every unique element of $\mathbb{R}^{n+n_2}$ a unique element of $\mathbb{R}^l$. An approximate inversion model that satisfies this requirement is required to guarantee that given a desired pseudo-control input $\nu \in \mathbb{R}^{n_2}$ a control command $u$ can be found by dynamic inversion as follows

$$u = \hat{f}^{-1}(x, \nu).$$ (1.31)

The model in Section 1.3.4.1 is an example of a differentially flat approximate inversion model. For the general system in Equation (1.30), the use of an approximate inversion model results in a model error of the form

$$\dot{x}_2 = \nu + \Delta(x, u)$$ (1.32)

where $\Delta$ is the modeling error. The modeling error captures the difference between the system dynamics and the approximate inversion model

$$\Delta(z) = f(z) - \hat{f}(z).$$ (1.33)

Note that if the control assignment function were known and invertible with respect to $u$, then an inversion model can be chosen such that the modeling error is only a function of the state $x$.

Often, UAV dynamics can be represented by models that are affine in control. In this case the existence of the approximate inversion model can be related directly to the invertibility of the control effectiveness matrix $B$ (e.g., see [50]). For example, let $G \in \mathbb{R}^{n_2 \times n_2}$, and let $B \in \mathbb{R}^{n_2 \times l}$ denote the control assignment matrix and consider the following system

$$\begin{align*}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= Gx_2(t) + B(\Theta(x) + u(t)),
\end{align*}$$ (1.34)

where $\Theta(x)$ is a nonlinear function. If $B^TB$ is invertible, and the pair $(G, B)$ is controllable, one approximate inversion model is: $\nu(t) = Bu(t)$, which results in a unique $u$ for a unique $\nu$: $u(t) = (B^TB)^{-1}B^T \nu(t)$. Adding and subtracting $\nu = Bu$ yields Equation (1.32), with $\Delta(x) =$
\[ Gx_2 + B(\Theta(x) + u) - Bu = Gx_2 + B\Theta(x). \]

A reference model is used to characterize the desired response of the system

\[
\begin{align*}
\dot{x}_{1_{rm}} &= x_{2_{rm}}, \\
\dot{x}_{2_{rm}} &= f_{rm}(x_{rm}, r),
\end{align*}
\]

where \( f_{rm}(x_{rm}(t), r(t)) \) denote the reference model dynamics which are assumed to be continuously differentiable in \( x_{rm} \) for all \( x_{rm} \in D_x \subset \mathbb{R}^n \). The command \( r(t) \) is assumed to be bounded and piecewise continuous, furthermore, \( f_{rm} \) is assumed to be such that \( x_{rm} \) is bounded for a bounded reference input.

The pseudo-control input \( \nu \) is designed by combining a linear feedback part \( \nu_{pd} = [K_1, K_2]e \) with \( K_1 \in \mathbb{R}^{n_2 \times n_2} \) and \( K_2 \in \mathbb{R}^{n_2 \times n_2} \), a linear feedforward part \( \nu_{rm} = \dot{x}_{2_{rm}} \), and an approximate feedback linearizing part \( \nu_{ai}(z) \)

\[
\nu = \nu_{rm} + \nu_{pd} - \nu_{ai}.
\]

Defining the tracking error \( e \) as \( e(t) = x_{rm}(t) - x(t) \), and using Equation (1.32) the tracking error dynamics can be written as

\[
\dot{e} = \dot{x}_{rm} - \begin{bmatrix} x_2 \\ \nu + \Delta \end{bmatrix}.
\]

Letting \( A = \begin{bmatrix} 0 & I_1 \\ -K_1 & -K_2 \end{bmatrix} \), \( B = [0, I_2]^T \) where \( 0 \in \mathbb{R}^{n_2 \times n_2} \), \( I_1 \in \mathbb{R}^{n_2 \times n_2} \), and \( I_2 \in \mathbb{R}^{n_2 \times n_2} \) are the zero and identity matrices, and using Equation (1.36) gives the following tracking error dynamics that are linear in \( e \)

\[
\dot{e} = Ae + B[\nu_{ai}(z) - \Delta(z)].
\]

The baseline linear full state feedback controller \( \nu_{pd} \) should be chosen such that \( A \) is a Hurwitz matrix. Furthermore, letting \( \nu_{ai} = \Delta(z) \) using Equation (1.33) ensures that the above tracking error dynamics is exponentially stable, and the states of the UAV track the reference model.

### 1.3.5 Model Reference Adaptive Control

The Model Reference Adaptive (MRAC) Control architecture has been widely studied for UAV control in presence of nonlinearities and modeling uncertainties (see e.g. [49,66,96,102]). MRAC attempts to ensure that the controlled states track the output of an appropriately chosen reference model (see e.g. [6,44,92,122]). Most MRAC methods achieve this by using a parameterized model of the uncertainty, often referred to as the adaptive element and its parameters referred to as adaptive weights. Aircraft dynamics can often be separated into a linear part whose mathematical model is fairly well known, and an uncertain part that may contain unmodeled linear or nonlinear effects. This representation is also helpful in representing nonlinear external disturbances affecting the system dynamics. Therefore, one typical technique for implementing adaptive controllers is to augment a baseline linear controller, designed and verified using techniques discussed in "Linear
Flight Control Techniques for Unmanned Aerial Vehicles" in this book, with an adaptive controller that deals with nonlinearities and modeling uncertainties. Let \( x(t) \in \mathbb{R}^n \) be the known state vector, let \( u \in \mathbb{R} \) denote the control input, and consider a following system of this type:

\[
\dot{x} = Ax(t) + B(u(t) + \Delta(z(t))), \tag{1.39}
\]

where \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), and \( \Delta(x, u) \in \mathbb{R}^m \) is a continuously differentiable function representing the uncertainty. It is assumed that the pair \((A, B)\) is controllable (see “Linear Flight Control Techniques for Unmanned Aerial Vehicles” in this book). For notational convenience, let \( z = [x, u] \), and note that the function \( \Delta(z) \) capturing the uncertain part of the dynamics is assumed to lie in the range space of the control effectiveness matrix \( B \). More general formulations of adaptive control also assume that the matrix \( A \) is completely unknown (see e.g. [6,44,92,122] and the Chapter on adaptive control in this book).

A reference model can be designed that characterizes the desired response of the system

\[
\dot{x}_{rm} = A_{rm}x_{rm}(t) + B_{rm}r(t), \tag{1.40}
\]

where \( A_{rm} \in \mathbb{R}^{n \times n} \) is such that all of its eigenvalues are in the complex left-half plane, the pair \((A, B)\) is controllable, and \( r(t) \) denotes a bounded exogenous reference signal. These conditions are sufficient to guarantee that \( x_{rm} \) is bounded for a bounded reference signal \( r(t) \). The tracking control law consists of a linear feedback part \( u_{pd} = K(x_{rm}(t) - x(t)) \), a linear feedforward part \( u_{rm} = K_r[x_{rm}^T, r(t)]^T \), and an adaptive part \( u_{ad}(x) \) and has the following form

\[
u = u_{rm} + u_{pd} - u_{ad}. \tag{1.41}\]

As before, define the tracking error \( e(t) = x_{rm}(t) - x(t) \); with an appropriate choice of \( A_{rm}, B_{rm}, \) and \( u_{rm} \) such that \( Bu_{rm} = (A_{rm} - A)x_{rm} + B_{rm}r(t) \) (conditions such as these are often required
in MRAC and are referred to as matching conditions, the tracking error dynamics simplify to

\[ \dot{e} = A_m e + B(u_{ad}(x,u) - \Delta(x,u)), \tag{1.42} \]

where the baseline full state feedback controller \( u_{pd} = Kx \) is assumed to be designed such that \( A_m = A - BK \) is a Hurwitz matrix. Hence for any positive definite matrix \( Q \in \mathbb{R}^{n \times n} \), a positive definite solution \( P \in \mathbb{R}^{n \times n} \) exists to the Lyapunov equation

\[ A_m^T P + P A_m = -Q. \tag{1.43} \]

MRAC architecture is depicted in Figure 1.5. Several MRAC approaches [6,16,44,67,92,122] assume that the uncertainty \( \Delta(z) \) can be linearly parameterized, that is, there exist a vector of constants \( W = [w_1, w_2, ..., w_m]^T \) and a vector of continuously differentiable functions \( \Phi(z) = [\phi_1(z), \phi_2(z), ..., \phi_m(z)]^T \) such that

\[ \Delta(z) = W^T \Phi(z). \tag{1.44} \]

The case when the basis of the uncertainty is known, that is the basis \( \Phi(x) \) is known, has been referred to as the case of structured uncertainty [18]. In this case letting \( W \) denote the estimate \( W^* \) the adaptive element is chosen as \( u_{ad}(x) = W^T \Phi(x) \). For this case it is known that the adaptive law

\[ \dot{W} = -\Gamma_W \Phi(z)e^TPB \tag{1.45} \]

where \( \Gamma_W \) is a positive definite learning rate matrix results in \( e(t) \to 0 \). However, it should be noted that Equation (1.50) does not guarantee the convergence (or even the boundedness) of \( W \) [91,122]. A necessary and sufficient condition for guaranteeing \( W(t) \to W \) is that \( \Phi(t) \) be persistently exciting (PE) [11,44,92,122].

Several approaches have been explored to guarantee boundedness of the weights without needing PE, these include the classic \( \sigma \)-modification [44], the \( e \)-modification [92], and projection based adaptive control [122]. Let \( \kappa \) denote the \( \sigma \) modification gain, then the \( \sigma \)-modification adaptive law is

\[ \dot{W} = -\Gamma_W (\Phi(z(t))e^TPB + \kappa W). \tag{1.46} \]

Thus it can be seen that the goal of \( \sigma \) modification is to add damping to the weight evolution. In \( e \) modification, this damping is scaled by the norm of the error \( ||e|| \). It should be noted that these adaptive laws guarantee the boundedness of the weights, however, they do not guarantee that the weights converge to their true values without PE.

A concurrent learning approach introduced in [18] guarantees exponential tracking error and weight convergence by concurrently using recorded data with current data without requiring PE. Other approaches that use recorded data include the \( Q \)-modification approach that guarantees convergence of the weights to a hyperplane where the ideal weights are contained [124], and the retrospective cost optimization approach [113]. Other approaches to MRAC favor instantaneous domination of uncertainty over weight convergence, one such approach is \( L_1 \) adaptive control [16,42].
Several other approaches to MRAC also exist, including the composite adaptive control approach in which direct and indirect adaptive control are combined [67], the observer based reference dynamics modification in adaptive controllers in which transient performance is improved by drawing on parallels between the reference model and a Luenberger observer [68]. The derivative-free MRAC in which a discrete derivative free update law is used in a continuous framework [127] and the optimal control modification [95]. Among approaches that deal with actuator time delays and constraints include the adaptive loop recovery modification which recovers nominal reference model dynamics in presence of time delays [13], and the pseudo control hedging which allows adaptive controllers to be implemented in presence of saturation [51].

In the more general case where the exact basis of the uncertainty is not a-priori, the adaptive part of the control law is often represented using a Radial Basis Function (RBF) Neural Network (NN) (see e.g. [14,19,49,75,112]). For ease of notation let \( z = [x, u] \), then the output of a RBF NN is given by

\[
\hat{W}^T \sigma(z).
\]

(1.47)

where \( \hat{W} \in \mathbb{R}^l \) and \( \sigma = [1, \sigma_2(z), \sigma_3(z), \ldots, \sigma_l(z)]^T \) is a vector of known radial basis functions. For \( i = 2, 3, \ldots, l \) let \( c_i \) denote the RBF centroid and \( \mu_i \) denote the RBF width then for each RBF, then RBFs can be expressed as

\[
\sigma_i(z) = e^{-\|z-c_i\|^2/\mu_i}. 
\]

(1.48)

This approach relies on the universal approximation property of Radial Basis Function Neural Networks [101] which asserts that given a fixed number of radial basis functions \( l \) there exists ideal weights \( W^* \in \mathbb{R}^l \) and a real number \( \tilde{\epsilon} \) such that

\[
\Delta(z) = W^*^T \sigma(z) + \tilde{\epsilon}, 
\]

(1.49)

where \( \tilde{\epsilon} \) can be made arbitrarily small given sufficient number of radial basis functions. For this case, adaptive laws can be obtained by replacing \( \Phi(z) \) in 1.50 by \( \sigma(z) \). For example, the adaptive law for an RBF NN adaptive element with \( \sigma \)-modification is

\[
\dot{\hat{W}} = \Gamma_W (\sigma(z)e^T PB + \kappa W). 
\]

(1.50)

RBFs have gained significant popularity in adaptive control research because they are linearly parameterized. However, in practice, it is difficult to determine a-priori how many RBFs should be chosen, and where their centers should lie. Online adaptation and selection of RBF centers is an active research area [60,90,120]. Another approach that has been successful in practice is to use nonlinearly parameterized neural networks, including single hidden layer neural networks (see e.g. [19,49,53,75]).
1.3.6 Approximate Model Inversion Based MRAC

The key issue with the approximate model inversion scheme introduced in Section 1.3.4.2 is that the system dynamics \( f(x, u) \) must be known in order to calculate \( \nu_{ai} \) to exactly cancel \( \Delta(z) \). As this is often not the case, the following presents the Approximate Model Inversion based MRAC architecture (AMI-MRAC), which extends the dynamic model inversion based control scheme of Section 1.3.4.2. Similar to the approach described there, the approach begins by choosing an approximate inversion model \( \hat{f} \) such given a desired pseudo-control input \( \nu \in \mathbb{R}^{n_2} \) a control command \( u \) can be found by dynamic inversion as in Equation (1.31). The parameters of the approximate inversion model do not have to be close to the real system model, however, in general, the mathematical structure should be similar to the real dynamical model. Typically it is sufficient to map the inputs to the states that they directly affect, for example, it is sufficient to map the desired pitch rate to elevator deflection. Given the desired acceleration \( \nu \) (pseudo-control input) required to track the reference model, the control input can be found by \( u = \hat{f}^{-1}(x, \nu) \). This results in a modeling error of the form

\[
\Delta(z) = f(z) - \hat{f}(z). \tag{1.51}
\]

The desired pseudo-control input can be formulated using the framework of dynamic model inversion as described in Section 1.3.4.2 as follows:

\[
\nu = \nu_{rm} + \nu_{pd} - \nu_{ad}. \tag{1.52}
\]

Note that the feedback linearizing part in equation (1.36) has been replaced here with the output of an adaptive element \( \nu_{ad} \). The tracking error dynamics of equation 1.37 now become

\[
\dot{e} = Ae + B[\nu_{ad}(z) - \Delta(z)]. \tag{1.53}
\]
This equation is similar to equation 1.42 of the MRAC architecture discussed in Section 1.3.5. Therefore, techniques similar to those discussed in Section 1.3.5 can be used. Particularly, if the uncertainty $\Delta(z)$ can be modeled using a linearly parameterized models such as in equation 1.44, then the adaptive element can take the form $\nu_{ad} = W(t)^T \Phi(z)$, and update law in equation 1.46 can be employed. It is also possible to use neural network adaptive elements in the same manner as equation 1.47. Note that with the formulation presented here the existence of a fixed point solution to $\nu_{ad} = \Delta(\cdot, \nu_{ad})$ needs to be assumed, sufficient conditions for guaranteeing this are also available [59]. AMI-MRAC based adaptive controllers have been extensively flight-test verified on several fixed-wing and rotorcraft UAVs (see e.g. [19,21,48,49] and also the chapter in this book titled Adaptive Control of Unmanned Aerial Vehicles - Theory and Flight Tests). Figure 1.6 depicts the framework of an approximate model inversion based MRAC architecture.

1.4 Model Based Fault Tolerant Control

Fault Tolerant Controllers (FTC) are designed to maintain aircraft stability or reduce performance deterioration in case of failures (see e.g. [103,119,128]). In general, there are mainly three types of failures (faults): sensor failure, actuator failure and structural damage. Fault tolerant control in presence of sensor and actuator failures with identified failure characteristics have been studied [45, 103]. On the other hand, fault tolerant control for structural damage, including partial loss of wing, vertical tail loss, horizontal tail loss, engine loss, are being explored [5,9,13,20,37,38,41,80,81,93,94].

Adaptive control has been widely studied for fault tolerant control in presence of modeling uncertainty brought about by structural damage or actuator degradation [20,43,65,66,119]. The idea has been to use adaptive control techniques similar to those described in Section 1.3.5 to adapt to changing dynamics. Adaptive control based fault tolerant controllers do not require prior system identification or modeling efforts. Developing adaptive fault tolerant controllers with associated quantifiable metrics for performance and without requiring restrictive matching conditions is an
open area of research. Alternatively, a model based approach can also be used for fault tolerant control. An overview of one such approach is presented here.

1.4.1 Control Design for Maximum Tolerance Range using Model Based Fault Tolerant Control

Fault tolerant control technique presented here is applicable to aircraft with possible damage that lies approximately in the body $x - z$ plane of symmetry, in other words vertical tail-damage as illustrated in Figure 1.7. A linear model of aircraft dynamics was derived in “Linear Flight Control Techniques for Unmanned Aerial Vehicles” in this book. In that model $A$ represents the matrix containing aerodynamic derivatives, and $B$ represents the matrix containing the control effectiveness derivatives. The states of the model considered here are $x(t) = [u, w, q, \theta, v, p, r, \phi]^T$, and the input given by $u(t) = [\delta e, \delta f, \delta a, \delta r]^T$ (see “Linear Flight Control Techniques for Unmanned Aerial Vehicles” in this book for definitions of these variables, details of the corresponding $A$ and $B$ matrices can be found in [76]). Variation to aerodynamic derivatives due to structural damage is approximately proportional to the percentage of loss in structure [114]. Therefore, the linearized motion of the damaged aircraft can be represented by a parameter-dependent model based on the baseline linear model of the aircraft, where the parameters provide a notion of the degree of damage:

$$\dot{x}(t) = (A - \mu \bar{A})x(t) + (B - \mu \bar{B})u(t), \quad \mu \in [0, 1]$$ (1.54)

where $\mu$ is the parameter representing the damage degree. Specifically, $\mu = 0$ represents the case of no damage, $\mu = 1$ represents complete tail loss, and $0 < \mu < 1$ represents partial vertical tail loss. The damage loss can be related to a maximum change in the aerodynamic derivatives (e.g. $\Delta C_{y\beta}$) and control effectiveness derivatives (e.g. $\Delta C_{y\delta}$) under damage in the following way:

$$\begin{bmatrix} \Delta C_{y\beta} & \Delta C_{n\beta} & \Delta C_{l\beta} \\ \Delta C_{yp} & \Delta C_{np} & \Delta C_{lp} \\ \Delta C_{yr} & \Delta C_{nr} & \Delta C_{lr} \\ \Delta C_{yrs} & \Delta C_{nrs} & \Delta C_{lrs} \end{bmatrix} = \mu \begin{bmatrix} \Delta C_{y\beta}^{\text{max}} & \Delta C_{n\beta}^{\text{max}} & \Delta C_{l\beta}^{\text{max}} \\ \Delta C_{yp}^{\text{max}} & \Delta C_{np}^{\text{max}} & \Delta C_{lp}^{\text{max}} \\ \Delta C_{yr}^{\text{max}} & \Delta C_{nr}^{\text{max}} & \Delta C_{lr}^{\text{max}} \\ \Delta C_{yrs}^{\text{max}} & \Delta C_{nrs}^{\text{max}} & \Delta C_{lrs}^{\text{max}} \end{bmatrix}$$ (1.55)

For a given linear controller gain $K$, let $J(K, \mu)$ denote the performance metric of an aircraft with possible vertical tail damage as a function of the degree of damage $\mu$. For example, a quadratic performance metric can be used

$$J = \int_0^\infty [x^T(t) Q x(t) + u^T(t) R u(t)] \, dt,$$ (1.56)

where $Q$ and $R$ are weighting positive definite matrices. In extreme damage cases, such as complete tail loss ($\mu = 1$), the aircraft may not be able to recover. The question then is under what level of damage a fault tolerant control could still be able to stabilize the aircraft. This notion of the maximum tolerance range, defined as the maximum allowable damage degree, presents a valuable
design criterion in fault tolerant control development. For most aircraft configurations, and without considering differential throttle as a way to control yaw motion, it is reasonable to expect that the maximum tolerance range would be less than 1. The notion of maximum tolerance, denoted by $\mu_m$, is captured using the following relationship,

$$
\mu_m := \min \left\{ 1, \max_K \{ \mu_u \geq 0 : J(K, \mu) \text{ is satisfied for } \mu \in [0, \mu_u] \} \right\}.
$$

This definition indicates that for damage degree $0 \leq \mu \leq \mu_m \leq 1$, the aircraft control system is able to guarantee the desired level of performance $J$ (see e.g. Equation (1.56)) with a certain controller $K$. In particular, $\mu_m = 0$ means that there is no tolerance for the desired performance, while $\mu_m = 1$ means that the control strategy can guarantee the performance requirement up to a total loss of vertical tail. The bigger $\mu_m$ is, the more tolerant the system becomes. Moreover, this notion implies a trade-off between damage tolerance and performance requirement. Since damage degree is unpredictable, a passive fault tolerant strategy is to design a controller to maintain the expected performance under possible damage.

This can be achieved through a robust control design technique in which an upper bound is established for a linear quadratic cost function for all the considered uncertainty [1,104]. Consider the parameterized system of Equation (1.54) describing damaged aircraft dynamics. Let $\Delta A = -\mu \bar{A}$ and $\Delta B = -\mu \bar{B}$ be uncertainty matrices expressed as

$$
\left[ \begin{array}{c} \Delta A & \Delta B \end{array} \right] = DF \left[ \begin{array}{c} E_1 & E_2 \end{array} \right].
$$

In these expression, $F$ satisfies $F^TF \leq I$. $D$ and $E$ are matrices containing the structural information of $\Delta A$ and $\Delta B$ and are assumed to be known a priori. For the given structure of vertical damage, $D = -I$, $F = \mu I$, $E_1 = \bar{A}$, and $E_2 = \bar{B}$.

One control design approach for the uncertain system in Equation (1.54) is presented here based on LMI. If the following LMI with respect to a positive matrix $X$, matrix $W$, and positive scalar $\varepsilon$
(\varepsilon > 0) is feasible \[104,126\]

\[
\begin{bmatrix}
AX + BW + XAT + WTB^T + \varepsilon DD^T \\
E_1X + E_2W \\
Q^{1/2}X \\
R^{1/2}W
\end{bmatrix}
\begin{bmatrix}
X E_1^T + WTE_2^T \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
X(Q^{1/2})^T \\
W(R^{1/2})^T
\end{bmatrix}
< 0
\]

(1.58)

then there exists a state feedback guaranteed cost control law \( K = WX^{-1} \) and the corresponding cost has an upper bound \( J \leq x_0^T X^{-1} x_0 \).

As described above, one can formulate the aircraft dynamic model as an uncertain system by defining

\[
\begin{bmatrix}
\Delta A & \Delta B
\end{bmatrix} = DF
\begin{bmatrix}
E_1 & E_2
\end{bmatrix} =: \mu_m(-\Delta) \begin{bmatrix}
\bar{A} & \bar{B}
\end{bmatrix}
\text{ with } |\Delta| < 1.
\]

The weighting matrices \( Q \) and \( R \) in the linear quadratic criterion in Equation (1.56) can be chosen to represent a desired performance criterion. They also serve as free design parameters that indirectly reduce the effect of control surface limits on maximum tolerance and performance.

The above result gives a method to design a guaranteed cost controller. A controller with the maximum tolerance \( \mu_m \) can be obtained by testing the feasibility of LMI via the bisection algorithm \[117,132\].

**Algorithm:**

1. Take \( \mu_d = 0 \) and \( \mu_t = 1 \). If the LMI (1.58) is feasible for \( \mu_t = 1 \), then \( \mu_m = 1 \), output the corresponding controller and stop.

2. Let \( \mu = \frac{\mu_d + \mu_t}{2} \).

3. If the LMI (1.58) is feasible for \( \mu \), then \( \mu_d = \mu \), otherwise, \( \mu_t = \mu \).

4. If \( \mu_t - \mu_d < \delta \) (\( \delta \) is a pre-determined scalar and is small enough), then \( \mu_m = \mu_t \), output the controller with respect to \( \mu_t \) and stop. Otherwise, go to 2.

### 1.5 Ongoing Research

Future UAV applications are expected to continue to leverage S&R capabilities of UAVs and go beyond. Some envisioned applications and some of the associated technical challenges in the field of autonomous control are listed below (see Figure 1.8):

**UCAV** Unmanned Combat Aerial Vehicles (UCAV) have been researched extensively. The goal is to create an autonomous multi-role fighter capable of air-to-air combat and targeted munition delivery. UCAVs are expected to be highly agile, and need to be designed to tolerate severe damage. Furthermore, the lack of human pilot onboard means that UCAVs can be designed to tolerate much higher flight loads in order to perform extremely agile maneuvers. The main
technical challenge here is to create autonomous controllers capable of reliably operating in nonlinear flight regimes for extended durations.

**Transport UAVs** These UAVs are expected to autonomously deliver valuable and fragile payload to forward operating units. The technical challenge here is to enable reliable automated waypoint navigation in presence of external disturbances and changed mass properties, and reliable vertical takeoff and landing capabilities in harsh terrains with significant uncertainties.

**HALE** High Altitude Long Endurance (HALE) aircraft are being designed to support persistent S&R missions at stratospheric altitude. The main differentiating factor here is the use of extended wing span equipped with solar panels for in-situ power generation. The technical challenges here are brought about by unmodeled flexible body modes.

**Optionally Manned Aircraft** Optionally manned aircraft are envisioned to support both piloted and autonomous modes. The idea is to extend the utility of existing successful aircraft by enabling autonomous operation in absence of an onboard human pilot. The main technical challenge here is to ensure reliability and robustness to uncertainties, considering specifically that these aircraft may share air-space with manned aircraft.

**Micro/Nano UAVs** UAVs with wingspan less than 30 cm are often referred to as Micro/Nano UAVs. These UAVs are characterized by high wing (or rotor blade) loading and are being designed for example to operate in the vicinity of human first-responders. Some of these aircraft are expected to operate in dynamic or hostile indoor environments, and are therefore required to be agile and low cost. Large scale collaborative swarms of low-cost micro UAVs have also been envisioned as flexible replacements to a single larger UAV. The key technical challenges here are creating low-cost, low-weight controllers capable of guaranteeing safe and agile operation in the vicinity of humans, and development of decentralized techniques for guidance and control.

In order to realize these visions, there is a need to develop robust and adaptive controllers that are able to guarantee excellent performance as the UAV performs agile maneuvers in nonlinear flight regime in presence of uncertainties. Furthermore, there is a significant thrust towards collaborative operation of manned and unmanned assets and towards UAVs sharing airspace with their manned counterparts. The predictable and adaptable behavior of UAVs as a part of such co-dependent networks is evermore important.

### 1.5.1 Integrated Guidance and Control under Uncertainty

Guidance and Control algorithms for autonomous systems are safety/mission critical. They need to ensure safe and efficient operation in presence of uncertainties such as unmodeled dynamics, damage, sensor failure, and unknown environmental effects. Traditionally control and guidance methods have often evolved separately. The ignored interconnections may lead to catastrophic
failure if commanded trajectories end up exciting unmodeled dynamics such as unmodelled flexible dynamics (modes). Therefore, there is a need to establish a feedback between the guidance and command loops to ensure that the UAV is commanded a feasible trajectories. This is also important in order to avoid actuator saturation, and particularly important if the UAV capabilities have been degraded due to damage [22]. Isolated works exist in the literature where authors have proposed a scheme to modify reference trajectories to accommodate saturation or to improve tracking performance (e.g. [51,68]), however, the results are problem specific and more general frameworks are required.

There has been significant recent research on UAV task allocation and planning algorithms. The algorithms developed here provide waypoints and reference commands to the guidance system in order to satisfy higher level mission goals. There may be value in accounting specifically for the UAV’s dynamic capabilities and health to improve planning performance.

1.5.2 Output Feedback Control in Presence of non-Gaussian noise and Estimation Errors

LQG theory provides a solution to designing optimal controllers for linear systems with Gaussian white measurement noise. However, such a generalized theory does not exist for nonlinear control. Furthermore, the choice of sensors on miniature UAVs in particular, is often restricted to low-cost, low-weight options. Several of the standard sensors employed on miniature aircraft such as
sonar altimeters, scanning lasers, and cameras do not have Gaussian noise properties. Therefore, even with an integrated navigation solution that fuses data from multiple sensors, the resulting state estimates may contain significant estimation error. One of the key future thrusts required therefore is a theory of output feedback control for nonlinear control implementations in presence of non-Gaussian noise.

1.5.3 Metrics on Stability and Performance of UAV controllers

A strong theory exists for computing performance and stability metrics for linear time invariant control laws. Examples of well accepted metrics include gain and phase margin which quantify the ability of the controller to maintain stability in presence of unforeseen control effectiveness changes, time delays, and disturbances. However, these metrics do not easily generalize to nonlinear systems. On the other hand, several different groups have established that nonlinear control techniques in general can greatly improve command tracking performance over linear techniques. However, the inability to quantify stability and robustness of nonlinear controllers may pose a significant hurdle in wide acceptance of nonlinear techniques. The problem is further complicated as controllers are implemented on digital computers, while they are often designed using a continuous framework. Any further work in this area should avail the rich literature on the theory of sampled data systems.

1.5.4 Agile and Fault-Tolerant Control

Agile flight refers to flight conditions which cannot be easily represented by linear models linearized around equilibrium flight conditions. Agile flight is often characterized by rapid transitions between flight domains (e.g. hover flight domain and forward flight domain for rotorcraft UAVs), high wing (blade) loading, actuator saturation, and nonlinear flight regimes. The lack of a human pilot onboard means that the allowable g-force is not a limiting factor on the aggressive maneuvers a UAV platforms can potentially perform. Significant progress has been made in creating UAV autopilots that perform highly aggressive maneuvers. Authors in [3,24,34,49] have established control methods that can be used to perform specific agile maneuvers under test conditions. Future work in this area could lead to responsive agile flight maneuvers to meet higher level mission requirements and quantifiable metrics of performance and stability during agile flight.

For some agile flight regimes, accurate modeling of the vehicle dynamics might be very difficult, requiring that models and/or control strategies be identified in real-time, in essence pushing the capabilities discussed in [23,47,87,89] to be on-line and for non-linear models. In this case, model identification (or learning) combined with on-line optimization (e.g., model predictive control) [30,31,125] might provide an important direction to consider. The key challenge here is to simultaneously guarantee stability and performance while guaranteeing on-line learning of reasonable model approximations.

Fault Tolerant Control (FTC) research for UAVs is focused on guaranteeing recoverable flight in presence of structural and actuator failures. Several directions exist in fault tolerant flight control, excellent reviews can be found in [119,128]. Authors in [21,52] have established through flight tests
the effectiveness of fault tolerant controllers in maintaining flight in presence of severe structural damage including loss of significant portions of the wing. Detection and diagnosis of structural faults is also being studied, along with detection of sensor anomalies.
Bibliography


