

Casimir effect in a two dimensional signature changing spacetime

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Abstract

In this paper we study the Casimir effect for free massless scalar fields propagating on a two-dimensional cylinder with a metric that admits a change of signature from Lorentzian to Euclidean.

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1 Introduction

The Casimir effect is regarded as one of the most striking manifestation of vacuum fluctuations in quantum field theory. The presence of reflecting boundaries alters the zero-point modes of a quantized field, and results in the shifts in the vacuum expectation values of quantities quadratic in the field, such as the energy density and stresses. In particular, vacuum forces arise acting on the constraining boundaries. The particular features of these forces depend on the nature of the quantum field, the type of spacetime manifold and its dimensionality, the boundary geometries and the specific boundary conditions imposed on the field. Since the original work by Casimir in 1948 [1] many theoretical and experimental works have been done on this problem (see, e.g., [3, 4, 5, 6, 7, 8, 9, 10] and references therein). There are several methods to calculate the Casimir energy. For instance, we can mention to the mode summation [2], Green's function method [4], heat kernel method [8] along with appropriate regularization schemes such as point separation [11, 12], dimensional regularization [13], zeta function regularization [14, 15, 16, 17, 18]. Recently, general new methods to compute renormalized one-loop quantum energies and energy densities are given in [19, 20].

On the other hand, the initial idea of signature change is due to Hartle, Hawking and Sakharov [21] which makes it possible to have a spacetime with Euclidean and Lorentzian regions in quantum gravity. It has been shown that the signature change may happen even in classical general relativity [22]. There are two different approaches to this problem: continuous and discontinuous signature changes. In the continuous approach, in passing from Euclidean region to Lorentzian one the signature of metric changes continuously. Hence the metric becomes degenerate at the border of these regions. In the discontinuous approach, however, the metric becomes nondegenerate everywhere and is discontinuous at the border of Euclidean and Lorentzian regions. The issue of propagation of quantum fields on signature-changing spacetimes has also been of some interest [23]. For example, Dray *et al* have shown that the phenomenon of particle production can happen for propagation of scalar particles in spacetime with heterotic signature. They have also obtained a rule for propagation of massless scalar fields on a two dimensional spacetime with signature change.

To the authors knowledge no attempt has yet been done to study the Casimir effect in the geometries with signature change. In this paper we study the Casimir effect for a free massless scalar field propagating on a two-dimensional cylinder with a metric that admits a change of signature from Lorentzian to Euclidean [24].

2 Casimir stress tensor in signature changing spacetime

We consider the free massless scalar field ϕ which propagates on a two-dimensional manifold $M = R \times S^1$ (the circle S^1 represents space and the real line R represents time) with the following metric:

$$ds^2 = -d\tau^2 + g(\sigma)d\sigma^2, \quad (1)$$

where τ is timelike coordinate and σ is a periodic spacelike coordinate with the period L , and $g(\sigma)$ is a periodic function of σ , which takes +1 for Lorentzian and -1 for Euclidean regions¹.

¹Notice that in this region the metric will be $g_{\alpha\beta} = \text{diag}(-1, -1)$

Let us introduce a segment σ_0 for which $g(\sigma)$ is given by

$$g(\sigma) = \begin{cases} -1 & 0 < \sigma < \sigma_0 + \text{mod } L \\ +1 & \sigma_0 < \sigma < L + \text{mod } L. \end{cases} \quad (2)$$

One can obtain the following expressions for the components of energy-momentum tensors associated with the real scalar field $\phi(\sigma, \tau)$ in both Euclidean and Lorentzian regions [24]

$$\begin{aligned} T_{00}^E &= [(\partial_\tau \Phi^E)^2 - (\partial_\sigma \Phi^E)^2], & T_{01}^E &= 2 \partial_\tau \Phi^E \partial_\sigma \Phi^E, \\ T_{00}^L &= [(\partial_\tau \Phi^L)^2 + (\partial_\sigma \Phi^L)^2], & T_{01}^L &= 2 \partial_\tau \Phi^L \partial_\sigma \Phi^L. \end{aligned} \quad (3)$$

By introducing new coordinates σ_+^E, σ_-^E in the Euclidean region and σ_+^L, σ_-^L in the Lorentzian region

$$\begin{aligned} \sigma_+^E &= \tau + i\sigma, & \sigma_+^L &= \tau + \sigma, \\ \sigma_-^E &= \tau - i\sigma, & \sigma_-^L &= \tau - \sigma, \end{aligned} \quad (4)$$

we obtain

$$\begin{aligned} T_{++}^E &= (T_{00}^E - iT_{01}^E)/2, & T_{++}^L &= (T_{00}^L + T_{01}^L)/2, \\ T_{--}^E &= (T_{00}^E + iT_{01}^E)/2, & T_{--}^L &= (T_{00}^L - T_{01}^L)/2, \\ T_{+-}^E &= T_{-+}^E = 0, & T_{+-}^L &= T_{-+}^L = 0. \end{aligned} \quad (5)$$

Then we have

$$\begin{aligned} T_{++}^E &= \frac{1}{2}[(\partial_\tau \Phi - i\partial_\sigma \Phi)]^2 \\ &= 2 \sum_{\omega\omega'} \tilde{f}_\omega^E(+)\tilde{f}_{\omega'}^E(+)\tilde{\alpha}_\omega\tilde{\alpha}_{\omega'}^\dagger, \\ T_{--}^E &= \frac{1}{2}[(\partial_\tau \Phi + i\partial_\sigma \Phi)]^2 \\ &= 2 \sum_{\omega\omega'} \tilde{f}_{-\omega}^E(-)\tilde{f}_{-\omega'}^E(-)\tilde{\alpha}_\omega\tilde{\alpha}_{\omega'}^\dagger, \\ T_{++}^L &= \frac{1}{2}[(\partial_\tau \Phi + \partial_\sigma \Phi)]^2 \\ &= 2 \sum_{\omega\omega'} \tilde{f}_\omega^L(+)\tilde{f}_{\omega'}^L(+)\tilde{\alpha}_\omega\tilde{\alpha}_{\omega'}^\dagger, \\ T_{--}^L &= \frac{1}{2}[(\partial_\tau \Phi - \partial_\sigma \Phi)]^2 \\ &= 2 \sum_{\omega\omega'} \tilde{f}_{-\omega}^L(-)\tilde{f}_{-\omega'}^L(-)\tilde{\alpha}_\omega\tilde{\alpha}_{\omega'}^\dagger, \end{aligned} \quad (6)$$

where

$$\tilde{f}_\omega^E(\sigma^\pm) = [(a/b)_\omega + 1] \exp(-i\omega\sigma_\pm^E)/\sqrt{4\pi \langle \Phi_\omega, \Phi_\omega \rangle}, \quad (7)$$

$$\tilde{f}_\omega^L(\sigma^\pm) = [(c/b)_\omega + (1/b)_{-\omega}] \exp(-i\omega\sigma_\pm^L)/\sqrt{4\pi \langle \Phi_\omega, \Phi_\omega \rangle},$$

and $(a/b)_\omega, (c/b)_\omega, (1/b)_\omega$ are given by

$$\begin{aligned} (a/b)_\omega &= \frac{\sin\omega(\theta-2\pi)}{\cosh\omega\theta + \sinh\omega\theta - \cos\omega(\theta-2\pi)}, \\ (c/b)_\omega &= \frac{(1+i)\exp(2\pi i\omega)(\sinh\omega\theta + \cosh\omega\theta - \exp(i\omega(\theta-2\pi)))}{2(\cosh\omega\theta + \sinh\omega\theta - \cos\omega(\theta-2\pi))}, \\ (1/b)_\omega &= \frac{(1-i)\exp(-2\pi i\omega)(\sinh\omega\theta + \cosh\omega\theta - \exp(-i\omega(\theta-2\pi)))}{2(\sinh\omega\theta - \cos\omega(\theta-2\pi) + \cosh\omega\theta)}. \end{aligned} \quad (8)$$

From above relations we find

$$\begin{aligned}
T_{00}^E &= T_{++}^E + T_{--}^E = 2 \sum_{\omega\omega'} (\tilde{f}_\omega^E(+)\tilde{f}_{\omega'}^E(+) + \tilde{f}_{-\omega}^E(-)\tilde{f}_{-\omega'}^E(-)) \tilde{\alpha}_\omega \tilde{\alpha}_{\omega'}^\dagger, \\
T_{01}^E &= i(T_{++}^E - T_{--}^E) = 2i \sum_{\omega\omega'} (\tilde{f}_\omega^E(+)\tilde{f}_{\omega'}^E(+) - \tilde{f}_{-\omega}^E(-)\tilde{f}_{-\omega'}^E(-)) \tilde{\alpha}_\omega \tilde{\alpha}_{\omega'}^\dagger, \\
T_{00}^L &= T_{++}^L + T_{--}^L = 2 \sum_{\omega\omega'} (\tilde{f}_\omega^L(+)\tilde{f}_{\omega'}^L(+) + \tilde{f}_{-\omega}^L(-)\tilde{f}_{-\omega'}^L(-)) \tilde{\alpha}_\omega \tilde{\alpha}_{\omega'}^\dagger, \\
T_{01}^L &= T_{++}^L - T_{--}^L = 2 \sum_{\omega\omega'} (\tilde{f}_\omega^L(+)\tilde{f}_{\omega'}^L(+) - \tilde{f}_{-\omega}^L(-)\tilde{f}_{-\omega'}^L(-)) \tilde{\alpha}_\omega \tilde{\alpha}_{\omega'}^\dagger.
\end{aligned} \tag{9}$$

Since we have

$$[\tilde{\alpha}_\omega, \tilde{\alpha}_{\omega'}^\dagger] = \omega \delta_{\omega+\omega',0} \tag{10}$$

and

$$\begin{aligned}
\tilde{\alpha}_\omega |0\rangle &= 0, \\
\langle 0 | \tilde{\alpha}_{\omega'}^\dagger &= 0,
\end{aligned}$$

then

$$\langle 0_L | \tilde{\alpha}_\omega \tilde{\alpha}_{\omega'} | 0_L \rangle = \omega \delta_{\omega+\omega',0} \tag{11}$$

and we obtain

$$\begin{aligned}
\langle 0_L | T_{00}^E | 0_L \rangle &= 2 \sum_{\omega\omega'} (\tilde{f}_\omega^E(+)\tilde{f}_{\omega'}^E(+) + \tilde{f}_{-\omega}^E(-)\tilde{f}_{-\omega'}^E(-)) \omega \delta_{\omega+\omega',0}, \\
\langle 0_L | T_{01}^E | 0_L \rangle &= 2i \sum_{\omega\omega'} (\tilde{f}_\omega^E(+)\tilde{f}_{\omega'}^E(+) - \tilde{f}_{-\omega}^E(-)\tilde{f}_{-\omega'}^E(-)) \omega \delta_{\omega+\omega',0}, \\
\langle 0_L | T_{00}^L | 0_L \rangle &= 2 \sum_{\omega\omega'} (\tilde{f}_\omega^L(+)\tilde{f}_{\omega'}^L(+) + \tilde{f}_{-\omega}^L(-)\tilde{f}_{-\omega'}^L(-)) \omega \delta_{\omega+\omega',0}, \\
\langle 0_L | T_{01}^L | 0_L \rangle &= 2 \sum_{\omega\omega'} (\tilde{f}_\omega^L(+)\tilde{f}_{\omega'}^L(+) - \tilde{f}_{-\omega}^L(-)\tilde{f}_{-\omega'}^L(-)) \omega \delta_{\omega+\omega',0}, \\
\langle 0_L | T_{11}^E | 0_L \rangle &= - \langle 0_L | T_{00}^E | 0_L \rangle, \\
\langle 0_L | T_{11}^L | 0_L \rangle &= \langle 0_L | T_{00}^L | 0_L \rangle,
\end{aligned} \tag{12}$$

where $|0_L\rangle$ denotes the vacuum state for the signature changing cylinder. By the following normalization

$$4\pi \langle \Phi_\omega, \Phi_\omega \rangle = \frac{1}{2}(L - \sigma_0), \tag{13}$$

and using the equations (7) and applying the delta function $\delta_{\omega+\omega',0}$ we obtain

$$\begin{aligned}
\langle 0_L | T_{00}^E | 0_L \rangle &= \frac{2}{L-\sigma_0} \sum_{\omega=0}^{\infty} [(\frac{a}{b})_{\omega} + 1][(\frac{a}{b})_{-\omega} + 1] \omega, \\
\langle 0_L | T_{01}^E | 0_L \rangle &= \langle 0_L | T_{10}^E | 0_L \rangle = 0, \\
\langle 0_L | T_{11}^E | 0_L \rangle &= - \langle 0_L | T_{00}^E | 0_L \rangle, \\
\langle 0_L | T_{00}^L | 0_L \rangle &= \frac{2}{L-\sigma_0} \sum_{\omega=0}^{\infty} [(\frac{c}{b})_{\omega} + (\frac{1}{b})_{-\omega}][(\frac{c}{b})_{\omega} + (\frac{1}{b})_{-\omega}]^* \omega, \\
\langle 0_L | T_{01}^L | 0_L \rangle &= \langle 0_L | T_{10}^L | 0_L \rangle = 0, \\
\langle 0_L | T_{11}^L | 0_L \rangle &= \langle 0_L | T_{00}^L | 0_L \rangle.
\end{aligned} \tag{14}$$

Now, we use the Heaviside distribution Θ^+, Θ^- with support in Euclidean and Lorentzian regions, respectively, such that

$$d\Theta^{\pm} = \pm \delta, \tag{15}$$

where δ is the hypersurface Dirac distribution with support on the hypersurface $\sigma = \sigma_0$. Then, we may write

$$\begin{aligned}
\langle 0_L | T_{00} | 0_L \rangle &= \Theta^+ \langle 0_L | T_{00}^E | 0_L \rangle + \Theta^- \langle 0_L | T_{00}^L | 0_L \rangle, \\
\langle 0_L | T_{11} | 0_L \rangle &= \Theta^+ \langle 0_L | T_{11}^E | 0_L \rangle + \Theta^- \langle 0_L | T_{11}^L | 0_L \rangle.
\end{aligned} \tag{16}$$

The normal ordered expressions are as follows

$$\begin{aligned}
\langle 0_L | : T_{00} : | 0_L \rangle &= \Theta^+ \langle 0_L | : T_{00}^E : | 0_L \rangle + \Theta^- \langle 0_L | : T_{00}^L : | 0_L \rangle, \\
\langle 0_L | : T_{11} : | 0_L \rangle &= \Theta^+ \langle 0_L | : T_{11}^E : | 0_L \rangle + \Theta^- \langle 0_L | : T_{11}^L : | 0_L \rangle,
\end{aligned} \tag{17}$$

such that

$$\begin{aligned}
\langle 0_L | : T_{00}^E : | 0_L \rangle &= \langle 0_L | T_{00}^E | 0_L \rangle - \lim_{L' \rightarrow \infty} \langle 0_{L'} | T_{00}^E | 0_{L'} \rangle, \\
\langle 0_L | : T_{00}^L : | 0_L \rangle &= \langle 0_L | T_{00}^L | 0_L \rangle - \lim_{L' \rightarrow \infty} \langle 0_{L'} | T_{00}^L | 0_{L'} \rangle, \\
\langle 0_L | : T_{11}^E : | 0_L \rangle &= \langle 0_L | T_{11}^E | 0_L \rangle - \lim_{L' \rightarrow \infty} \langle 0_{L'} | T_{11}^E | 0_{L'} \rangle, \\
\langle 0_L | : T_{11}^L : | 0_L \rangle &= \langle 0_L | T_{11}^L | 0_L \rangle - \lim_{L' \rightarrow \infty} \langle 0_{L'} | T_{11}^L | 0_{L'} \rangle.
\end{aligned} \tag{18}$$

By introducing the cut-off to the sums we obtain

$$\begin{aligned}
\langle 0_L | T_{00}^E | 0_L \rangle_{cut-off} &= \frac{2}{L-\sigma_0} \sum_{\omega=0}^{\infty} [(\frac{a}{b})_{\omega} + 1][(\frac{a}{b})_{-\omega} + 1] \omega e^{-\alpha\omega}, \\
\langle 0_L | T_{00}^L | 0_L \rangle_{cut-off} &= \frac{2}{L-\sigma_0} \sum_{\omega=0}^{\infty} [(\frac{c}{b})_{\omega} + (\frac{1}{b})_{-\omega}][(\frac{c}{b})_{\omega} + (\frac{1}{b})_{-\omega}]^* \omega e^{-\alpha\omega}.
\end{aligned} \tag{19}$$

We now break each sum into two separate sums

$$\begin{aligned}
& \langle 0_L | T_{00}^E | 0_L \rangle_{cut-off} = \frac{2}{L-\sigma_0} \sum_{\omega=0}^{N-1} [(\frac{a}{b})_{\omega} + 1][(\frac{a}{b})_{-\omega} + 1] \omega \\
& + \frac{2}{L-\sigma_0} \sum_{\omega=N}^{\infty} [(\frac{a}{b})_{\omega} + 1][(\frac{a}{b})_{-\omega} + 1] \omega e^{-\alpha\omega} \\
& \langle 0_L | T_{00}^L | 0_L \rangle_{cut-off} = \frac{2}{L-\sigma_0} \sum_{\omega=0}^{N-1} [(\frac{c}{b})_{\omega} + (\frac{1}{b})_{-\omega}][(\frac{c}{b})_{\omega} + (\frac{1}{b})_{-\omega}]^* \omega \\
& + \frac{2}{L-\sigma_0} \sum_{\omega=N}^{\infty} [(\frac{c}{b})_{\omega} + (\frac{1}{b})_{-\omega}][(\frac{c}{b})_{\omega} + (\frac{1}{b})_{-\omega}]^* \omega e^{-\alpha\omega},
\end{aligned} \tag{20}$$

where we have just introduced the cut-off in the second sums, because the first sums are finite. As have been shown in [24], the continuous spectrum of ω approaches the integer spectrum at large values of ω , provided σ_0 is small. We assume that σ_0 is sufficiently small so that $\omega < N$ is continuous and $\omega > N$ is almost integer, where $N \gg 0$. We first focus on the second sum in $\langle 0_L | T_{00}^E | 0_L \rangle_{cut-off}$. Since each of the terms $[(\frac{a}{b})_{\omega} + 1]$ and $[(\frac{a}{b})_{-\omega} + 1]$ approaches 1 for large $\omega \geq N$ (because $(\frac{a}{b})_{\pm\omega} \simeq 0$) then the second sum goes like

$$\frac{2}{L-\sigma_0} \sum_{\omega=N}^{\infty} \omega e^{-\alpha\omega}.$$

In the same way for $\langle 0_L | T_{00}^L | 0_L \rangle_{cut-off}$, we obtain that the second sum goes like

$$\frac{2}{L-\sigma_0} \sum_{\omega=N}^{\infty} \omega e^{-\alpha\omega},$$

which is the same for both Euclidean and Lorentzian regions. Therefore, we calculate this second sum for both regions. We know that $\omega \geq N$ denotes for integers, so we may redefine $\omega = N$ to $\Omega = 0$ (Ω integer). To this end, we note that $\omega \geq N$ is the root of $\cos \omega(L - \sigma_0) = 0$ [24], from which we obtain $\omega \geq \frac{n+1/2}{L-\sigma_0} \pi$ and so $N = \frac{n+1/2}{L-\sigma_0} \pi$. This is equal to $n = \frac{N(L-\sigma_0)}{\pi} - \frac{1}{2}$ (n integer). Therefore, we may define $\Omega = n - \frac{N(L-\sigma_0)}{\pi} + \frac{1}{2}$, with $\Omega = 0, 1, 2, \dots$. We also obtain ω in term of Ω as $\omega = N + \frac{\pi\Omega}{L-\sigma_0}$.

Therefore, the second sum in both Euclidean and Lorentzian regions is written as

$$\frac{2}{L-\sigma_0} \sum_{\Omega=0}^{\infty} (N + \frac{\pi\Omega}{L-\sigma_0}) e^{-\alpha(N + \frac{\pi\Omega}{L-\sigma_0})},$$

which after some calculations we obtain

$$\frac{2}{L-\sigma_0} \sum_{\Omega=0}^{\infty} (N + \frac{\pi\Omega}{L-\sigma_0}) e^{-\alpha(N + \frac{\pi\Omega}{L-\sigma_0})} = \frac{2}{L-\sigma_0} e^{-\alpha N} \frac{(N + \frac{\pi\Omega}{L-\sigma_0}) e^{-\alpha(N + \frac{\pi\Omega}{L-\sigma_0})} - N}{(e^{-\alpha(N + \frac{\pi\Omega}{L-\sigma_0})} - 1)^2}. \tag{21}$$

For the Euclidean region we find

$$\Theta^+ \left\{ \frac{2}{L-\sigma_0} \left[\sum_{\omega=0}^{N-1} [(\frac{a}{b})_{\omega} + 1][(\frac{a}{b})_{-\omega} + 1] \omega + e^{-\alpha N} \frac{(N + \frac{\pi\Omega}{L-\sigma_0}) e^{-\alpha(N + \frac{\pi\Omega}{L-\sigma_0})} - N}{(e^{-\alpha(N + \frac{\pi\Omega}{L-\sigma_0})} - 1)^2} \right] \right\}$$

$$-\frac{2}{L-\sigma_0} \left[\sum_{\omega=0}^{N-1} \omega + \lim_{\sigma_0 \rightarrow \infty} e^{-\alpha N} \frac{(N + \frac{\pi\Omega}{L-\sigma_0})e^{-\alpha(N + \frac{\pi\Omega}{L-\sigma_0})} - N}{(e^{-\alpha(N + \frac{\pi\Omega}{L-\sigma_0})} - 1)^2} \right],$$

or

$$\Theta^+ \left\{ \frac{2}{L-\sigma_0} \left[\sum_{\omega=0}^{N-1} \left[\left(\frac{a}{b} \right)_\omega + 1 \right] \left[\left(\frac{a}{b} \right)_{-\omega} + 1 \right] - 1 \right] \omega \right. \\ \left. + \frac{2}{L-\sigma_0} \left[e^{-\alpha N} \frac{(N + \frac{\pi\Omega}{L-\sigma_0})e^{-\alpha(N + \frac{\pi\Omega}{L-\sigma_0})} - N}{(e^{-\alpha(N + \frac{\pi\Omega}{L-\sigma_0})} - 1)^2} - \lim_{\sigma_0 \rightarrow \infty} e^{-\alpha N} \frac{(N + \frac{\pi\Omega}{L-\sigma_0})e^{-\alpha(N + \frac{\pi\Omega}{L-\sigma_0})} - N}{(e^{-\alpha(N + \frac{\pi\Omega}{L-\sigma_0})} - 1)^2} \right] \right\}.$$

We now expand the $e^{-\alpha N} \frac{(N + \frac{\pi\Omega}{L-\sigma_0})e^{-\alpha(N + \frac{\pi\Omega}{L-\sigma_0})} - N}{(e^{-\alpha(N + \frac{\pi\Omega}{L-\sigma_0})} - 1)^2}$ terms in the second bracket about $\alpha = 0$. We obtain

$$\frac{2}{\alpha^2 \pi} - \left(\frac{\alpha^2 N^2}{2} - \alpha N + 1 \right) \left[\frac{\pi}{6(L-\sigma_0)^2} + \frac{13}{6(L-\sigma_0)} N \right] - \frac{N^2}{\pi} (1 - \alpha N),$$

which leads to

$$\frac{2}{L-\sigma_0} \left[e^{-\alpha N} \frac{(N + \frac{\pi\Omega}{L-\sigma_0})e^{-\alpha(N + \frac{\pi\Omega}{L-\sigma_0})} - N}{(e^{-\alpha(N + \frac{\pi\Omega}{L-\sigma_0})} - 1)^2} - \lim_{\sigma_0 \rightarrow \infty} e^{-\alpha N} \frac{(N + \frac{\pi\Omega}{L-\sigma_0})e^{-\alpha(N + \frac{\pi\Omega}{L-\sigma_0})} - N}{(e^{-\alpha(N + \frac{\pi\Omega}{L-\sigma_0})} - 1)^2} \right] \\ = -\frac{\pi}{6(L-\sigma_0)^2} - \frac{13}{6(L-\sigma_0)} N.$$

Finally, we have the following expression for the Euclidean region

$$\langle 0_L | : T_{00}^E : | 0_L \rangle = \left\{ \frac{2}{L-\sigma_0} \sum_{\omega=0}^{N-1} \left[\left(\frac{a}{b} \right)_\omega + 1 \right] \left[\left(\frac{a}{b} \right)_{-\omega} + 1 \right] - 1 \right\} \omega - \frac{\pi}{6(L-\sigma_0)^2} - \frac{13}{6(L-\sigma_0)} N. \quad (22)$$

In the same way, the calculations for the Lorentzian region leads to

$$\langle 0_L | : T_{00}^L : | 0_L \rangle = \left\{ \frac{2}{L-\sigma_0} \sum_{\omega=0}^{N-1} \left[\left(\frac{c}{b} \right)_\omega + \left(\frac{1}{b} \right)_{-\omega} \right] \left[\left(\frac{c}{b} \right)_\omega + \left(\frac{1}{b} \right)_{-\omega} \right]^* - 1 \right\} \omega - \frac{\pi}{6(L-\sigma_0)^2} - \frac{13}{6(L-\sigma_0)} N, \quad (23)$$

where in this case we have used $\lim_{(L-\sigma_0) \rightarrow \infty}$ instead of $\lim_{\sigma_0 \rightarrow \infty}$ in the cut-off calculations. For the other components we have

$$\langle 0_L | : T_{11}^E : | 0_L \rangle = - \langle 0_L | T_{00}^E | 0_L \rangle, \quad (24) \\ \langle 0_L | : T_{11}^L : | 0_L \rangle = \langle 0_L | T_{00}^L | 0_L \rangle.$$

Therefore, the state $|0_L\rangle$ contains the finite energy density and pressure in the Euclidean and Lorentzian regions as follows:

$$\begin{cases} \rho^E = \langle 0_L | : T_{00}^E : | 0_L \rangle, \\ \rho^L = \langle 0_L | : T_{00}^L : | 0_L \rangle, \end{cases} \quad (25)$$

$$\begin{cases} p^E = \langle 0_L | : T_{11}^E : | 0_L \rangle = -\rho^E, \\ p^L = \langle 0_L | : T_{11}^L : | 0_L \rangle = \rho^L. \end{cases} \quad (26)$$

Therefore, the total pressure acting on the signature changing hypersurface $\sigma = \sigma_0$ is nonzero and given by

$$p^T = p^L - p^E = \rho^L + \rho^E. \quad (27)$$

It is seen that in the special case $N = 0$ the first sums and the last terms vanish in (22), (23) and these results lead to the standard result for the pure Lorentzian metric $\sigma_0 = 0$ on the cylinder as $-\frac{\pi}{6L^2}$ [2]. In fact, $N = 0$ means that ω starts to be integer from $\omega = N = 0$ which occurs only in the pure Lorentzian region.

3 Conclusion

We have considered a two dimensional model in which spacetime is a cylinder $M = R \times S^1$ where the circle represents “space” and the real line represents “time”. Moreover, we have supposed that this manifold admits a signature change of the type which had already been reported in [24].

We were interested in studying the Casimir effect for the real massless scalar fields propagating over this manifold. To this end, we have considered the expressions for the components of energy-momentum tensors associated with the real scalar field and have calculated the corresponding vacuum expectation values. These expressions are found to be infinite and a regularization is needed to make them finite. By introducing a convenient cut-off and a regularization scheme, after some calculations we have obtained the finite expressions for the vacuum expectation values of the energy momentum tensor. These provide us with the finite energy densities and pressures in both Euclidean and Lorentzian regions so that the net pressure on the signature changing hypersurface is obtained. These results approach to those of standard Lorentzian case, at higher energies of the scalar field.

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