Irregular subset of the Grassman manifolds

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Abstract

The book is devoted to study irregular subsets of the Grassman manifolds. This class of sets was introduced by author. We consider the group of bijective maps of $\mathbb{G}_k^n$ onto $\mathbb{G}_k^n$ which preserve the class of irregular sets. In what follows maps belonging to this group will be called regular. We say that two irregular subsets of $\mathbb{G}_k^n$ are similar if there exists a regular map transfering one of them to another. We want to discuss the question: how many nonsimilar irregular sets exist?

Contents

Introduction

0.1. Irregular subsets and Dimension Theory. Irregular subsets of the Grassman manifolds were introduced in the author's paper [12] (see also [13]) where we studied orthogonal projections of k-dimensional subsets of $\mathbb{R}^n$ onto k-dimensional planes (a set is called k-dimensional if it's topological dimension is equal to k [**?**, 7]). For a set $X \subset \mathbb{R}^n$ we considered the set of all k-dimensional planes such that the orthogonal projections of X onto these planes are "regular".

Now we give the strong definition of these terms. Let $X \subset \mathbb{R}^n$. Then the orthogonal projection

$$p_l : X \to \mathbb{R}^k$$

of the set X onto some k-dimensional plane l is called

— *d-regular* if $\dim p_l(X) = k$ (i.e. the interior of the set $p_l(X)$ is not empty),

— *c-regular* if $p_l(X)$ is a set of second category in $\mathbb{R}^k$,

— *m-regular* if $p_l(X)$ has a non-zero Lebesgue measure.

Consider the sets

$$DR^n_k(X) = \{ \ l \in \mathbb{G}^n_k \mid p_l \text{ is } d\text{-regular } \} \ ,$$

$$CR^n_k(X) = \{ \ l \in \mathbb{G}^n_k \mid p_l \text{ is } c\text{-regular } \} \ ,$$

$$MR^n_k(X) = \{ \ l \in \mathbb{G}^n_k \mid p_l \text{ is } m\text{-regular } \} \ .$$

It is trivial that

$$DR^n_k(X) \subset CR^n_k(X) \text{ and } DR^n_k(X) \subset MR^n_k(X) \ .$$

Moreover, if X is an F_σ-subset of $\mathbb{R}^n$ then the set $DR^n_k(X)$ and $CR^n_k(X)$ coincide (recall that any continuable union of closed subsets of $\mathbb{R}^n$ is called an F_σ-subset of $\mathbb{R}^n$).

It was proved in [12, 13] that if the set X setisfies the condition $\dim X \geq k$ then the sets

$$\mathbb{G}^n_k \setminus CR^n_k(X) \text{ and } \mathbb{G}^n_k \setminus CR^n_k(X)$$

are irregular. Moreover, if X is an k-dimensional F_σ-subset then the set

$$\mathbb{G}^n_k \setminus DR^n_k(X)$$

is irregular.

Note that for k-dimensional subsets of $\mathbb{R}^n$ which are not F_σ-sets the last statement does not hold. We constructed an $(n-1)$-dimensional set $X \subset \mathbb{R}^n$ such that the complement to the set $DR^n_k(X)$ is not irregular for any k setisfying the condition

$$k \geq \left[\frac{n}{2}\right] + 1 \ .$$

These results generalize the Nobeling Theorem [10] which states that for any k-dimensional compact set $X \subset \mathbb{R}^n$ the set $DR^n_k(X)$ is not empty.

In the paper [13] we considered the following natural quation: how large may be an irregular set? In the cases $k = 1, n-1$ our problem is trivial, each irregular subset is nowhere dense in $\mathbb{G}^n_k$. For the general case $(1 < k < n-1)$ the similar statement is not proved. However, there are some results supporting our conjecture. One of them states that any irregular subset of $\mathbb{G}^n_k$ has an empty interior (i.e. it's complement is everywhere dense subset of $\mathbb{G}^n_k$).

This implies that for any set $X \subset \mathbb{R}^n$ setisfying the condition $\dim X \geq k$ the sets $CR^n_k(X)$ and $MR^n_k(X)$ are everywhere dense in $\mathbb{G}^n_k$; moreover, in the cases $k = 1, n-1$ the complements to these sets are nowhere dense in $\mathbb{G}^n_k$. If X is an F_σ-set then the set $DR^n_k(X)$ is everywhere dense in $\mathbb{G}^n_k$ and in the cases $k = 1, n-1$ it's complement is a nowhere dense subset of $\mathbb{G}^n_k$.

0.2. Chogoshvili's Conjecture. The results obtained in [12, 13] are connected with well-known Chogoshvili's Conjecture (see [1] or [4]). Now we recall it's formulation.

Let $f : X \to Y$ be a continuous map of some topological space X into a matric spase Y. We say that $y \in f(X) \subset Y$ is an *unstable* value for the map f if for any $\varepsilon > 0$ there exists a continuous map $g_\varepsilon : X \to Y$ such that

$$d(f(x), g_\varepsilon(x)) < \varepsilon \quad \forall\, x \in X$$

and $y \notin g_\varepsilon(X)$. Otherwise, y is called a *stable* value. A map will be called a *stable* if it has a a stable value.

It is well-known that a searable matric space X setisfies the condition $\dim X \geq k$ if and only if there exists a stable map of X into $\mathbb{R}^k$ (see, for example, [6, 7]). In [1] G. Chogoshvili formulated the theorem wich is known now as the Chogoshvili Conjecture. It states that for any k-dimensional set $X \subset \mathbb{R}^n$ there exists a k-dimensional plane such that the orthogonal projection of X onto this plane is a stable map.

The Chogoshvili Conjecture can be formulated in other terms. Let X and Y be a subsets of $\mathbb{R}^n$. We say that X has *unstable* intersection with Y if for any $\varepsilon > 0$ there exists a continuous map $f_\varepsilon : X \to \mathbb{R}^n$ such that

$$|x - f_\varepsilon(x)| < \varepsilon \quad \forall\, x \in X$$

and $f_\varepsilon(X) \cap Y = \emptyset$. In other words X can be removed from Y by arbitrarily small move. Otherwise, we say that X has *stable* intersection with Y.

It is not difficult to see (we do not prove it here) that for any set $X \subset \mathbb{R}^n$ the following two conditions are equivalent:

(i) the orthogonal projection of X onto some k-dimensional plane l is an unstable map;

(ii) the set X has unstable intersection with any $(n-k)$-dimensional plane orthogonal to l.

Now we can formulated the Chogoshvili Conjecture in the next form: any k-dimensional subset of $\mathbb{R}^n$ has a stable intersection with some $(n-k)$-dimensional plane.

It is easy to see that for any stable map $f : X \to Y$ the image $f(X)$ has a non-empty interior in Y. Moreover, there exists $\varepsilon > 0$ such that the similar condition holds for any continuous map $g : X \to Y$ setisfying the inequality

$$|f(x) - g(x)| < \varepsilon \quad \forall\, x \in X \ .$$

This implies that the Nobeling theorem [10] and our results [12, 13] support the Chogoshvili Conjecture.

However, the Sitnikov example [14] shows that the Conjecture fails for non-compact sets. Recently A. N. Dranishnikov [5] disproved the Chogoshvili Conjecture in the compact case.

0.3. Brief rewiev of the contents. This book is devoted to study the following two problems.

– What is the class of bijective maps of $\mathbb{G}_k^n$ onto $\mathbb{G}_k^n$ which preserve the class of irregular sets? In what follows maps belonging to this class will be called *regular*.

– We will say that two irregular subsets of $\mathbb{G}_k^n$ are *similar* if there exists a regular map transfering one of them to another. We want to discuss the question: how many nonsimilar irregular sets exist? We restrict ourself only to the case of maximal irregular sets indeed for any irregular set there exists a maximal irregular set containing it.

Now we would say some words about the contents of the book. In Chapter 1 we recall the definition and basic properties of the Grassman manifolds. In Chapter 2 we give the definition, prove some elementary properties and consider examples of irregular sets. The main results will be formulated in Chapter 3. Their proofs will be given in the next part (Chapters 4 – 7).

0.4. Thanks. We wish to express our deep gratitude to my teacher V. V. Sharko and to my collegues I. Yu. Vlasenko, S. I. Maximenko, E. A. Polulyakh for their interest in our research.

Chapter 1

Grassman manifolds

In this chapter we recall the definition and some basic properties of the Grassman manifolds. We show that the Grassman manifold $\mathbb{G}_k^n$ is a $k(n-k)$-dimensional compact topological manifold and the manifolds $\mathbb{G}_k^n$, $\mathbb{G}_{n-k}^n$ are homeomorphic.

1.1 Definition and basic properties

1.1.1. First of all we consider one open subset of Euclidean space which is called the Stiefel manifold and use it to defined the Grassman manifolds.

Definition 1.1.1. We say that a collection of k vectors $(x_1, ..., x_k)$ in the Euclidean space $\mathbb{R}^n$ is a *k-frame* if the vectors $x_1, ..., x_n$ are linearly independent. The set of all k-frames in $\mathbb{R}^n$ is called the *Stiefel manifold* and denoted by $\mathbb{V}_k^n$.

It is easy to see that $\mathbb{V}_k^n$ is an open subset of $\mathbb{R}^{nk}$.

Denote by $\mathbb{G}_k^n$ the set of all k-dimensional planes in $\mathbb{R}^n$ passing through the origin of the coordinates (in what follows the origin of the coordinates will be denoted by O). Consider the map

$$p_k^n : \mathbb{V}_k^n \to \mathbb{G}_k^n$$

transfering each k-frame to the plane generated by it. This map could be exploit to give $\mathbb{G}_k^n$ the following topology: a set $U \subset \mathbb{G}_k^n$ is open if and only if $(p_k^n)^{-1}(U)$ is an open subset of $\mathbb{V}_k^n$. It is trivial that in this topology the map p_k^n is continuous.

1.1.2. Now we recall the following two well-known definition.

Definition 1.1.2. We say that a topological space X is a *Hausdorff space* if for any two points $x_1 \in X$ and $x_2 \in X$ there exist disjoint open subsets U_1 and U_2 of X containing x_1 and x_2 respectively.

Definition 1.1.3. A Hausdorff space X is called *n-dimensional topological manifold* if for each point $x \in X$ there exists an open subset of X containing x and homeomorphic to $\mathbb{R}^n$.

Proposition 1.1.1. *The topological space $\mathbb{G}^n_k$ is a $k(n-k)$-dimensional compact topological manifold.*

Remark 1.1.1. The space $\mathbb{G}^n_k$ has the structure of a smooth manifold [9]. However we do not consider it in what follows.

Definition 1.1.4. The spaces $\mathbb{G}^n_k$ $(k = 1, ..., n-1)$ are called the *Grassman manifolds*. The space $\mathbb{G}^n_1$ is called the $(n-1)$-*dimensional projective space* and denoted by $\mathbb{P}^{n-1}$.

Consider the map

$$\varphi^n_{k\,n-k} : \mathbb{G}^n_k \to \mathbb{G}^n_{n-k} \ ,$$
$$\varphi^n_{k\,n-k}(l) = l^\perp \quad \forall\, l \in \mathbb{G}^n_k \ ,$$

where $l^\perp$ is the orthogonal complement to l.

Proposition 1.1.2. *The map $\varphi^n_{k\,n-k}$ is a homeomorphism of $\mathbb{G}^n_k$ onto $\mathbb{G}^n_{n-k}$.*

Propositions 1.1.1 and 1.1.2 will be proved in the next two sections. In Subsection 1.2.3 we construct an open subset of $\mathbb{G}^n_k$ homeomorphic to $\mathbb{R}^{k(n-k)}$. This construction will be exploited in what follows.

1.2 Proof of Proposition 1.1.1

1.2.1. Denote by $\mathbb{O}^n_k$ the set of all orthonormal k-frames in $\mathbb{R}^n$. Then $\mathbb{O}^n_k$ is a compact subset of $\mathbb{R}^{nk}$ (it is easy to see that this set is closed and bounded). The map p^n_k is continuous and

$$p^n_k(\mathbb{O}^n_k) = \mathbb{G}^n_k \ .$$

Therefore the topological space $\mathbb{G}_k^n$ is compact. It is easy to see that the map

$$p_k^n : \mathbb{O}_k^n \to \mathbb{G}_k^n$$

transfers any open subset of $\mathbb{O}_k^n$ to an open subset of $\mathbb{G}_k^n$ (recall that a map setisfying this condition is called *open*).

1.2.2. Show that the topological space $\mathbb{G}_k^n$ is Hausdorff. For any two planes l_1 and l_2 belonging to $\mathbb{G}_k^n$ we construct a continuous function $f : \mathbb{G}_k^n \to \mathbb{R}$ such that $f(l_1) \neq f(l_2)$. There exist disjoin open subsets U_1 and U_2 of $\mathbb{R}$ containing $f(l_1)$ and $f(l_2)$ respectively. Then the sets $f^{-1}(U_1)$ and $f^{-1}(U_2)$ setisfy the required conditions.

Fix a point $x \in \mathbb{R}^n$. For any plane $l \in \mathbb{G}_k^n$ denote by $d_x(l)$ the square of the distance between x and l. If this plane is generated by an orthonormal k-frame $(x_1, ..., x_k)$ then

$$d_x(l) = |x|^2 - (x, x_1)^2 - ... - (x, x_k)^2 \ .$$

This implies that the function

$$d'_x : \mathbb{O}_k^n \to \mathbb{R} \ ,$$

comparing to each orthonormal k-frame the number $d_x(l)$ (where l is the plane generated by this k-frame) is continuous. Then the following simple lemma shows that

$$d_x : \mathbb{G}_k^n \to \mathbb{R}$$

is a continuous function.

Lemma 1.2.1. *Let $f : X \to Y$ and $g : Y \to Z$ be a maps of topological spaces. If f is open and gf is continuous then g is continuous.*

Proof. An immediate verification shows that for any set $U \subset Z$ we have

$$g^{-1}(U) = f((gf)^{-1}(U)) \ .$$

This implies that if the set U is open then $g^{-1}(U)$ is an open subset of Y. ■

It is easy to see that

$$d'_x = d_x p_k^n \ .$$

The map p_k^n is open and d'_x is a continuous function. Then Lemma 1.2.1 shows that d_x is continuous.

For any two planes l_1 and l_2 belonging to $\mathbb{G}_k^n$ there exists a point $x \in \mathbb{R}^n$ such that l_1 contains x and l_2 does not contain x. Then $d_x(l_1) \neq d_x(l_2)$.

1.2.3. In this subsection we construct an open subset of $\mathbb{G}_k^n$ homeomorphic to $\mathbb{R}^{k(n-k)}$. Fix a plane $s \in \mathbb{G}_{n-k}^n$ and consider an orthogonal coordinate system $\{x_i\}_{i=1}^n$ for $\mathbb{R}^n$ such that the point O is the origin of the coordinates for this system and s is the plane generated by the coordinate axes $x_{k+1}, ..., x_n$. Denote by $U_k^n(s)$ the set of all planes belonging to $\mathbb{G}_k^n$ and intersecting s only in the point O (in other words $U_k^n(s)$ contains a plane $l \in \mathbb{G}_k^n$ if and only if the planes l and s are transverse). It is not difficult to see that $U_k^n(s)$ is an open subset of $\mathbb{G}_k^n$ and any plane l belonging to it could be considered as a graph of some linear map of $\mathbb{R}^k$ into $\mathbb{R}^{n-k}$. Let $A_k^n(s)(l)$ be the matrix of this map in the coordinates $\{x_i\}_{i=1}^k$ and $\{x_i\}_{i=k+1}^n$. Then the map

$$A_k^n(s) : U_k^n(s) \to \mathbb{R}^{k(n-k)}$$

is a bijection of the set $U_k^n(s)$ onto $\mathbb{R}^{k(n-k)}$. In the next subection we show that this map is a homeomorphism.

For any plane $l \in \mathbb{G}_k^n$ there exists a plane $s \in \mathbb{G}_{n-k}^n$ such that the set $U_k^n(s)$ contains l. Therefore $\mathbb{G}_k^n$ is a $k(n-k)$-dimensional topological manifold.

1.2.4. Define

$$\hat{U}_k^n(s) = (p_k^n)^{-1}(U_k^n(s)) \cap \mathbb{O}_k^n \ .$$

Then $\hat{U}_k^n(s)$ is an open subset of $\mathbb{O}_k^n$. Consider the map

$$\hat{A}_k^n(s) = A_k^n(s) p_k^n : \hat{U}_k^n(s) \to \mathbb{R}^{k(n-k)} \ .$$

Let $l \in U_k^n(s)$ and

$$A_k^n(s)(l) = (x_{i\,j})_{i=1\ \, j=1}^{n-k\ k} \ .$$

Then the linear map defined by the matrix $A_k^n(s)(l)$ transfers the k vectors

$$(1, 0, ..., 0), ..., (0, ..., 0, 1)$$

to the vectors

$$(x_{1\,1}, ..., x_{n-k\,1}), ..., (1, x_{1\,k}, ..., x_{n-k\,k}) \ .$$

This implies that the plane l is generated by the vectors

$$(1, 0, ..., 0, x_{1\,1}, ..., x_{n-k\,1}), ..., (0, ..., 0, 1, x_{1\,k}, ..., x_{n-k\,k}) \ .$$

Therefore the map $\hat{A}_k^n(s)$ transfers each orthonormal k-frame

$$(x_{1\,1}, ..., x_{n\,1}), ..., (x_{1\,k}, ..., x_{n\,k})$$

to a k-frame

$$(1, 0, ..., 0, x'_{1\,1}, ..., x'_{n-k\,1}), ..., (0, ..., 0, 1, x'_{1\,k}, ..., x'_{n-k\,k})$$

such that the planes generated by these k-frames are coincident. In other words the map $\hat{A}^n_k(s)$ could be considered as the reduction of a matrix

$$(x_{i\,j})^{n}_{i=1}{}^{k}_{j=1}$$

to the next form

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ x'_{1\,1} & x'_{1\,2} & \dots & x'_{1\,k} \\ x'_{2\,1} & x'_{2\,2} & \dots & x'_{2\,k} \\ \vdots & \vdots & \ddots & \vdots \\ x'_{n-k\,1} & x'_{n-k\,2} & \dots & x'_{n-k\,k} \end{pmatrix} .$$

This implies that $\hat{A}^n_k(s)$ is a continuous map. The map

$$p^n_k : \hat{U}^n_k(s) \to U^n_k(s)$$

is open and Lemma 1.2.1 shows that the map $A^n_k(s)$ is continuous.

The map $(A^n_k(s))^{-1}$ transfers eahc matrix

$$(x_{i\,j})^{n-k}_{i=1}{}^{k}_{j=1}$$

to the plane generated by the k-frame

$$(1, 0, ..., 0, x_{1\,1}, ..., x_{n-k\,1}), ..., (0, ..., 0, 1, x_{1\,k}, ..., x_{n-k\,k}) .$$

It is trivial that this map is continuous and we have proved that $A^n_k(s)$ is a homeomorphism.

1.3 Proof of Proposition 1.1.2

It is not difficult to see that $\varphi^n_{k\,n-k}$ transfers any two planes intersecting only in the point O to planes intersecting only in the point O. This implies that

$$\varphi^n_{k\,n-k}(U^n_k(s)) = U^n_{n-k}(s^{\perp})$$

for any $s \in \mathbb{G}^n_{n-k}$.

Fix a plane $s \in \mathbb{G}^n_{n-k}$ and consider an orthogonal coordinate system $\{x_i\}_{i=1}^n$ such that the point O is the origin of the coordinates for this system and s is the plane generated by the coordinate axes $x_{k+1}, ..., x_n$. Show that the map

$$\varphi^n_{k\,n-k}(s) = A^n_{n-k}(s^\perp)\varphi^n_{k\,n-k}(A^n_k(s))^{-1} : \mathbb{R}^{k(n-k)} \to \mathbb{R}^{k(n-k)}$$

is continuous.

Recall that if $l \in U^n_k(s)$ and

$$A^n_k(s)(l) = (x_{i\,j})_{i=1\ j=1}^{n-k k}$$

then the plane l is generated by the vectors

$$(1, 0, ..., 0, x_{1\,1}, ..., x_{n-k\,1}), ..., (0, ..., 0, 1, x_{1\,k}, ..., x_{n-k\,k})$$

(see Subsection 1.2.4). The similar arguments show that if $l \in U^n_{n-k}(s^\perp)$ and

$$A^n_{n-k}(s^\perp)(l) = (x_{i\,j})_{i=1 j=1}^{k\ \ n-k}$$

then the plane l is generated by the vectors

$$(x_{1\,1}, ..., x_{k\,1}, 1, 0, ..., 0), ..., (x_{1\,n-k}, ..., x_{k\,n-k}, 0, ..., 0, 1)\ .$$

This implies that $\varphi^n_{k\,n-k}(s)$ could be considered as the map transfering any k-frame

$$x_1 = (1, 0, ..., 0, x_{1\,1}, ..., x_{n-k\,1})\ ,$$

$$...\ ,$$

$$x_k = (0, ..., 0, 1, x_{1\,k}, ..., x_{n-k\,k})$$

to the $(n-k)$-frame

$$y_1 = (y_{1\,1}, ..., y_{k\,1}, 1, 0, ..., 0)\ ,$$

$$...\ ,$$

$$y_{n-k} = (y_{1\,n-k}, ..., y_{k\,n-k}, 0, ..., 0, 1)$$

such that the vectors x_i and y_j are orthogonal for any $i = 1, ..., k$ and $j = 1, ..., n-k$. An immediate verification shows that

$$y_{j\,i} = -x_{i\,j} \quad \forall\ i = 1, ..., k\ \ \forall\ j = 1, ..., n-k\ .$$

Then $\varphi^n_{k\,n-k}(s)$ transfers a matrix A to the matrix transposed to $-A$ and $\varphi^n_{k\,n-k}(s)$ is a continuous map.

We have proved above that for any $s \in \mathbb{G}^n_{n-k}$ the map $\varphi^n_{k\,n-k}(s)$ is continuous. This implies that $\varphi^n_{k\,n-k}$ is a continuous map. Ricall that $\varphi^n_{k\,n-k}$ is a bijection of $\mathbb{G}^n_k$ onto $\mathbb{G}^n_k$ and the space $\mathbb{G}^n_k$ is compact. Therefore $\varphi^n_{k\,n-k}$ is a homeomorhism.

Chapter 2

Regular and irregular subsets of the Grassman manifolds

In Section 2.1 we introduce so-called regular subsets of the Grassman manifolds. In the next section they will be exploited to define irregular sets and maximal irregular sets. In Sections 2.3 we show that the homeomorphism $\varphi^n_{k\,n-k}$ transfers any regular subset and any irregular subset of $\mathbb{G}^n_k$ to a regular subset or an irregular subset of $\mathbb{G}^n_{n-k}$ respectively. Section 2.4 is devoted to simple examples; more complicated examples will be considered in Chapter 7. In Section 2.5 we discuss the following quation: how large may be an irregular set? There we prove that any irregular set $V \subset \mathbb{G}^n_k$ has the everywhere dense complement $\mathbb{G}^n_k \setminus V$.

2.1 Regular subsets of the Grassman manifolds

We begin with the following definition.

Definition 2.1.1. A set $R \subset \mathbb{G}^n_k$ is called *regular* ($R \in \mathfrak{R}^n_k$) if there exists a coordinate system for $\mathbb{R}^n$ such that any plane belonging to R is a coordinate plane for this system.

In other words a set R is regular if there exists a collection of n linearly independent lines belonging to $\mathbb{G}^n_1$ and such that each plane belonging to R is generated by lines from this collection. Any linearly independent lines belonging to $\mathbb{G}^n_1$ generate a regular subset of $\mathbb{G}^n_1$.

Each coordinate system for $\mathbb{R}^n$ has

$$c_k^n = \frac{n!}{k!(n-k)!}$$

distinct k-dimensional coordinate planes. Therefore a regular subset of $\mathbb{G}_k^n$ contains at most c_k^n elements.

Definition 2.1.2. We say that a regular set $R \subset \mathbb{G}_k^n$ is *maximal* ($R \in \mathfrak{MR}_k^n$) if any regular subset of $\mathbb{G}_k^n$ containing R coincides with it.

Proposition 2.1.1. *For each regular subset of $\mathbb{G}_k^n$ there exists a maximal regular subset of $\mathbb{G}_k^n$ containing it. A regular subset of $\mathbb{G}_k^n$ is maximal if and only if it contains c_k^n elements.*

Proof. Let R be a regular subset of $\mathbb{G}_k^n$. Consider a coordinate system for $\mathbb{R}^n$ such that any plane belonging to R is a coordinate plane for this system. In what follows any coordinate system setisfying this condition will be called *associated* with the regular set R. Denote by R' the set of all k-dimensional coordinate planes. Then R' is a maximal regular subset of $\mathbb{G}_k^n$ containing R. Moreover the regular set R is maximal if and only if it coincides with R'. ■

For a regular set $R \subset \mathbb{G}_k^n$ consider a coordinate system associated with it. If our regular set R is maximal then the coordinate system is uniquely defined. Denote by $r_{k\,m}^n(R)$ the set of all m-dimensional coordinate plane for this system. Then $r_{k\,m}^n(R)$ is a maximal regular subset of $\mathbb{G}_m^n$ and the map

$$r_{k\,m}^n : \mathfrak{MR}_k^n \to \mathfrak{MR}_m^n$$

defines an one-to-one correspondence between the classes $\mathfrak{MR}_k^n$ and $\mathfrak{MR}_m^n$. It is trivial that $r_{m\,i}^n r_{k\,m}^n = r_{k\,i}^n$.

2.2 Irregular and maximal irregular subsets of the Grassman manifolds

2.2.1. Now we give the definition of irregular sets.

Definition 2.2.1. A set $V \subset \mathbb{G}_k^n$ is called *irregular* ($V \in \mathfrak{I}_k^n$) if it is not regular and does not contain maximal regular subsets of $\mathbb{G}_k^n$.

It is trivial that if a subset of some irregular set is not regular then it is irregular.

Definition 2.2.2. We say that an irregular set $V \subset \mathbb{G}_k^n$ is *maximal* ($V \in \mathfrak{MI}_k^n$) if any irregular subset of $\mathbb{G}_k^n$ containing V coincides with it.

Proposition 2.2.1. *For any irregular subset of $\mathbb{G}_k^n$ there exists a maximal irregular subset of $\mathbb{G}_k^n$ containing it.*

The proof of Proposition 2.2.1 will be given in Subsection 2.2.3. Now we shall prove one simple lemma which will be often exploited in what follows.

Lemma 2.2.1. *An irregular set $V \subset \mathbb{G}_k^n$ is maximal if and only if for any plane l belonging to $\mathbb{G}_k^n \setminus V$ there exists a set $R \subset V$ such that $R \cup \{l\}$ is a maximal regular subset of $\mathbb{G}_k^n$.*

Proof. The irregular set V is maximal if and only if for any $l \in \mathbb{G}_k^n \setminus V$ the set $V \cup \{l\}$ is not irregular. ■

2.2.2. Zorn's Lemma. Now we formulate the Zorn Lemma. In the next subsection we show that Proposition 2.3.1 is a consequence of it.

Let $\mathcal{I}$ be a family of subsets of some set X and $\mathcal{J} \subset \mathcal{I}$.

Definition 2.2.3. We say that the family $\mathcal{J}$ is *linearly ordered* if for any two sets U_1 and U_2 belonging to it one of the following inclusions

$$U_1 \subset U_2 \text{ or } U_2 \subset U_1$$

holds true.

Definition 2.2.4. The family $\mathcal{J}$ is called *bounded above* in $\mathcal{I}$ if there exists a set belonging to the family $\mathcal{I}$ and containing each set $U \in \mathcal{J}$.

Definition 2.2.5. We say that a set $U \in \mathcal{I}$ is *maximal* in the family $\mathcal{I}$ if any set belonging to this family and containing U coinsides with it.

The Zorn Lemma. (see, for example, [8]) *If any linearly ordered family $\mathcal{J} \subset \mathcal{I}$ is bounded above in $\mathcal{I}$ then the family $\mathcal{I}$ has a maximal set.*

2.2.3. Proof of Proposition 2.2.1. Let V be an irregular subset of $\mathbb{G}_k^n$. Denote by $\mathcal{I}$ the family of all irregular subsets of $\mathbb{G}_k^n$ containing V. We want to show that for each linearly ordered family $\mathcal{J} \subset \mathcal{I}$ the set

$$U(\mathcal{J}) = \bigcup_{U \in \mathcal{J}} U$$

is irregular; i.e. the family $\mathcal{J}$ is bounded above. Then Proposition 2.2.1 is a consequence of the Zorn Lemma.

Assume that the set $U(\mathcal{J})$ is not irregular. Then it contains a maximal regular set R. For any plane l belonging to R there exists a set $U(l) \in \mathcal{J}$ containing it. The family $\mathcal{J}$ is linearly ordered and there exists a plane $l' \in R$ such that

$$U(l) \subset U(l') \quad \forall\, l \in R\,.$$

It is easy to see that $U(l')$ contains the maximal regular set R and is not irregular. ■

2.3 Our sets and the canonical homeomorphism

2.3.1. In this subsection we show that the canonical homeomorphism $\varphi^n_{k\,n-k}$ transfers the classes

$$\mathfrak{R}^n_k\,,\ \mathfrak{MR}^n_k\,,\ \mathfrak{I}^n_k\,,\ \mathfrak{MI}^n_k$$

to the classes

$$\mathfrak{R}^n_{n-k}\,,\ \mathfrak{MR}^n_{n-k}\,,\ \mathfrak{I}^n_{n-k}\,,\ \mathfrak{MI}^n_{n-k}$$

respectively. First of all we prove the following statement.

Proposition 2.3.1. *For any $R \subset \mathbb{G}_k^n$ the set $\varphi^n_{k\,n-k}(R)$ is a regular subset of $\mathbb{G}^n_{n-k}$ if and only the set R is regular. Moreover $\varphi^n_{k\,n-k}(R)$ is a maximal regular subset of $\mathbb{G}^n_{n-k}$ if and only if R is a maximal regular set.*

Proof. We want to show that for any regular set $R \subset \mathbb{G}_k^n$ the set $\varphi^n_{k\,n-k}(R)$ is a regular subset of $\mathbb{G}^n_{n-k}$ and if the regular set R is maximal then $\varphi^n_{k\,n-k}(R)$ is a maxmal regular set. This statement and the equality

$$\varphi^n_{n-k\,k} = (\varphi^n_{k\,n-k})^{-1}$$

imply Proposition 2.3.1.

Let R' be a maximal regular subset of $\mathbb{G}_k^n$ containing R. In the case $k = n-1$ we have $R' = \{l_i\}_{i=1}^n$ and

$$\varphi_{n-1\,1}^n(R') = \{l_i^\perp\}_{i=1}^n \,.$$

Assume that the set $\varphi_{n-1\,1}^n(R')$ is not regular and there exists an $(n-1)$-dimensional plane s containing each line $l_i^\perp$ $(i = 1, ..., n)$. Then any plane belonging to R' contains the line $s^\perp$ and the set R' is not regular. This implies that our hypothesis fails and $\varphi_{n-1\,1}^n(R')$ is a maximal regular subset of $\mathbb{G}_1^n$ containing $\varphi_{n-1\,1}^n(R)$.

In the general case consider the maximal regular set

$$r_{k\,1}^n(R') = \{l_i\}_{i=1}^n \subset \mathbb{G}_1^n \,.$$

For any $i = 1, ..., n$ denote by s_i the $(n-1)$-dimensional plane generated by the lines

$$l_1, ..., l_{i-1}, l_{i+1}, ..., l_n \,.$$

Then

$$r_{k\,n-1}^n(R') = \{s_i\}_{i=1}^n$$

and

$$\varphi_{n-1\,1}^n(r_{k\,n-1}^n(R')) = \{s_i^\perp\}_{i=1}^n$$

is a maximal regular subset of $\mathbb{G}_1^n$. For any plane l belonging to R' consider the lines $l_{i_1}, ..., l_{i_k}$ generating it. It is easy to see that l is the intersection of the planes $s_{j_1}, ..., s_{j_{n-k}}$, where

$$\{j_1, ..., j_{n-k}\} = \{1, ..., n\} \setminus \{i_1, ..., i_k\}$$

Then the plane $l^\perp$ is generated by the lines $s_{j_1}^\perp, ..., s_{j_{n-k}}^\perp$. This implies that

$$\varphi_{k\,n-k}^n(R') = r_{1\,n-k}^n(\varphi_{n-1\,1}^n(r_{k\,n-1}^n(R')))$$

is a maximal regular subset of $\mathbb{G}_{n-k}^n$ containing $\varphi_{k\,n-k}^n(R)$. ■

Corollary 2.3.1. *For any $V \subset \mathbb{G}_k^n$ the set $\varphi_{k\,n-k}^n(V)$ is an irregular subset of $\mathbb{G}_{n-k}^n$ if and only the set V is irregular. Moreover $\varphi_{k\,n-k}^n(V)$ is a maximal irregular subset of $\mathbb{G}_{n-k}^n$ if and only if V is a maximal irregular set.*

Proof. Proposition 2.3.1 shows that V contains some maximal regular set if and only $\varphi^n_{k\,n-k}(V)$ contains a maximal regular subset of $\mathbb{G}^n_{n-k}$. Therefore $\varphi^n_{k\,n-k}(V)$ is an irregular set if and only if the set V is irregular.

The homeomorphism $\varphi^n_{k\,n-k}$ transfers any irregular subset of $\mathbb{G}^n_k$ containing V to an irregular subset of $\mathbb{G}^n_{n-k}$ containing $\varphi^n_{k\,n-k}(V)$ and $\varphi^n_{n-k\,k}$ transfers any irregular subset of $\mathbb{G}^n_{n-k}$ containing $\varphi^n_k(V)$ to an irregular set containing V. This implies that $\varphi^n_{k\,n-k}(V)$ is a maximal irregular subset of $\mathbb{G}^n_{n-k}$ if and only if V is a maximal irregular set. ■

2.3.2. For any plane $s \in \mathbb{G}^n_m$ define

$$\mathbb{G}^n_k(s) = \begin{cases} \{\, l \in \mathbb{G}^n_k \mid l \subset s \,\} & \text{if } m \geq k \,, \\ \{\, l \in \mathbb{G}^n_k \mid s \subset l \,\} & \text{if } m \leq k \,. \end{cases}$$

It is not difficult to see that the following equality

$$\varphi^n_{k\,n-k}(\mathbb{G}^n_k(s)) = \mathbb{G}^n_{n-k}(s^{\perp}) \tag{2.3.1}$$

holds true.

In the case $m \geq k$ there exists the natural homeomorphism

$$\varphi_s : \mathbb{G}^n_k(s) \to \mathbb{G}^m_k \,.$$

If $m \leq k$ then the maps

$$\varphi'_s = \varphi_{s^{\perp}} \varphi^n_{k\,n-k} : \mathbb{G}^n_k(s) \to \mathbb{G}^{n-m}_{n-k}$$

and

$$\varphi_s = \varphi^{n-m}_{n-k\,k-m} \varphi'_s : \mathbb{G}^n_k(s) \to \mathbb{G}^{n-m}_{k-m}$$

are homeomorphisms (in what follows we shall use only second homeomorphism).

Lemma 2.3.1. *For any $R \subset \mathbb{G}^n_k(s)$ the set $\varphi_s(R)$ is regular if and only if R is regular.*

Proof. In the case $m \geq k$ Lemma 2.3.1 is trivial. In the case $m \leq k$ it is a consequense of the results of Subsection 2.3.1. ■

Let R be a maximal regular subset of $\mathbb{G}^n_k$ and s be an m-dimensional coordinate plane for the coordinate system associated with R. Consider the set

$$R(s) = \mathbb{G}^n_k(s) \cap R \,.$$

The homeomorphism φ_s transfers it to a maximal regular set. Therefore

$$|R(s)| = \begin{cases} c_k^m & \text{if } m \geq k \ , \\ c_{k-m}^{n-m} & \text{if } m \leq k \ . \end{cases}$$

This implies that any regular set contained in $\mathbb{G}_k^n(s)$ has less than c_k^n elements. Therefore the set $\mathbb{G}_k^n(s)$ is irregular.

2.4 Examples

2.4.1. Maximal irregular subsets of $\mathbb{G}_k^n$ in the cases $k = 1, n-1$. In the previous section we considered the irregular set $\mathbb{G}_k^n(s)$, where s is some fixed plane belonging to $\mathbb{G}_m^n$. Lemma 2.2.1 implies that in the case $k = 1$ the irregular set $\mathbb{G}_k^n(s)$ is maximal for any plane $s \in \mathbb{G}_{n-k}^n$. Corollary 2.3.1 shows that the similar statement holds for the case $k = n-1$. Now we show that in these cases there are not other maximal irregular sets.

Proposition 2.4.1. *Let V be a maximal irregular subset of $\mathbb{G}_k^n$ and $k = 1$ or $k = n-1$. Then there exists a plane $s \in \mathbb{G}_{n-k}^n$ such that $V = \mathbb{G}_k^n(s)$.*

Proof. First of all consider the case $k = 1$. Fix a line l belonging to $\mathbb{G}_1^n \setminus V$. Lemma 2.2.1 implies the existence of a set

$$R = \{l_i\}_{i=1}^{n-1} \subset V$$

such that $R \cup \{l\}$ is a maximal regular subset of $\mathbb{G}_1^n$. Denote by s the $(n-1)$-dimensional plane generated by the lines $l_1, ..., l_{n-1}$. It is not difficult to see that the set V does not contain lines transverse to s. Then $V \subset \mathbb{G}_1^n(s)$. Recall that the set $\mathbb{G}_1^n(s)$ is irregular and V is a maximal irregular set. Therefore the inverse inclusion holds true.

In second case $\varphi_{n-1\,1}^n(V)$ is a maximal irregular subset of $\mathbb{G}_1^n$ and our statement is a consequence of the equation (2.3.1). ■

2.4.2. For any plane $s \in \mathbb{G}_m^n$ consider the sets

$$X_k^n(s) = \bigcup_{t \in \mathbb{G}_1^n(s)} \mathbb{G}_k^n(t)$$

and

$$Y_k^n(s) = \bigcup_{t \in \mathbb{G}_{n-1}^n(s)} \mathbb{G}_k^n(t) \ .$$

It is easy to see that

$$X_k^n(s) = \{\, l \in \mathbb{G}_k^n \mid \dim l \cap s \geq 1 \,\} \tag{2.4.1}$$

and

$$Y_k^n(s) = \varphi_{k\,n-k}^n(X_{n-k}^n(s^\perp)) \; . \tag{2.4.2}$$

Moreover the set $X_k^n(s)$ coincides with $\mathbb{G}_k^n(s)$ if $m = 1$ and the set $Y_k^n(s)$ coincides with $\mathbb{G}_k^n(s)$ if $m = n - 1$.

Proposition 2.4.2. *The following statemnts are fulfilled:*

(i) *if $m > n - k$ then $\mathbb{G}_k^n = X_k^n(s)$;*

(ii) *if $m < n - k$ then $\mathbb{G}_k^n = Y_k^n(s)$;*

(iii) *if $m \leq n - k$ then the set $X_k^n(s)$ is irregular;*

(iv) *if $m \geq n - k$ then the set $Y_k^n(s)$ is irregular;*

(v) *the irregular set $X_k^n(s)$ is maximal if and only if $m = n - k$; moreover in this case it coincides with the set $Y_k^n(s)$.*

2.4.3. Proof of Proposition 2.4.2. First of all note that the statements (ii), (iv) are consequences of the statements (i), (iii) and the equation (2.4.2). In the case $m > n - k$ the condition

$$\dim l \cap s \geq 1 \tag{2.4.3}$$

holds for any plane l belonging to $\mathbb{G}_k^n$ and the equation (2.4.1) implies the statement (i).

To prove the statements (iii) and (v) we exploit the following lemma.

Lemma 2.4.1. *For any coordinate system for $\mathbb{R}^n$ and any m-dimensional plane s passing through of the origin of the coordinates there exists an $(n-m)$-dimensional coordinate plane intersecting s only in the origin of the coordinates.*

Proof. In the case $m = 1$ this statement is trivial. Assume that it holds for any number m such that $m < m_0$ ($m_0 > 1$) and consider the case $m = m_0$. Let s' be an $(m-1)$-dimensional plane contained in s. The inductive hypothesis implies the existence of $(n-m+1)$-dimensional coordinate plane t' intersecting s' only in the origin of the coordinates. The intersection $s \cap t'$ is a line passing through of the origin of the coordinates . There exists an $(n-m)$-dimensional coordinate plane t contained in t' and intersecting this line only in the origin of the coordinates. It is not difficult to see that the plane t satisfies the required condition. ■

Proof of the statement (iii). Let R be a regular set contained in $X_k^n(s)$. Consider a coordinate system associated with it. It is not difficult to see that s is a coordinate plane for this system. Then Lemma 2.4.1 implies the existence of some $(n-m)$-dimensional coordinate plane t intersecting s only in the origin of the coordinates. In the case $m \leq n-k$ there exist $c_k^{n-m} \geq 1$ distinct k-dimensional coordinate planes contained in t. These planes intersect s only in the origin of the coordinates and $X_k^n(s)$ does not contain them. Therefore in this case the regular set R is not maximal and the set $X_k^n(s)$ is irregular. ■

Proof of the statement (v). It was proved above that in the case $m < n-k$ any regular set R contained in $X_k^n(s)$ setisfies the following condition

$$|R| \leq c_k^n - c_k^{n-m} < c_k^n - 1 \ .$$

Lemma 2.2.1 guarantees that any maximal irregular subset of $\mathbb{G}_k^n$ contains a regular set R such that

$$|R| = c_k^n - 1 \ .$$

This implies that in the case $m < n-k$ the irregular set $X_k^n(s)$ is not maximal.

Let $m = n-k$ and l' be a plane belonging to the set $\mathbb{G}_k^n \setminus X_k^n(s)$. This plane intersects s only in the origin of the coordinates and there exists a coordinate system for $\mathbb{R}^n$ such that l' and s are coordinate planes for it. Denote by R the set of all k-dimensional coordinate planes. The equality $m = n-k$ shows that the inequality (2.4.3) holds for each plane l belonging to the set $R \setminus \{l'\}$. Then the equation (2.4.1) implies the inclusion

$$R \setminus \{l'\} \subset X_k^n(s)$$

and Lemma 2.2.1 shows that in this case the irregular set $X_k^n(s)$ is maximal.

Any plane l belonging to $X_k^n(s)$ setisfies the condition (2.4.3). The equality $m = n-k$ implies the existence of an $(n-1)$-dimensional plane containing l and s. Therefore $l \in Y_k^n(s)$ and we have proved the inclusion $X_k^n(s) \subset Y_k^n(s)$.

Prove the inverse inclusion. For any plane l belonging to $Y_k^n(s)$ there exists an $(n-1)$-dimensional plane containing l and s. The equality $m = n-k$ guarantees that the inequality (2.4.3) holds. Then the required inclusion is a consequence of the equation (2.4.1). ■

2.5 How large may be an irregular set?

2.5.1. For any k-dimensional set $X \subset \mathbb{R}^n$ the set $I_k^n(X)$ of all "irregular projection" is an irregular subset of $\mathbb{G}_k^n$ (see Introduction). It seems to be natural to ask how large may be an irregular set? Now we discuss this quastion.

For the cases $k = 1, n-1$ our problem is trivial. Propositions 2.2.1 and 2.4.1 imply that in these cases any irregular subset of $\mathbb{G}_k^n$ is nowhere dense. For the case $1 < k < n-1$ the similar statement is not proved. The regular location of a collection of c_k^n distinct k-dimensional planes in $\mathbb{R}^n$ is not in the general position (see Subsection 2.5.3.). Now we prove some statement supporting our conjecture.

Theorem 2.5.1. *Any irregular set $V \subset \mathbb{G}_k^n$ has an empty interior in $\mathbb{G}_k^n$ and $\mathbb{G}_k^n \setminus V$ is an everywhere dense subset of $\mathbb{G}_k^n$.*

In Chapter 6 we consider one class of irregular subsets which elements are sets of first category.

2.5.2. Now we consider some subsets of the space

$$\mathbf{G}_k^n = \underbrace{\mathbb{G}_k^n \times ... \times \mathbb{G}_k^n}_{c_k^n} \ .$$

In the next subsection we exploit their properties to prove Theorem 2.5.1.

Let

$$\mathbf{R}_k^n = \{(l_1, ..., l_{c_k^n}) \in \mathbf{G}_k^n \mid \{l_i\}_{i=1}^{c_k^n} \in \mathfrak{M}\mathfrak{R}_k^n\}$$

and

$$\mathbf{I}_k^n = \mathbf{G}_k^n \setminus \mathbf{R}_k^n \ .$$

Now we prove one statement showing that in the case $1 < k < n-1$ the regular location of a collection of c_k^n distinct k-dimensional planes in $\mathbb{R}^n$ is not in the general position.

Proposition 2.5.1. *The next two statements hold true.*

(i) *if* $k = 1, n-1$ *then* $\mathbf{R}_k^n$ *is an open everywhere dense subset of* $\mathbf{G}_k^n$*;*

(ii) *if* $1 < k < n-1$ *then* $\overline{\mathbf{R}_k^n}$ *is a nowhere dense subset of* $\mathbf{G}_k^n$*.*

To prove Proposition 2.5.1 we exploit the following lemma.

Lemma 2.5.1. *A point*

$$\mathbf{l} = (l_1, ..., l_{c_k^n}) \in \mathbf{G}_k^n$$

is a limit point of the set $\mathbf{R}_k^n$ *if and only if there exists a collection of* n *lines belonging to* $\mathbb{G}_1^n$ *and such that the planes* $l_1, ..., l_{c_k^n}$ *are generated by lines from this collection.*

Proof. Let $\mathbf{l} \in \mathbf{R}_k^n$ and there exists a family

$$L^j = \{l_i^j\}_{i=1}^{c_k^n} \ \ j \in \mathbb{N}$$

of maximal regular subsets of $\mathbb{G}_k^n$ such that the sequence

$$\{(l_1^j, ..., l_{c_k^n}^j)\}_{j=1}^{\infty}$$

converges to $\mathbf{l}$. For any $j \in \mathbb{N}$ consider the maximal regular subset

$$C^j = \{c_i^j\}_{i=1}^n = r_{k\,1}^n(L^j)\ .$$

The space $\mathbf{G}_1^n$ is compact and there exists a sequence $\{j_m\}_{m=1}^{\infty}$ such that the sequences

$$\{c_1^{j_m}\}_{m=1}^{\infty}, ..., \{c_n^{j_m}\}_{m=1}^{\infty}$$

converge to some lines $c_1, ..., c_n$ respectively. It is easy to see that the planes $l_1, ..., l_{c_k^n}$ are generated by the lines $c_1, ..., c_n$.

Prove the inverse statement. Assume that the planes $l_1, ..., l_{c_k^n}$ are generated by some lines $c_1, ..., c_n$ belonging to $\mathbb{G}_1^n$. Consider a family

$$C^j = \{c_i^j\}_{i=1}^n \ \ (j \in \mathbb{N})$$

of maximal regular subsets of $\mathbb{G}_1^n$ such that the sequence $\{c_i^j\}_{j=1}^{\infty}$ converges to c_i for any $i = 1, ..., n$. For each $j \in \mathbb{N}$ define

$$L^j = \{l_i^j\}_{i=1}^{c_k^n} = r_{1,k}^n(C^j)\ .$$

Then the sequence

$$\{\mathbf{l}^j = (l_1^j, ..., l_{c_k^n}^j)\}_{j=1}^{\infty}$$

converges to $\mathbf{l}$. ■

Proof of Proposition 2.5.1. Lemma 2.5.1 implies that $\mathbf{R}_1^n$ is an everywhere dense subset of $\mathbf{G}_1^n$. It is not difficult to see that this set is open. The results of Section 1.4 show that the similar statement holds for the set $\mathbf{R}_{n-1}^n$.

Let $L = \{l_i\}_{i=1}^{c_k^n}$ be a maximal regular subset of $\mathbb{G}_k^n$ and

$$r_{k\,1}^n(L) = \{c_i\}_{i=1}^n \ .$$

Assume that the plane l_1 is generated by the lines $c_1, ..., c_k$. Consider a sequence $\{l_1^j\}_{j=1}^{\infty}$ of planes belonging to $\mathbb{G}_k^n$ and setisfying the following conditions:

(i) the sequence $\{l_1^j\}_{j=1}^{\infty}$ converges to l_1;

(ii) each l_1^j does not contain the lines $c_1, ..., c_k$.

Lemma 2.5.1 shows that

$$\mathbf{l}^j = (l_1^j, l_2, ..., l_{c_k^n}) \notin \overline{\mathbf{R}_k^n}$$

for any $j \in \mathbb{N}$ and the sequence $\{\mathbf{l}^j\}_{j=1}^{\infty}$ converges to

$$\mathbf{l} = (l_1, ..., l_{c_k^n}) \in \mathbf{R}_k^n \ .$$

This implies that $\mathbf{G}_k^n \setminus \overline{\mathbf{R}_k^n}$ is an everywhere dense subset of $\mathbf{G}_k^n$ and $\overline{\mathbf{R}_k^n}$ has an empty interior. ■

2.5.3. Proof of Theorem 2.5.1. For any irregular set $V \subset \mathbb{G}_k^n$ consider the set

$$\mathbf{V} = \underbrace{V \times ... \times V}_{c_k^n} \ .$$

In the cases $k = 1, n-1$ Proposition 2.6.1 shows that $\mathbf{I}_k^n$ is a nowhere dense subset of $\mathbf{G}_k^n$. Then the inclusion $\mathbf{V} \subset \mathbf{I}_k^n$ shows that $\mathbf{V}$ is a nowhere dense subset of $\mathbf{G}_k^n$ (the inclusion holds indeed the set V is irregular). This implies that V is a nowhere dense subset of $\mathbb{G}_k^n$.

In the case $1 < k < n-1$ the set $\mathbf{I}_k^n$ is not nowhere dense in $\mathbf{G}_k^n$ (see Proposition 2.5.1) and we can not prove the similar statement. We show that any irregular set $V \subset \mathbb{G}_k^n$ has an empty interior.

Denote by U the interior of the set V in $\mathbb{G}_k^n$. Then

$$\mathbf{U} = \underbrace{U \times ... \times U}_{c_k^n} \subset \mathbf{I}_k^n$$

is the interior of $\mathbf{V}$ in $\mathbf{G}_k^n$. Assume that the set U is not empty and consider the plane l belonging to it. Then $\mathbf{U}$ contains the point

$$\mathbf{l} = \underbrace{(l, ..., l)}_{c_k^n} .$$

There exist n lines $c_1, ..., c_n$ contained in the plane l and setisfying the following condition: for any set

$$I = \{i_1, ..., i_k\} \subset \{1, ..., n\}$$

the lines $c_{i_1}, ..., c_{i_k}$ generate the plane l. Lemma 2.5.1 shows that $\mathbf{l}$ is a limit point of the set $\mathbf{R}_k^n$. This implies that $\mathbf{l} \notin \mathbf{U}$; i.e. the sets $\mathbf{U}$ and $\mathbf{V}$ are empty. ■

Chapter 3

Regular maps of the Grassman manifolds

In the chapter we consider the group $\mathfrak{R}(\mathbb{G}_k^n)$ of all bijective maps of $\mathbb{G}_k^n$ onto $\mathbb{G}_k^n$ preserving the classes of regular and irregular sets. In what follows these maps will be called *regular*. Any non-singular linear map of $\mathbb{R}^n$ onto $\mathbb{R}^n$ induces a regular map of $\mathbb{G}_k^n$ onto $\mathbb{G}_k^n$ (these maps will be considered in Section 3.2). In the case $n = 2k$ there exists another example of a regular map; it is the homeomorphism $\varphi_{k\,k}^{2k} : \mathbb{G}_k^{2k} \to \mathbb{G}_k^{2k}$.

One of the main results of the book (Theorem 3.3.1) states that in the case $n \neq 2k$ each regular map of $\mathbb{G}_k^n$ onto $\mathbb{G}_k^n$ is induced by a non-singular linear map of $\mathbb{R}^n$ onto $\mathbb{R}^n$; if $n = 2k$ then the group $\mathfrak{R}(\mathbb{G}_k^n)$ is generated by maps induced by non-singular linear maps of $\mathbb{R}^n$ onto $\mathbb{R}^n$ and the homeomorphism $\varphi_{k\,k}^{2k}$. For the case $k = 1$ this statement is known as the Fundamental Theorem of Projective Geometry (Section 5.1). In the general case it was proved by author. The proof of Theorem 3.3.1 is sufficiently complicated and we give it in Chapter 5. In this chapter we consider some corollaries of Theorem 3.3.1 and introduce the notion of similar subsets of the Grassman manifolds.

3.1 Definition and trivial examples

3.1.1. Defimition.

Definition 3.1.6. A bijective map f of $\mathbb{G}_k^n$ onto $\mathbb{G}_k^n$ is called *regular* ($f \in \mathfrak{R}(\mathbb{G}_k^n)$) if it preserves the class of all regular sets; i. e. for any $R \subset \mathbb{G}_k^n$ the set $f(R)$ is regular if and only if R is a regular set.

Lemma 3.1.1. *Let f be a bijective map of $\mathbb{G}_k^n$ onto $\mathbb{G}_k^n$. Then the following conditions are equivalent:*

(i) *f is regular;*

(ii) *f preserves the class of all maximal regular sets;*

(iii) *f preserves the class of all irregular sets;*

(iv) *f preserves the class of all maximal irregular sets.*

Proof. The implication (i) $\Leftrightarrow$ (ii) is trivial. The proof of the implications (ii) $\Leftrightarrow$ (iii) and (iii) $\Leftrightarrow$ (iv) is similar to the proof of Corollary 2.3.1. $\blacksquare$

All regular maps of $\mathbb{G}_k^n$ onto $\mathbb{G}_k^n$ constitute a group. We denote it by $\mathfrak{R}(\mathbb{G}_k^n)$.

3.1.2. Examples. For any two lines l_1 and l_2 ($l_1 \neq l_2$) belonging to $\mathbb{G}_1^2$ the set $\{l_1, l_2\}$ is regular (moreover this regular set is maximal). Therefore any bijective map of $\mathbb{G}_1^2$ onto $\mathbb{G}_1^2$ is regular. In the general case the similar statement fails.

Proposition 2.3.1 shows that $\varphi_{k\,k}^{2k} : \mathbb{G}_k^{2k} \to \mathbb{G}_k^{2k}$ is a regular map. Moreover the canonical homeomorphism $\varphi_{k\,n-k}^n$ induces the isomorphism

$$\Phi_{k\,n-k}^n : \mathfrak{R}(\mathbb{G}_k^n) \to \mathfrak{R}(\mathbb{G}_{n-k}^n)$$

$$\Phi_{k\,n-k}^n(f) = \varphi_{k\,n-k}^n f \varphi_{n-k\,k}^n \quad \forall\, f \in \mathfrak{R}(\mathbb{G}_k^n)$$

between the groups $\mathfrak{R}(\mathbb{G}_k^n)$ and $\mathfrak{R}(\mathbb{G}_{n-k}^n)$.

Other examples of regular maps will be considered in the next section.

3.2 Linear maps

3.2.1. Linear maps of $\mathbb{R}^n$ onto $\mathbb{R}^n$. Let $\mathfrak{GL}(n)$ be the group of all nonsingular linear maps of $\mathbb{R}^n$ onto $\mathbb{R}^n$. It is easy to see that

$$\mathfrak{R}(n) = \{\ \alpha \mathrm{Id} \mid \alpha \in \mathbb{R}\ ,\ \alpha \neq 0\ \}$$

(where Id is the identity map) is a normal subgroup of $\mathfrak{GL}(n)$. Moreover

$$\mathfrak{GL}(n)/\mathfrak{R}(n) \approx \mathfrak{SL}(n) = \{\ l \in \mathfrak{GL}(n) \mid \det l = 1\ \} \text{ if } n \neq 2i \qquad (3.2.1)$$

and

$$\mathfrak{GL}(n)/\mathfrak{R}(n) \approx \mathfrak{SL}'(n) = \{\ l \in \mathfrak{GL}(n) \mid |\det l| = 1\ \} \text{ if } n = 2i\ . \qquad (3.2.2)$$

The groups $\mathfrak{SL}(n)$ and $\mathfrak{SL}'(n)$ are the kernels of the homomorphisms

$$\det \text{ and } |\det|$$

of $\mathfrak{GL}(n)$ into the multiplicative group $\mathbb{R}\setminus\{0\}$. This implies that $\mathfrak{SL}(n)$ and $\mathfrak{SL}'(n)$ are normal subgroups of $\mathfrak{GL}(n)$. An immediate verification shows that

$$\mathfrak{GL}(n)/\mathfrak{SL}(n) \approx \mathfrak{R}(n) \text{ if } n \neq 2i$$

and

$$\mathfrak{GL}(n)/\mathfrak{SL}'(n) \approx \mathfrak{R}(n) \text{ if } n = 2i\ .$$

3.2.2. Linear maps of $\mathbb{G}_k^n$ onto $\mathbb{G}_k^n$. Any non-singular linear map of $\mathbb{R}^n$ onto $\mathbb{R}^n$ induces a regular map of $\mathbb{G}_k^n$ onto $\mathbb{G}_k^n$.

Definition 3.2.7. A regular map of $\mathbb{G}_k^n$ onto $\mathbb{G}_k^n$ induced by a non-singular linear map will be called *linear*.

All linear maps of $\mathbb{G}_k^n$ onto $\mathbb{G}_k^n$ form a group. Denote it by $\mathfrak{L}(\mathbb{G}_k^n)$.

Consider the homomorphism

$$L_k^n : \mathfrak{GL}(n) \to \mathfrak{L}(\mathbb{G}_k^n)$$

transfering any non-singular linear map of $\mathbb{R}^n$ onto $\mathbb{R}^n$ in the linear map of $\mathbb{G}_k^n$ onto $\mathbb{G}_k^n$ induced by it. The image of this homomorphism coinsides with $\mathfrak{L}(\mathbb{G}_k^n)$ and

$$Ker L_k^n = \mathfrak{R}(n)\ .$$

Then

$$\mathfrak{L}(\mathbb{G}_k^n) \approx \mathfrak{GL}(n)/Ker L_k^n\ .$$

The equations (3.2.1) and (3.2.2) show that

$$\mathfrak{L}(\mathbb{G}_k^n) \approx \begin{cases} \mathfrak{SL}(n) \text{ if } n \neq 2i\ , \\ \mathfrak{SL}'(n) \text{ if } n = 2i\ . \end{cases}$$

Let f be a linear map of $\mathbb{G}_k^n$ onto $\mathbb{G}_k^n$. Consider a linear map f' belonging to the set $(L_k^n)^{-1}(f)$. For each $i = 1, ..., n-1$ define

$$L_{k\,i}^n(f) = L_i^n(f')\ .$$

For other linear map f'' belonging to $(L^n_k)^{-1}(f)$ we have

$$f'(f'')^{-1} \in Ker L^n_k = Ker L^n_i = \mathfrak{R}(n) \ .$$

Therefore

$$L^n_i(f') = L^n_i(f'')$$

and the map

$$L^n_{k\,i} : \mathfrak{L}(\mathbb{G}^n_k) \to \mathfrak{L}(\mathbb{G}^n_i)$$

is well-defined. It is not difficult to see that $L^n_{k\,i}$ is an isomorphism between the groups $\mathfrak{L}(\mathbb{G}^n_k)$ and $\mathfrak{L}(\mathbb{G}^n_i)$ (in the case $k = i$ this isomorphism is identity). An immediate verification shows us that

$$L^n_{k\,m} = L^n_{i\,m} L^n_{k\,i}$$

and

$$(L^n_{k\,m})^{-1} = L^n_{m\,k} \ .$$

3.2.3. Let us consider the regular map $\varphi^{2k}_{k\,k} : \mathbb{G}^{2k}_k \to \mathbb{G}^{2k}_k$. It is not difficult to see that

$$\varphi^{2k}_{k\,k}(\mathbb{G}^{2k}_k(s)) = \mathbb{G}^{2k}_k(s^{\perp}) \quad \forall\, s \in \mathbb{G}^{2k}_m \ . \tag{3.2.3}$$

For any linear map f of $\mathbb{G}^n_k$ onto $\mathbb{G}^n_k$ we have the following equality

$$f(\mathbb{G}^n_k(s)) = \mathbb{G}^n_k(L^n_{k\,m}(f)(s)) \quad \forall\, s \in \mathbb{G}^n_m \ . \tag{3.2.4}$$

This implies that the regular map $\varphi^{2k}_{k\,k}$ is not linear.

3.3 Structure of the group $\mathfrak{R}(\mathbb{G}^n_k)$

3.3.1. The following statement is one of the main results of the book.

Theorem 3.3.1. *If $n \geq 3$ and $n \neq 2k$ then*

$$\mathfrak{R}(\mathbb{G}^n_k) = \mathfrak{L}(\mathbb{G}^n_k) \ .$$

The group $\mathfrak{R}(\mathbb{G}^{2k}_k)$ $(k \geq 2)$ is generated by the group $\mathfrak{L}(\mathbb{G}^{2k}_k)$ and the regular map $\varphi^{2k}_{k\,k}$.

Remark 3.3.1. The theorem states that for any regular map f of $\mathbb{G}_k^{2k}$ onto $\mathbb{G}_k^{2k}$ we have the next two cases:

(i) the map f is linear;

(ii) there exists a linear map f' of $\mathbb{G}_k^{2k}$ onto $\mathbb{G}_k^{2k}$ such that one of the the following equalities

$$f = \varphi_{k\,k}^{2k} f' \text{ or } f = f' \varphi_{k\,k}^{2k}$$

holds true.

Theorem 3.3.1 will be proved in Chapter 5.

3.3.2. Remarks to the structure of the group $\mathbb{G}_k^{2k}$. The equations (3.2.3) and (3.2.4) show that for any linear map f of $\mathbb{G}_k^{2k}$ onto $\mathbb{G}_k^{2k}$ the map $\varphi_{k\,k}^{2k} f \varphi_{k\,k}^{2k}$ is linear. This implies the following statement.

Proposition 3.3.1. *The group $\mathfrak{L}(\mathbb{G}_k^{2k})$ is normal subgroup of $\mathfrak{R}(\mathbb{G}_k^{2k})$ and*

$$\mathfrak{R}(\mathbb{G}_k^{2k})/\mathfrak{L}(\mathbb{G}_k^{2k}) \approx \{Id,\ \varphi_{k\,k}^{2k}\}\ .$$

Note that the subgroup $\{Id,\ \varphi_{k\,k}^{2k}\}$ is not normal. There exist linear maps f of $\mathbb{G}_k^{2k}$ onto $\mathbb{G}_k^{2k}$ such that the regular maps $f\varphi_{k\,k}^{2k} f$ and $\varphi_{k\,k}^{2k}$ do not coincide.

3.4 Similar subsets of the Grassman manifolds

3.4.1.

Definition 3.4.8. We say that two subsets V_1 and V_2 of $\mathbb{G}_k^n$ are *similar* if there exists a regular map f of $\mathbb{G}_k^n$ onto $\mathbb{G}_k^n$ such that $f(V_1) = V_2$.

Lemma 3.4.1. *Subsets V_1 and V_2 of $\mathbb{G}_k^n$ are similar if and only if the sets $\varphi_{k\,n-k}^n(V_1)$ and $\varphi_{k\,n-k}^n(V_2)$ are similar.*

Proof. An immediate verification shows that the equality $f(V_1) = V_2$ holds for some regular map f of $\mathbb{G}_k^n$ onto $\mathbb{G}_k^n$ if and only if

$$\Phi_{k\,n-k}^n(f)(\varphi_{k\,n-k}^n(V_1)) = \varphi_{k\,n-k}^n(V_2)\ .$$

Note that $\Phi^n_{k\,n-k}(f)$ is a regular map of $\mathbb{G}^n_{n-k}$ onto $\mathbb{G}^n_{n-k}$. ■

If one of two similar subsets of $\mathbb{G}^n_k$ is a regular set or a maximal regular set then other set is a regular set or a maximal regular set respectively. Any two maximal regular subsets of $\mathbb{G}^n_k$ are similar. Regular subsets of $\mathbb{G}^n_1$ containing the same numbers of elements are similar. Lemma 3.4.1 guarantees that the analogous statement holds for regular subsets of $\mathbb{G}^n_{n-1}$. In Chapter 4 we show that for the case $1 < k < n-1$ this fails.

If one of two similar subsets of $\mathbb{G}^n_k$ is an irregular set or a maximal irregular set then other set is an irregular set or a maximal irregular set respectively. Proposition 2.4.1 show that any two maximal irregular subsets of $\mathbb{G}^n_k$ ($k = 1, n-1$) are similar. In the general case this statement does not hold.

Theorem 3.4.1. *If $1 < k < n-1$ then there exist maximal irregular subsets of $\mathbb{G}^n_k$ which are not similar.*

Theorem 3.4.1 will be proved in Chapter 6. There we construct a maximal irregular subset of $\mathbb{G}^n_k$ which are not similar to $X^n_k(s)$ ($s \in \mathbb{G}^n_{n-k}$).

Remark 3.4.1. Any linear map of $\mathbb{G}^n_k$ onto $\mathbb{G}^n_k$ transfers $X^n_k(s)$ ($s \in \mathbb{G}^n_{n-k}$) to $X^n_k(s')$ ($s' \in \mathbb{G}^n_{n-k}$). The homeomorphism $\varphi^{2k}_{k\,k}$ transfers $X^{2k}_k(s)$ ($s \in \mathbb{G}^{2k}_k$) to

$$Y^{2k}_k(s^\perp) = X^{2k}_k(s^\perp)$$

(see Proposition 2.4.2). This implies that any two maximal irregular sets $X^n_k(s_1)$ and $X^n_k(s_2)$ ($s_1 \in \mathbb{G}^n_{n-k}$, $s_2 \in \mathbb{G}^n_{n-k}$) are similar. Moreover for any set $V \subset \mathbb{G}^n_k$ which is similar to $X^n_k(s)$ ($s \in \mathbb{G}^n_{n-k}$) there exists $s' \in \mathbb{G}^n_{n-k}$ such that $V = X^n_k(s')$.

Chapter 4

Degree of inexactness of regular sets

The chapter is devoted to study some combinatorical properties of regular subsets of the Grassmannian manifolds. We introduce one number characteristic of regular sets (so-called the degree of inexactness and use it to ivestigate the structure of regular sets having a lot of elements). In the next chapter these results will be exploited to prove of Theorem 3.3.1.

4.1 Definitions and examples

4.1.1. A regular set $R \subset \mathbb{G}_k^n$ is called *exact* ($R \in \mathfrak{ER}_k^n$) if there exists unique maximal regular set containing it. In other words the regular set R is exact if planes belonging to it are coordinates planes only for one coordinate system; i.e. a coordinate system associated with R is uniquely defined.

For any regular set $R \subset \mathbb{G}_k^n$ define

$$\deg(R) = \min_{\hat{R} \in \mathfrak{ER}_k^n \; R \subset \hat{R}} \{|\hat{R}| - |R|\} \, .$$

The number $\deg(R)$ is called the *degree of inexactness* of the regular set R. It is trivial that the equality $\deg(R) = 0$ holds if and only if the regular set R is exact.

Proposition 4.1.1. *The canonical homeomorphism $\varphi_{k\,n-k}^n$ and any regular map f of $\mathbb{G}_k^n$ onto $\mathbb{G}_k^n$ preserve the degree of inexactness; i.e. for any regular* *set* $R \subset \mathbb{G}_k^n$ *we have*

$$\deg(\varphi^n_{k\,n-k}(R)) = \deg(R)$$

and

$$\deg(f(R)) = \deg(R)\ .$$

Proof. The regular map f transfers any maximal regular set containing R to a maximal regular set containing $f(R)$ and the regular map f^{-1} transfers any maximal regular set containing $f(R)$ to a maximal regular set containing R. Therefore f transfers any exact regular set containing R to an exact regular set containing $f(R)$ and f^{-1} transfers any exact regular set containing $f(R)$ to an exact regular set containing R. This implies second equality. The proof of first equality is similar. ∎

Proposition 4.1.1 shows that the equality $\deg(R_1) = \deg(R_2)$ holds for any similar regular sets R_1 and R_2.

4.1.2. Examples I. Clearly, a maximal regular set is exact. In the case $k = 1$ for any regular set $R \subset \mathbb{G}_k^n$ we have $\deg(R) = n - |R|$ and the regular set R is exact if and only if R is a maximal regular set. Proposition 4.1.1 implies that the similar statement holds for the case $k = n - 1$.

The general case is more complicated. This statement does not holds. In the next section we find a number s_k^n such that $s_k^n < c_k^n$ and any regular set $R \subset \mathbb{G}_k^n$ $(1 < k < n - 1)$ containing greater than s_k^n elements is exact.

There are other examples. Consider a coordinate system $\{x_i\}_{i=1}^n$ for $\mathbb{R}^n$. Denote by l_i $(i = 1, ..., n-1)$ the plane generated by x_i, x_{i+1} and denote by l_n the plane generated by x_n, x_1. Then $\{l_n\}_{i=1}^n$ is an exact regular subset of $\mathbb{G}_2^n$ which is not maximal.

4.1.3. Examples II. In this subsection we consider examples of regular sets which are not exact.

Let R be a maximal regular subset of $\mathbb{G}_k^n$ $(1 < k < n - 1)$ and s be an m-dimensional coordinate plane for the coordinate system associated with R. Recall that

$$R(s) = R \cap \mathbb{G}_k^n(s)$$

and the homeomorphism φ_s transfers this set to a maximal regular subset of $\mathbb{G}_k^m$ $(m \geq k)$ or $\mathbb{G}_{k-m}^{n-m}$ $(m \leq k)$. Therefore

$$|R(s)| = \begin{cases} c_k^m & m \geq k \\ c_{k-m}^{n-m} & m \leq k \end{cases}$$

(see Subsection 2.3.2). In what follows we show that in the case when $k \geq n-k$ and $m = 1$

$$\deg(R(s)) = 2\ . \tag{4.1.1}$$

Proposition 4.1.1 and the equation (2.3.1) implies that the equality (4.1.1) holds for the case when $k \leq n-k$ and $m = n-1$.

Let s_1 and s_2 be an $(n-1)$-dimensional and a two-dimensional coordinate planes for the coordinate system associated with R (here R is the maximal regular set introduced above). It is not defficul to see that if s_1 does not contain s_2 then

$$R(s_1) \cap R(s_2) = \emptyset$$

and

$$|R(s_1) \cup R(s_2)| = |R(s_1)| + |R(s_2)| = c_k^{n-1} + c_{k-2}^{n-2}\ .$$

In what follows we prove that

$$\deg(R(s_1) \cup R(s_2)) = 1\ .$$

4.2 Degree of inexactness for regular sets containing a lot of elements

Here we show that a regular set containing a lot of elements has a small degree of inexactness. Next we find a number s_k^n such that any regular subset of $\mathbb{G}_k^n$ containing greater than s_k^n elements is exact.

Theorem 4.2.1. *For any regular set $R' \subset \mathbb{G}_k^n$ the following statements are fulfilled:*

(i) *if $n-k < k < n-1$ and R' contains not less than c_{k-1}^{n-1} elements then* $\deg(R') \leq 2$ *and the equality* $\deg(R') = 2$ *holds if and only if there exist a maximal regular set R and line $s \in \mathbb{G}_1^n$* (which is a coordinate axis for the coordinate system associated with R) *such that $R' = R(s)$;*

(ii) *if $n = 2k$ and R' contains not less than $c_k^{n-1} = c_{k-1}^{n-1}$* (see Remark 4.2.1) *elements then* $\deg(R') \leq 2$ *and the equality* $\deg(R') = 2$ *holds if and only if there exist a maximal regular set R and a plane $s \in \mathbb{G}_m^n$* ($m = 1$ or $n-1$ and s is a coordinate plane for the coordinate system associated with R) *such that $R' = R(s)$;*

(iii) *if* $1 < k < n-k$ *and* R' *contains not less than* c_k^{n-1} *elements then* $\deg(R') \le 2$ *and the equality* $\deg(R') = 2$ *holds if and only if there exist a maximal regular set* R *and a plane* $s \in \mathbb{G}_{n-1}^n$ (which is a coordinate plane for the coordinate system associated with R) *such that* $R' = R(s)$.

Remark 4.2.1. The equalities

$$c_k^{n-1} = \frac{n!}{(k)!(n-k-1)!} = \frac{n!(n-k)}{k!(n-k)!}$$

and

$$c_{(n-k)-1}^{n-1} = \frac{n!}{((n-k)-1)!(n-k)!} = \frac{n!(n-k)}{k!(n-k)!}$$

show that

$$c_k^{n-1} = c_{(n-k)-1}^{n-1} . \tag{4.2.1}$$

This implies that

$$c_k^{n-1} = c_{k-1}^{n-1} \text{ if and only if } n = 2k .$$

The equations (2.3.1) implies that for any maximal regular set $R \subset \mathbb{G}_k^n$ and any plane s which is a coordinate plane for the coordinate system associated with R we have

$$\varphi_{k\,n-k}^n(R(s)) = R'(s^{\perp})$$

where $R' = \varphi_{k\,n-k}^n(R)$. Therefore the statement (iii) is a consequence of Proposition 4.1.1, the equation (4.2.1) and the statement (i).

Corollary 4.2.1. *For any regular set* $R' \subset \mathbb{G}_k^n$ *the following two statements hold true:*

— *if* $n-k \le k < n-1$ *and* R' *contains greater than* c_{k-1}^{n-1} *elements then* $\deg(R') \le 1$*;*

— *if* $1 < k \le n-k$ *and* R' *contains greater than* c_k^{n-1} *elements then* $\deg(R') \le 1$.

Define

$$s_k^n = c_k^{n-1} + c_{k-2}^{n-2} .$$

Then we have the following statement.

Theorem 4.2.2. *If a regular set $R' \subset \mathbb{G}_k^n$ contains not less than s_k^n elements then $\deg(R') \le 1$ and the equality $\deg(R') = 1$ holds if and only if there exist a maximal regular set R, planes $s_1 \in \mathbb{G}_{n-1}^n$ and $s_2 \in \mathbb{G}_2^n$ (which are coordinate planes for the coordinate system associated with R) such that s_1 does not contain s_2 and*

$$R' = R(s_1) \cup R(s_2) \ .$$

Corollary 4.2.2. *Any regular subset of $\mathbb{G}_k^n$ containing greater than s_k^n elements is exact.*

Let us find the number $c_k^n - s_k^n$. The well–known equality

$$c_j^i = c_{j-1}^{i-1} + c_j^{i-1} \tag{4.2.2}$$

shows that

$$c_k^n = c_{k-1}^{n-1} + c_k^{n-1} = c_{k-1}^{n-2} + c_{k-2}^{n-2} + c_k^{n-1} = s_k^n + c_{k-1}^{n-2} \ .$$

This implies that

$$s_k^n = c_k^n - c_{k-1}^{n-2} \ .$$

Therefore we could obtain an exact regular subset of $\mathbb{G}_k^n$ by removing any $c_{k-1}^{n-2} - 1$ planes from a maximal regular set.

Remark 4.2.2. The equation (4.2.2) could be proved by an immediate verification. We propose other method. Consider a coordinate system $\{x_i\}_{i=1}^n$ associated with a maximal regular set $R \subset \mathbb{G}_k^n$. Denote by s_i the $(n-1)$-dimensional coordinate plane transverse to the axis x_i. Then

$$R = R(x_i) \cup R(s_i)$$

and

$$R(x_i) \cap R(s_i) = \emptyset \ .$$

This implies that

$$c_k^n = |R| = |R(x_i)| + |R(s_i)| = c_{k-1}^{n-1} + c_k^{n-1}$$

and we have the required equality.

4.3 Proof of Theorems 4.2.1 and 4.2.2

4.3.1. Notation. In this subsection we introduce a notation which will be exploited to prove Theorems 4.2.1 and 4.2.2.

Let R'' be an exact regular subset of $\mathbb{G}_k^n$ containing R' and setisfying the following condition

$$\deg(R') = |R''| - |R'| \ .$$

Denote by R the maximal regular subset of $\mathbb{G}_k^n$ containing the exact regular set R''. It is trivial that the set R is unequely defined.

Let $\{x_i\}_{i=1}^n$ be the coordinate system associated with R. For any $i = 1, ..., n$ consider the set

$$R_i = R' \cap R(x_i)$$

and the plane

$$s_i = \bigcap_{l \in R_i} l \ .$$

An immediate verification shows that

$$R_i = R' \cap R(s_i) \ .$$

Define

$$n_i = \begin{cases} \dim s_i & R_i \neq \emptyset \\ 0 & R_i = \emptyset; . \end{cases}$$

It is easy to see that the regular set R' is exact if and only if $n_i = 1$ for any $i = 1, ..., n$.

4.3.2. Lemmas. The subsection is devoted to prove a few lemmas which will be used to prove Theorem 4.2.1. We consider the case when $n - k < k < n - 1$ and $|R| \geq c_{k-1}^{n-1}$.

Lemma 4.3.1. *If $n - k < k < n - 1$ then $n_i > 0$ for any $i = 1, ..., n$. The condition $n_i = 0$ holds for some number i if and only if $n = 2k$ and there exists $(n-1)$-dimensional coordinate plane s such that $R' = R(s)$.*

Proof. Assume that there exists a number i such that $n_i = 0$; i.e. the set R_i is empty. Then any plane belonging to R' is contained in the $(n-1)$-dimensional coordinate plane s transverse to the axis x_i. In other words $R' \subset R(s)$. This implies that

$$c_{k-1}^{n-1} \leq |R'| \leq |R(s)| = c_k^{n-1} \ . \tag{4.3.1}$$

In Remark 4.2.1 we have proved that the inequality (4.3.1) holds if and only if $n = 2k$. Moreover in this case we have

$$c_{k-1}^{n-1} = c_k^{n-1} \ .$$

Then (4.3.1) and the inclusion $R' \subset R(s)$ show that $R' = R(s)$. ■

Remark 4.3.1. We have proved above that if $n_i = 0$ for some number i then $R' = R(s)$, where s is the $(n-1)$-dimensional coordinate plane transverse to the axis x_i. It is easy to see that $\varphi_s(R' = R(s))$ is a maximal regular subset of $\mathbb{G}_k^{n-1}$. This implies that $n_j = 1$ for any $j \neq i$.

Lemma 4.3.2. *If $n - k \leq k < n - 1$ then the inequality $n_i \leq n - k$ holds for any $i = 1, ..., n$.*

Remark 4.3.2. In the general case the inequality $n_i < n - k$ does not hold. Consider, for example, the set

$$R' = R(s) \cup \{l\} \ ,$$

where $n - k = k$, s is the $(n-1)$-dimensional coordinate plane transverse to some coordinate axis x_i and l is a plane belonging to R and containing the axis x_i. It is not difficult to see that $R_i = \{l\}$ and $n_i = n - k$.

Proof. The case $n_i = 0$ is trivial. In the case $n_i > 0$ there exists a plane l belonging to R' and containing the axis x_i. Therefore $n_i \leq k$ and for the case $n - k = k$ our statement is proved.

In the case $n - k < k$ denote by s the $(n-1)$-dimensional coordinate plane transverse to the axis x_i and consider a plane l belonging to R'. If $l \in R_i$ then $l \in R(s_i)$. If $l \notin R_i$ then the plane l does not contain the axis x_i; i.e. l is contained in the plane s and we have $l \in R(s)$. These arguments show that

$$R' \subset R(s_i) \cup R(s) \ .$$

Then the equality

$$R(s_i) \cap R(s) = \emptyset$$

implies that

$$c_{k-1}^{n-1} \leq |R'| \leq |R(s_i)| + |R(s)| = c_{k-n_i}^{n-n_i} + c_k^{n-1} \ . \qquad (4.3.2)$$

An immediate verification shows us that for any two number k_1 and k_2 such that $k_1 \leq k_2 \leq k$ we have

$$c_{k-k_1}^{n-k_1} \geq c_{k-k_2}^{n-k_2} \, .$$

Therefore if $n_i \geq n-k+1$ then

$$c_{k-n_i}^{n-n_i} \leq c_{2k-n-1}^{k-1}$$

and (4.3.2) implies that

$$c_{k-1}^{n-1} - c_k^{n-1} \leq c_{2k-n-1}^{k-1} \, . \tag{4.3.3}$$

Prove that the inequality (4.3.3) fails. We have

$$c_{k-1}^{n-1} - c_k^{n-1} = \frac{(n-1)!}{(k-1)!(n-k)!} - \frac{(n-1)!}{k!(n-k-1)!} = \frac{(n-1)!(2k-n)}{k!(n-k)!} =$$

$$(2k-n)\underbrace{k(k+1)...(n-2)(n-1)}_{n-k}\frac{(k-1)!}{k!(n-k)!}$$

and

$$c_{2k-n-1}^{k-1} = \frac{(k-1)!}{(2k-n-1)!(n-k)!} =$$

$$(2k-n)\underbrace{(2k-n+1)...(k-1)k}_{n-k}\frac{(k-1)!}{k!(n-k)!} \, .$$

The condition $n-k < k < n-1$ shows that

$$(2k-n)\underbrace{k(k+1)...(n-2)(n-1)}_{n-k} \geq (2k-n)\underbrace{(2k-n+1)...(k-1)k}_{n-k} \, .$$

and the equations (4.3.3), (4.3.2) do not hold. Therefore $n_i \leq n-k$. ■

Lemma 4.3.3. *If $n-k \leq k < n-1$ and there exists a number i setisfying the condition $n_i \geq 3$ then $n_j = 1$ for any number j such that $j \neq i$.*

Proof. Denote by s' and s'' the $(n-1)$-dimensional coordinate plane transverse to the axis x_i and the $(n-2)$-dimensional coordinate plane transverse to the plane generated by the axes x_i, x_j. Consider a plane l belonging to the set R'. We have the following three cases.

(i). If $l \in R_i$ then $l \in R(s_i)$.

(ii). If $l \notin R_i$ and $l \in R_j$ then the plane l does not contain the axis x_i and $l \in R(s')$. Then the condition $l \in R_j$ implies that

$$l \in R(s') \cap R(s_j) \ .$$

(iii). If $l \notin R_i$ and $l \notin R_j$ then l does not contain the axes x_i and x_j. Therefore l is contained in the plane s''; i.e. $l \in R(s'')$.

This implies that

$$R' \subset R(s_i) \cup (R(s') \cap R(s_j)) \cup R(s'') \ .$$

The sets

$$R(s_i) \ , \ (R(s') \cap R(s_j)) \ , \ R(s'')$$

are mutually disjoint. Therefore

$$|R'| \leq |R(s_i)| + |R(s') \cap R(s_j)| + |R(s'')| \ . \tag{4.3.4}$$

Note that if the plane s_j contains the axis x_i (i.e. $R_j \subset R_i$) then

$$R(s') \cap R(s_j) = \emptyset \ .$$

Otherwise s_j is contained in the plane s' and the homeomorphism $\varphi_{s'}$ transfers this set in the intersection of some maximal irregular subset of $\mathbb{G}_k^{n-1}$ with $\mathbb{G}_k^{n-1}(\varphi_{s'}(s_j))$; this implies that

$$|R(s') \cap R(s_j)| = c_{k-n_j}^{n-n_j-1} \ .$$

The equation (4.3.4) shows that

$$c_{k-1}^{n-1} \leq c_{k-n_i}^{n-n_i} + c_{k-n_j}^{n-n_j-1} + c_k^{n-2} \ . \tag{4.3.5}$$

Assume that $n_i \geq 3$ and $n_j \geq 2$. Then

$$c_{k-n_i}^{n-n_i} \leq c_{k-3}^{n-3}$$

and

$$c_{k-n_j}^{n-n_j-1} \leq c_{k-2}^{n-3}$$

(see the proof of Lemma 4.3.2). Then (4.3.5) implies the inequality

$$c_{k-1}^{n-1} \leq c_{k-3}^{n-3} + c_{k-2}^{n-3} + c_{k}^{n-2} \ .$$

The equality (4.2.2) shows that

$$c_{k-1}^{n-1} = c_{k-1}^{n-2} + c_{k-2}^{n-2}$$

and

$$c_{k-2}^{n-2} = c_{k-3}^{n-3} + c_{k-2}^{n-3} \ .$$

Then the last inequality could be rewrited in the following form

$$c_{k-1}^{n-2} \leq c_{k}^{n-2} \ . \qquad (4.3.6)$$

We have

$$c_{k-1}^{n-2} = \frac{(n-2)!}{(k-1)!(n-k-1)!} = \frac{(n-2)!k}{k!(n-k-1)!}$$

and

$$c_{k}^{n-2} = \frac{(n-2)!}{k!(n-k-2)!} = \frac{(n-2)!(n-k-1)}{k!(n-k-1)!} \ .$$

The condition $n-k \leq k$ and the two last equalities imply that the inequalities (4.3.6) and (4.3.5) fail. The condition $n_i \geq 3$ guarantees the fulfillement of the inequality $n_j < 1$. Remark 4.3.1 shows that $n_j > 0$ and we have the required equality. ■

Lemma 4.3.4. *Let $n-k \leq k < n-1$ and the condition $n_i = 2$ holds for some number i. Then there exists unique axis x_j $(j \neq i)$ lying in the plane s_i and such that $n_j = 1$.*

Proof. The condition $n_i = 2$ guarantees the existence of unique axis x_j $(j \neq i)$ lying in the plane s_i. Assume that $n_j \geq 2$. Then Lemma 4.3.3 shows that the inequality $n_j \geq 3$ does not hold. Therefore $n_j = 2$ and the planes s_i, s_j coinside.

Denote by s is the $(n-2)$-dimensional coordinate plane transverse to s_i and consider a plane l belonging to the set R'. If $l \in R_i = R_j$ then $l \in R(s_i)$. In the case when $l \notin R_i = R_j$ the plane l does not contain the axes x_i, x_j and we have i.e. $l \in R(s)$. This implies that

$$R' \subset R(s) \cup R(s_i) \ .$$

It is easy to see that
$$R(s) \cap R(s_i) = \emptyset \ .$$
Then
$$c_{k-1}^{n-1} \le |R'| \le |R(s)| + |R(s_i)| = c_k^{n-2} + c_{k-2}^{n-2}$$
and (4.2.2) shows that
$$c_{k-1}^{n-1} = c_{k-1}^{n-2} + c_{k-2}^{n-2} \le c_k^{n-2} + c_{k-2}^{n-2} \ .$$
Therefore
$$c_{k-1}^{n-2} \le c_k^{n-2} \ .$$
We have proved above (see the proof of Lemma 4.3.3) that in the case $n-k \le k$ the last inequality does not hold. Then the inequality $n_j \ge 2$ fails. The set R_j is not empty and $n_j > 0$; i.e. $n_j = 1$. ■

Denote by $n(R')$ the number of all $i = 1, ..., n$ setisfying the condition $n_i = 1$. Then the following statement is fulfilled.

Lemma 4.3.5. *If $n-k \le k \le n-1$ and $0 < n_i \le 2$ for any $i = 1, ..., n$ then*
$$n(R') \ge k \ \text{or} \ n(R') = 1$$
and the condition $n(R') = 1$ holds if and only if there exists a coordinate axis x_i such that $R' = R(x_i)$.

Remark 4.3.3. It is trivial that if $R' = R(x_i)$ then $n_i = 1$. For any j such that $j \ne i$ the plane s_j is generated by the axes x_i and x_j; i.e. $n_j = 2$. Therefore $n(R') = 1$.

4.3.3. Proof of Lemma 3.3.5. Fix a number i such that $n_i = 2$ and denote by s the $(n-1)$-dimensional coordinate plane transverse to the axis x_i. Then the inclusion
$$R' \subset R(s_i) \cup R(s) \qquad (4.3.7)$$
and the equality
$$R(s_i) \cap R(s) = \emptyset \qquad (4.3.8)$$
show that
$$|R' \cap R(s)| \ge |R'| - |R(s_i)| = c_{k-1}^{n-1} - c_{k-2}^{n-2} = c_{k-1}^{n-2}$$

(see the proof of Lemma 4.3.2). Therefore

$$R'_s = \varphi_s(R' \cap R(s))$$

is a regular subset of $\mathbb{G}_k^{n-1}$ containing not less than $c_{k-1}^{(n-1)-1}$ elements (recall that φ_s is the natural homeomorphism of $\mathbb{G}_k^n(s)$ onto $\mathbb{G}_k^{n-1}$).

Lemma 4.3.6. *If the set R'_s contains c_{k-1}^{n-2} elements then $R_i = R(s_i)$.*

Proof. The equations (4.3.7) and (4.3.8) show that

$$R_i = R' \cap R(s_i) = R' \setminus (R' \cap R(s)) \ .$$

Then

$$|R_i| = |R'| - |R'_s| \geq c_{k-1}^{n-1} - c_{k-1}^{n-2} = c_{k-2}^{n-2} \ . \tag{4.3.9}$$

The condition $n_i = 2$ implies that

$$|R(s_i)| = c_{k-2}^{n-2}$$

and the required equality is a consequence of the equation (4.3.9) and the trivial inclusion $R_i \subset R(s_i)$. ■

The set

$$R_s = \varphi_s(R(s))$$

is a maximal regular subset of $\mathbb{G}_k^{n-1}$ containing R'_s and $\{x_j\}_{j=1,j\neq i}^n$ is the coordinate system for $\mathbb{R}^{n-1}$ associated with it. Let s' be a coordinate plane for this system. Define

$$R_s(s') = \mathbb{G}_k^{n-1}(s') \cap R_s \ .$$

For any number j such that $j \neq i$ denote by R_s^j the set of all planes belonging to R'_s and containing the coordinate axis x_j. Let

$$s'_j = \bigcap_{l \in R_s^j} l$$

and

$$n'_j = \begin{cases} \dim s'_j & \text{if } R_s^j \neq \emptyset \\ 0 & \text{if } R_s^j = \emptyset \ . \end{cases}$$

Then

$$R_s^j = R_s \cap R_s(s'_j) \ .$$

Denote by $n(R'_s)$ the number of all j ($j \neq i$) such that $n'_j = 1$. It is not difficult to see that $s_j \subset s'_j$ for any j such that $j \neq i$. Therefore $n(R') \geq n(R'_s)$.

Define $m = n - k$ and prove Lemma 4.3.5 for $m = 1$. In this case $c_{k-1}^{n-1} = n - 1$ and R' is a regular subset of $\mathbb{G}_{n-1}^n$ containing not less than $n - 1$ elements. If R' contains n elements then it is a maximal regular set and $n(R') = n$. If R' contains $n - 1$ elements then there exists a plane $l \in R$ such that $R' = R \setminus \{l\}$. In other words R' is the set of all $(n-1)$-dimensional coordinate planes containing the coordinate axis transverse to l and $n(R') = 1$ (see Remark 4.3.3).

Assume that our statement holds for any m such that $m < m_0$ ($m_0 > 1$) and consider the case $m = m_0$. Consider the set R'_s. We have the following two cases:

(i) there exists a number j' such that $n'_{j'} \neq 1, 2$;

(ii) $0 < n'_j \leq 2$ for any number j such that $j \neq i$.

Lemma 4.3.3 and Remark 4.3.1 show that in the case (i) the equality $n'_j = 1$ holds for any j such that $j \neq i, j'$. In our case $m = n - k \geq 2$. Therefore

$$n(R'_s) = n - 2 \geq k$$

and

$$n(R') \geq n(R'_s) \geq k \ .$$

If the set R' satisties the condition (ii) then the inductive hypothesis guarantees that we have the next two cases:

(i') $n(R'_s) \geq k$;

(ii') $n(R'_s) = 1$ and there exists a number j' such that $R'_s = R_s(x_{j'})$.

The case (i') is trivial indeed $n(R') \geq n(R'_s)$.

In the case (ii') consider unique coordinate axis $x_{j''}$ ($j'' \neq i$) lying in the plane s_i. Lemma 4.3.4 shows that $n_{j''} = 1$. Lemma 4.3.6 and the equation

$$|R'_s = R_s(x_{j'})| = c_{k-1}^{n-2}$$

imply that $R_i = R(s_i)$. Then

$$R' = (R' \cap R(s)) \cup R_i = (R(s) \cap R(x_{j'})) \cup R(s_i) \ . \qquad (4.3.10)$$

Show that $R' = R(x_{j'})$ if $j' = j''$. Consider a plane l belonging to $R(x_{j'})$. If $l \notin R(s_i)$ then l does not contain the axis x_i and $l \in R(s)$. Therefore

$$l \in R(s) \cap R(x_{j''})$$

and (4.3.10) implies the inclusion $R(x_{j'}) \subset R'$. In our case $x_{j'} = x_{j''}$ and the plane s_i is generated by the axes x_i and $x_{j''}$. Then the equation (4.3.10) shows that any plane belonging to R' contains the axis $x_{j'}$; i.e. $R' \subset R(x_{j'})$.

In the case when $j' \neq j''$ for any number j satisfying the conditions $j \neq i$ and $j \neq j'$ we have $n'_j = 2$. Lemma 4.3.4 shows that the plane s'_j is generated by the axes $x_{j'}$ and x_j. Consider a plane l belonging to $R(s_i)$ which contains the axis x_j and does not contain the axis $x_{j'}$. The equality $R_i = R(s_i)$ shows that $l \in R'$. It is not difficult to see that the intersection $l \cap s'_j$ is the axis x_j. Therefore $n_j = 1$ for any number j such that $j \neq i$ and $n(R') = n - 1$. ■

4.3.4. Proof of Theorem 4.2.1. Let $n - k \leq k < n - 1$. Then we have the following four cases.

(i) The condition $n_i > 0$ holds for any $i = 1, ..., n$ and there exists a number j such that $n_j > 2$. Then Lemma 4.3.3 shows that $n_i = 1$ for any number i such that $i \neq j$.

(ii) We have $0 < n_i \leq 2$ for any $i = 1, ..., n$ and $n(R') \geq k$.

(iii) We have $0 < n_i \leq 2$ for any $i = 1, ..., n$ and $n(R') = 1$. Lemma 4.3.5 implies the existence of a number j such that $R' = R(x_j)$.

(iv) The condition $n_i = 0$ holds for some number i. Lemma 4.3.1 shows that in this case $n = 2k$ and there exists an $(n-1)$-dimensional coordinate plane s such that $R' = R(s)$.

Consider the case (i). Lemma 3.3.2 implies that $n_j \leq n - k$ and here exist numders $i_1, ..., i_{k-1}$ such that the plane s_j does not contain the axis $x_{i_1}, ..., x_{i_{k-1}}$. Denote by l the k-dimensional plane generated by the axes

$$x_{i_1}, ..., x_{i_{k-1}} \text{ and } x_j \ .$$

Consider the regular set $R'' = R' \cup \{l\}$. It is trivial that

$$n(R'') \geq n(R') = n - 1$$

and the intersection $l \cap s_j$ is the axis x_j. Therefore $n(R'') = n$ and the regular set R'' is exact. This implies that $\deg(R') = 1$.

In the case (ii) consider all numbers $i_1, ..., i_m$ such that

$$n_{i_1} = ... = n_{i_m} = 2 \ .$$

Then $n(R') = n - m$ and the condition $n(R') \geq k$ shows that $m \leq n - k \leq k$. Denote by s the plane generated by the planes $s_{i_1}, ..., s_{i_m}$. It is easy to see that $\dim s \leq 2m$ and

$$n - 2m = (n - m) - m \geq k - m \geq 0 \ .$$

The last inequality guarantees the existence of $k - m$ numbers $j_1, ..., j_{k-m}$ such that

$$n_{j_1} = ... = n_{j_{k-m}} = 1$$

and the plane s does not contain the axes $x_{j_1}, ..., x_{j_{k-m}}$. Denote by l the k-dimensional plane generated by the axes

$$x_{i_1}, ..., x_{i_m}, x_{j_1}, ..., x_{j_{k-m}}$$

(note that in the case $k = m$ the plane l is generated by the axes $x_{i_1}, ..., x_{i_m}$). Consider the regular set $R'' = R' \cup \{l\}$. It is trivial that

$$n(R'') \geq n(R') = n - m \ .$$

For any number $j = 1, ..., m$ the plane s_{i_j} is generated by two axes. One of them coicides with x_{i_j} and other axis $x_{i(j)}$ satisfies the conditions $n_{i(j)} = 1$ (see Lemma 4.3.4) and

$$i(j) \neq j_1, ..., j_{n-k} \ .$$

This implies that the intersection $l \cap s_{i_j}$ is the axis x_{i_j} for any $j = 1, ..., m$. Therefore $n(R'') = n$ and the regular set $R' \cup \{l\}$ is exact; i.e. $\deg(R') = 1$.

In the case (iii) there are $n - 1$ numbers j satisfying the condition $n_j = 2$. Recall that $k \leq n - 2$. Therefore for any plane l belonging to R there exists a number j such that $n_j = 2$ and l intersect the plane s_j only in the origin of the coordinates. This implies that the regular set $R' \cup \{l\}$ is not exact and $\deg(R') \geq 2$.

Now we prove the inverse inequality. Fix k numbers $i_1, ..., i_k$ such that

$$n_{i_1} = ... = n_{i_k} = 2$$

and denote by l the plane generate by axes $x_{i_1}, ..., x_{i_k}$. For any number j such that $j \neq i$ (here i is the number satisfying the condition $R' = R(x_i)$) the plane s_j is generated by the axes x_i and x_j. Then for any $j = 1, ..., k$ the intersection $l \cap s_{i_j}$ is the axis x_{i_j}. This implies that

$$n(R' \cup \{l\}) = k + 1$$

and the set $R' \cup \{l\}$ satisfies the conditions of the case (ii). Then

$$\deg(R' \cup \{l\}) = 1$$

and $\deg(R') = 2$.

The equation (1.4.1) shows that in the case (iv) the set $\varphi_k^{2k}(R')$ satisfies the conditions of the case (iii) and the equality $\deg(R') = 2$ is a consequence of Proposition 4.1.1.

Theorem 4.2.1 is proved for $k \geq n - k$. Remark 4.2.1 shows that it holds for the case $k < n - k$.

4.3.5. Proof of Theorem 4.2.2. Now we consider the case when the regular set R' setisfies the condition

$$|R'| \geq s_k^n = c_k^{n-1} + c_{k-2}^{n-2} \,.$$

The trivial inequality $s_k^n > c_k^{n-1}$ and Lemma 4.3.1 show that $n_i > 0$ for any $i = 1, ..., n$. If the regular set R' is not exact then there exists a number i such that $n_i > 1$. Denote by s the $(n-1)$-dimensional plane transverse to the axis x_i. Then

$$R' = (R(s) \cap R') \cup (R(s_i) \cap R') \tag{4.3.11}$$

and

$$R(s) \cap R(s_i) = \emptyset$$

(see the proof of Lemma 4.3.5). It is trivial that

$$|R(s) \cap R'| \leq |R(s)| = c_{k-1}^{n-1} \tag{4.3.12}$$

and

$$|R(s_i) \cap R'| \leq |R(s_i)| = c_{k-n_i}^{n-n_i} \leq c_{k-2}^{n-2} \tag{4.3.13}$$

(see the proof of Lemma 4.3.2). The equations (4.3.11) – (4.3.13) imply that

$$|R'| = |R(s) \cap R'| + |R(s_i) \cap R'| < c_{k-1}^{n-1} + c_{k-2}^{n-2} = s_k^n \,.$$

Therefore the condition $|R'| \geq s_k^n$ holds if and only if (4.3.12) and (4.3.13) are equivalents and $n_i = 2$. This implies that

$$R(s) \cap R' = R(s)$$

and

$$R(s_i) \cap R' = R(s_i) \ .$$

Then Theorem 4.2.2 is a consequence of the equation (4.3.11). ■

Chapter 5

Bijective maps of the Grassman manifolds. Proof of Theorem 3.3.1

In this chapter we give the proof of Theorem 3.3.1. First of all we prove the Fundamental Theorem of Projective Geometry (Section 5.1). In the cases $k = 1, n-1$ Theorem 3.3.1 is a trivial concequence of it. To prove our statement in the general case (Section 5.4) we exploit the Chow Theorem (Sections 5.2, 5.3) and the results of Chapter 4.

5.1 Fundamental Theorem of Projective Geometry. Proof of Theorem 5.3.1 for the cases $k = 1, n-1$

5.1.1. In this section we prove the Fundamental Theorem of Projective Geometry. In acord with our terms it can be formulated in the following form: *if* $n \geq 3$ *then*

$$\mathfrak{R}(\mathbb{G}_1^n) = \mathfrak{L}(\mathbb{G}_1^n) \ .$$

Our proof of this statement is a modifecation of the proof from [3].

We want to show that any regular map f of $\mathbb{G}_1^n$ onto $\mathbb{G}_1^n$ is linear. Fix a maximal regular subset $R = \{l_i\}_{i=1}^n$ of $\mathbb{G}_1^n$ and consider a linear map f_0 of $\mathbb{G}_1^n$ onto $\mathbb{G}_1^n$ setisfying the following condition

$$f(l_i) = f_0(l_i) \quad \forall \ i = 1, ..., n \ .$$

Define $f_1 = (f_0)^{-1} f$. Then $f = f_0 f_1$ and

$$f_1(l_i) = l_i \quad \forall\, i = 1, ..., n\ .$$

Therefore the Fundamental Theorem of Projective Geometry is a consequence of the following statement: *if for a regular map f of $\mathbb{G}_1^n$ onto $\mathbb{G}_1^n$ there exists a maximal regular subset $R = \{l_i\}_{i=1}^n$ of $\mathbb{G}_1^n$ such that*

$$f(l_i) = l_i \quad \forall\, i = 1, ..., n \tag{5.1.1}$$

then the map f is identity.

5.1.2. In the next subsection we consider the case $n = 3$. To prove the theorem in this case we exploit the following statement.

Proposition 5.1.1. *Let f be a bijection map of $\mathbb{R}^2$ onto $\mathbb{R}^2$ transfering each parallel lines in parallel lines and setisfying the condition $f(0) = 0$. Then the map f is linear.*

Proof. Let x and y be points x and y of $\mathbb{R}^2$ setisfying the condition $y \neq \alpha x$. The map f transfers the parallelogram generated by the points

$$0, x, y, x + y$$

to the parallelogram generated by the points

$$0, f(x), f(y), f(x + y)$$

(Fig.1). Moreover, f also transfers the parallelogram generated by the points

$$x, y, -x, -y$$

to the parallelogram generated by the points

$$f(x), f(y), f(-x), f(-y)$$

(Fig.2). This implies that

$$f(x + y) = f(x) + f(y)$$

and

$$f(x) = -f(x)\ .$$

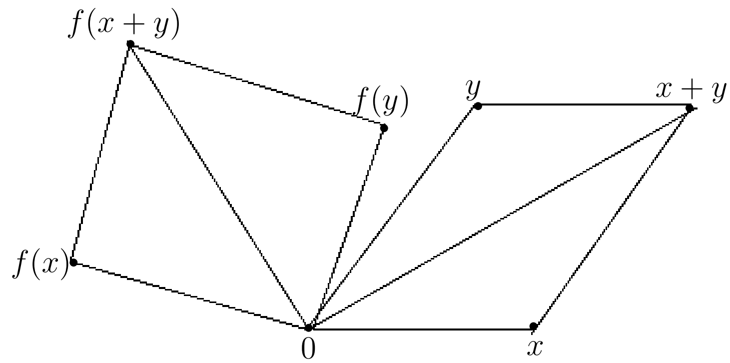

Fig. 1

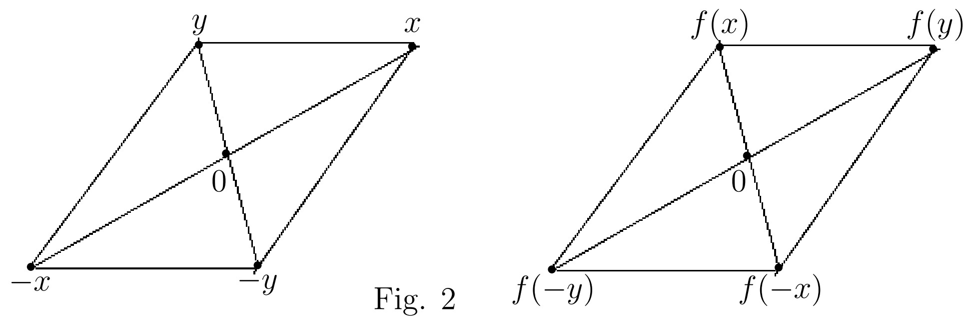

Fig. 2

Now consider the case when x and y are points lying on some line l passing through the origin of the coordinates and $x \neq -y$. Then there exists $\alpha \in \mathbb{R}$ ($\alpha \neq -1$) such that $y = \alpha x$. Let z be a point of $\mathbb{R}^2$ which does not lie on the line l. Assume that there exists $\beta \in \mathbb{R}$ such that

$$y - z = \beta(x + z) \ .$$

Then

$$y - \beta x = (\alpha - \beta)x = (\beta - 1)z$$

and the last equality shows that the point z lies on the line l. Therefore the points $x + z$ and $y - z$ do not lie on a line passing through the origin of the coordinates (note that in the case $\alpha = -1$ it does not hold). Then

$$f(x + y) = f((x + z) + (y - z)) = f(x + z) + f(y - z) =$$

$$f(x) + f(z) + f(y) - f(z) = f(x) + f(y) \ .$$

We have proved that the equality

$$f(x+y) = f(x) + f(y)$$

holds for any two points x and y of $\mathbb{R}^2$.

The map f transfers any line passing through the origin of the coordinates in a line passing through the origin of the coordinates. Therefore for any $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^2$ there exicts $f'(\alpha, x) \in \mathbb{R}$ such that

$$f(\alpha x) = f'(\alpha, x) f(x) \ .$$

Consider two points x and y of $\mathbb{R}^2$ setisfying the condition $y \neq \alpha x$. Then the triangles

$$(0, x, y) \text{ and } (0, \alpha x, \alpha y)$$

are similar. The map f transfers thes triangles to the similar triangles

$$(0, f(x), f(y)) \text{ and } (0, f(\alpha x), f(\alpha y))$$

(Fig. 3). This implies that

$$f'(\alpha, x) = f'(\alpha, y) \ .$$

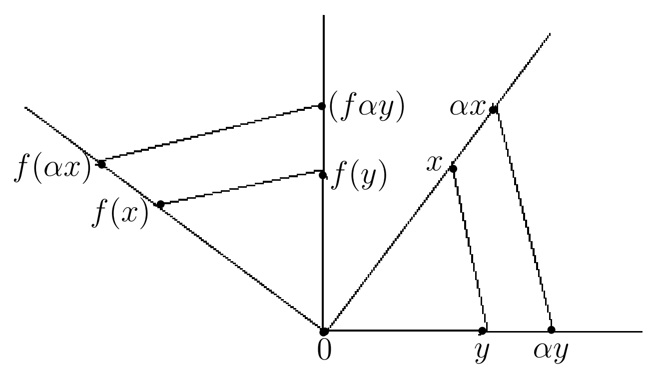

Fig. 3

In other words there exists a map $f' : \mathbb{R} \to \mathbb{R}$ such that

$$f(\alpha x) = f'(\alpha) f(x)$$

for any $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^2$. We have

$$f'(\alpha\beta)f(x) = f(\alpha\beta x) = f'(\alpha)f(\beta x) = f'(\alpha)f'(\beta)f(x)$$

and

$$f'(\alpha + \beta)f(x) = f((\alpha + \beta)x) = f(\alpha x) + f(\beta x) = (f'(\alpha) + f'(\beta))f(x)\ .$$

These equalities imply that

$$f'(\alpha\beta) = f'(\alpha)f'(\beta)$$

and

$$f'(\alpha + \beta) = f'(\alpha) + f'(\beta)$$

for any $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$. Therefore the map f' is linear. The condition $f'(1) = 1$ guarantees that f is the identical map and we have proved that

$$f(\alpha x) = \alpha f(x)$$

for any $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^2$. ∎

5.1.3. Proof of the Fundamental Theorem of Projective Geometry for the case n=3. Let f be a regular map of $\mathbb{G}_1^3$ onto $\mathbb{G}_1^3$ and there exists a maximal regular subset $R = \{l_i\}_{i=1}^3$ of $\mathbb{G}_1^3$ such that the equation (5.1.1) holds if $n = 3$. Denote by e_i $(i = 1, 2, 3)$ the unit vector lying on the line l_i and consider the two-dimensional plane s generated by l_1 and l_2. For any line l belonging to the set $\mathbb{G}_1^3 \setminus \mathbb{G}_1^3(s)$ there exist unique numbers $a_1 \in \mathbb{R}$ and $a_2 \in \mathbb{R}$ such that the vector

$$a_1e_1 + a_2e_2 + e_3$$

lies on l. Denote this vector by $e(l)$. Then the map

$$F : \mathbb{G}_1^3 \setminus \mathbb{G}_1^3(s) \to \mathbb{R}^2$$

$$f(l) = (a_1, a_2)$$

is a homeomorphism of $\mathbb{G}_1^3 \setminus \mathbb{G}_1^3(s)$ onto $\mathbb{R}^2$.

Remark 5.1.1. It is not difficult to see that

$$U_1^3(s) = \mathbb{G}_1^3 \setminus \mathbb{G}_1^3(s)$$

and F coincides with the homeomorphism $A_1^3(s)$ considered in Section 1.2.

Lemma 5.1.1. *The homeomorphism F setisfies the following conditions:*

(i) $F(l_3) = 0$;

(ii) *for any plane $t \in \mathbb{G}_2^3$ which does not coincides with the plane s the homeomorphism F transfers the set $\mathbb{G}_1^3(t) \setminus \mathbb{G}_1^3(s)$ to a line;*

(iii) *if for some planes t and p belonging to $\mathbb{G}_2^3$ the line $l = t \cap p$ is contained in the plane s then the lines*

$$F(\mathbb{G}_1^3(t) \setminus \mathbb{G}_1^3(s)) \text{ and } F(\mathbb{G}_1^3(p) \setminus \mathbb{G}_1^3(s))$$

are parallel.

Proof. First condition is trivial.

Show that the condition (ii) holds. Let l'_1 and l'_2 be lines belonging to $\mathbb{G}_1^3 \setminus \mathbb{G}_1^3(s)$ and generating the plane t. Let also

$$e(l'_1) = a_1^1 e_1 + a_2^1 e_2 + e_3$$

and

$$e(l'_2) = a_1^2 e_1 + a_2^2 e_2 + e_3 \ .$$

Consider a line l belonging to the set $\mathbb{G}_1^3(t) \setminus \mathbb{G}_1^3(s)$ and the vector

$$e(l) = xe_1 + ye_2 + e_3 \ .$$

There exist $a \in \mathbb{R}$ and $b \in \mathbb{R}$ such that

$$e(l) = ae(l'_1) + be(l'_2) \ .$$

Then

$$x = aa_1^1 + ba_1^2 \ ,$$
$$y = aa_2^1 + ba_2^2$$

and $a + b = 1$. This implies that

$$x = (a_1^1 - a_1^2)a + a_1^2 \ ,$$

$$y = (a_2^1 - a_2^2)a + a_2^2 \ .$$

An immediate verification implies the existence of $a' \in \mathbb{R}$ and $b' \in \mathbb{R}$ such that

$$y = a'x + b' \ . \tag{5.1.2}$$

We have proved that F transfers the set $\mathbb{G}_1^3(t) \setminus \mathbb{G}_1^3(s)$ to the line defined by the condition (5.1.2).

Consider the condition (iii). If s contains the line $l = t \cap p$ then the sets

$$\mathbb{G}_1^3(t) \setminus \mathbb{G}_1^3(s) \text{ and } \mathbb{G}_1^3(p) \setminus \mathbb{G}_1^3(s)$$

are disjoint. Therefore the lines

$$F(\mathbb{G}_1^3(t) \setminus \mathbb{G}_1^3(s)) \text{ and } F(\mathbb{G}_1^3(p) \setminus \mathbb{G}_1^3(s))$$

are disjoint; i.e. they are parallel. ■

Define $f' = FfF^{-1}$. It is a bijection map of $\mathbb{R}^2$ onto $\mathbb{R}^2$. Now we exploit Proposition 5.1.1 and prove that the map f' is linear. The equalities

$$f(l_3) = l_3 \text{ and } F(l_3) = 0$$

guarantee that $f'(0) = 0$. Proposition 2.4.1 and Lemma 3.1.1 show that for any plane $t \in \mathbb{G}_2^3$ there exists a plane $t' \in \mathbb{G}_2^3$ such that

$$f(\mathbb{G}_1^3(t)) = \mathbb{G}_1^3(t') \ .$$

Then the condition (ii) implies that the map f' transfers any line in a line. The equation

$$f(\mathbb{G}_1^3(s)) = \mathbb{G}_1^3(s)$$

(which is a trivial consequence of the equation (5.1.1) for $n = 3$) and the condition (iii) show that f' transfers any parallel lines in parallel lines. Then the required statement is a consequence of Proposition 5.1.1.

Let

$$f' = (f_1', f_2')$$

be the coordinate representation of the map f'. Consider the linear map g of $\mathbb{G}_1^3$ onto $\mathbb{G}_1^3$ induced by the linear map transfering each vector

$$a_1e_1 + a_2e_2 + a_3e_3$$

to the vector

$$f_1'(a_1, a_2)e_1 + f_2'(a_1, a_2)e_2 + a_3e_3 \ .$$

It is not difficult to see that

$$g(l) = f(l) \quad \forall\, l \in \mathbb{G}^3_1 \setminus \mathbb{G}^3_1(s)\ . \tag{5.1.3}$$

The following lemma implies that this equality holds for any line l belonging to $\mathbb{G}^3_1$.

Lemma 5.1.2. *Let f and g be regular maps of $\mathbb{G}^3_1$ onto $\mathbb{G}^3_1$. If there exists a plane $s \in \mathbb{G}^3_2$ such that the equation (5.1.3) holds and*

$$f(\mathbb{G}^3_1(s)) = g(\mathbb{G}^3_1(s)) = \mathbb{G}^3_1(s)\ . \tag{5.1.4}$$

Then the maps f and g coincide.

Lemma 5.1.1 shows that the map f is linear. The eqiality (5.1.1) ($n = 3$) guarantees that f is identity.

Proof of Lemma 5.1.2. For any line l' belonging to $\mathbb{G}^3_1(s)$ consider a plane $t \in \mathbb{G}^3_2$ such that $l' = s \cap t$. Proposition 2.4.1 and Lemma 3.1.1 guarantee the existence of planes t_1, t_2 belonging to $\mathbb{G}^3_2$ and satisfying the following conditions

$$f(\mathbb{G}^3_1(t)) = \mathbb{G}^3_1(t_1)\ ,$$
$$g(\mathbb{G}^3_1(t)) = \mathbb{G}^3_1(t_2)\ .$$

The equation (5.1.4) shows that

$$f(l') = s \cap t_1 \text{ and } g(l') = s \cap t_2\ . \tag{5.1.5}$$

Cosider lines l'_1 and l'_2 belonging to $\mathbb{G}^3_1 \setminus \mathbb{G}^3_1(s)$ and generating t. The planes t_1 and t_2 are generated by the lines $f(l'_1)$, $f(l'_2)$ and the lines $g(l'_1)$, $g(l'_2)$ respectively. The equality (5.1.3) implies that

$$f(l'_1) = g(l'_1) \text{ and } f(l'_2) = g(l'_2)\ .$$

Therefore the planes t_1 and t_2 coincide. Then the equality $f(l') = g(l')$ is a consequence of the equation (5.1.5). ■

5.1.4. Proof of the Fundamental Theorem of Projective Geometry for the general case. Let $n > 3$. Assume that our statement is proved for the case $n = k - 1$ and consider the case $n = k$.

Denote by s_i ($i = 1, ..., n$) the $(n-1)$-dimensional plane generated by the lines

$$l_1, ..., l_{i-1}, l_{i+1}, ..., l_n\ .$$

The condition (5.1.1) guarantees that

$$f(\mathbb{G}_1^n(s_i)) = \mathbb{G}_1^n(s_i) \quad \forall\, i = 1, ..., n \ .$$

Consider the regular map

$$f_i = \varphi_{s_i} f (\varphi_{s_i})^{-1}$$

of $\mathbb{G}_1^{n-1}$ onto $\mathbb{G}_1^{n-1}$. The inductive hipothesis implies that it is the identity for any $i = 1, ..., n$. In other words we have

$$f(l) = l \quad \forall\, l \in \bigcup_{i=1}^{n} \mathbb{G}_1^n(s_i) \ . \tag{5.1.6}$$

Let l be a line belonging to the set

$$\mathbb{G}_1^n \setminus \bigcup_{i=1}^{n} \mathbb{G}_1^n(s_i) \ .$$

For any two numbers i and j such that $i \neq j$ denote by s_{ij} the $(n-1)$-dimensional plane generated by l and

$$l_1, ..., l_{i-1}, l_{i+1}, ..., l_{j-1}, l_{j+1}, ..., l_n \ .$$

The intersection of the plane $s_{i\,j}$ with the plane generated by l_i and l_j is a line belonging to the set

$$\bigcup_{i=1}^{n} \mathbb{G}_1^n(s_i) \ .$$

Denote it by $l_{i\,j}$. It is not difficult to see that the plane $s_{i\,j}$ is generated by the lines $l_{i\,j}$ and

$$l_1, ..., l_{i-1}, l_{i+1}, ..., l_{j-1}, l_{j+1}, ..., l_n \ .$$

Lemma 3.1.1 shows that

$$f(\mathbb{G}_1^n(s_{i\,j})) = \mathbb{G}_1^n(s'_{i\,j}) \ ,$$

where s'_{ij} is a plane containing the lines $f(l_{i\,j})$ and

$$f(l_1), ..., f(l_{i-1}), f(l_{i+1}), ..., f(l_{j-1}), f(l_{j+1}), ..., f(l_n) \ .$$

The equation (5.1.6) implies that the planes $s_{i\,j}$ and $s'_{i\,j}$ coincide. An immediate verification shows that

$$\{l\} = \bigcap_{ij=1\; i\neq j}^{n} \mathbb{G}_1^n(s_{i\,j})\ ,$$

$$\{f(l)\} = \bigcap_{ij=1\; i\neq j}^{n} \mathbb{G}_1^n(s'_{i\,j}))$$

and we have $f(l) = l$. ■

5.1.5. Proof Theorem 3.3.1 for the case $k = n-1$. Now we use the Fundamental Theorem of Projective Geometry to prove Theorem 3.1.1 for the case $k = n-1$.

Proposition 2.4.1 and Lemma 3.1.1 guarantee that for any regular map f of $\mathbb{G}_{n-1}^n$ onto $\mathbb{G}_{n-1}^n$ there exists a map $f_1 : \mathbb{G}_1^n \to \mathbb{G}_1^n$ setisfying the following condition

$$f(\mathbb{G}_{n-1}^n(s)) = \mathbb{G}_{n-1}^n(f_1(s)) \quad \forall s \in \mathbb{G}_1^n\ .$$

It is not difficult to see that f_1 is a bijection of $\mathbb{G}_1^n$ to $\mathbb{G}_1^n$ (otherwise the map f is not bijective) and

$$f_1(\mathbb{G}_1^n(s)) = \mathbb{G}_1^n(f(s)) \quad \forall s \in \mathbb{G}_{n-1}^n\ . \tag{5.1.7}$$

Then Proposition 2.4.1 and Lemma 3.1.1 guarantee that the map f is regular. Therefore this map is linear and the equations (5.2.4), (6.1.7) implies that

$$L_{1,n-1}^n(f_1) = f\ ;$$

i.e. f is a linear map and the required statement is proved. ■

5.2 Chow's Theorem

For any two planes l' and l belonging to $\mathbb{G}_k^n$ $(1 < k < n-1,\ n > 3)$ the number

$$d(l,l') = k - \dim l \cap l'$$

is called the *distance* between l_1 and l_2. It is easy to see that $d(l,l') = i$ if and only if the dimension of the plane generated by l and l' is $k+i$. Moreover, we have the following inequality

$$d(l,l') \le \alpha_k^n = \begin{cases} k & \text{if } k \le n-k \\ n-k & \text{if } k \ge n-k \end{cases}$$

and there exist planes l and l' such that $d(l, l') = \alpha_k^n$.

We say that the planes l and l' are *adjacent* if $d(l, l') = 1$. An immediate verification shows that for any two planes l and l' the equality the $d(l, l') = i$ holds if and only if there exists a collection of planes l_0,..., l_i belonging to $\mathbb{G}_k^n$ and setisfying the following conditions:

(i) $l_0 = l$ and $l_i = l'$

(ii) the planes l_j, l_{j+1} are adjacent for any $j = 0, ..., i-1$.

A bijective map f of $\mathbb{G}_k^n$ onto $\mathbb{G}_k^n$ preserving the distance will be called a *Chow* map; i.e. f is a Chow map if

$$d(f(l), f(l')) = d(l, l')$$

for any two planes l and l' belonging to $\mathbb{G}_k^n$. Denote by $\mathfrak{C}(\mathbb{G}_k^n)$ the class of all Chow maps of $\mathbb{G}_k^n$ onto $\mathbb{G}_k^n$. It is not difficult to see tha t $\mathfrak{C}(\mathbb{G}_k^n)$ is a group and the following condition are equivalent:

(i) a bijective map f of $\mathbb{G}_k^n$ onto $\mathbb{G}_k^n$ is a Chow map;

(ii) for any two planes l and l' belonging to $\mathbb{G}_k^n$ the planes $f(l)$ and $f(l')$ are adjacent if and only if the planes l and l' are adjacent.

Lemma 5.2.1. *The canonical homeomorphism $\varphi_{k\,n-k}^n$ preserves the distance between planes.*

Proof. We want to show that planes l and l' belonging to $\mathbb{G}_k^n$ are adjacent if and only if $\varphi_{k\,n-k}^n(l)$ and $\varphi_{k\,n-k}^n(l')$ are adjacent planes. Assume that l and l' are adjacent planes and prove that the planes $\varphi_{k\,n-k}^n(l)$ and $\varphi_k^n(l')$ are adjacent. The inverse statement is a consequence of the equality

$$(\varphi_{k\,n-k}^n)^{-1} = \varphi_{n-k\,k}^n \ .$$

Consider the intersection $s = l \cap l'$. The equality $d(l, l') = 1$ guarantees that $\dim s = k-1$. Then $\dim s^\perp = n-k+1$ and the plane $s^\perp$ contains the planes $\varphi_{k\,n-k}^n(l)$ and $\varphi_{k\,n-k}^n(l')$; therefore these planes are adjacent. ■

Lemma 5.2.1 shows that $\varphi_{k\,k}^{2k}$ are a Chow map of $\mathbb{G}_k^{2k}$ onto $\mathbb{G}_k^{2k}$ and the map

$$\Phi_{k\,n-k}^n : \mathfrak{C}(\mathbb{G}_k^n) \to \mathfrak{C}(\mathbb{G}_{n-k}^n) \ ,$$

$$\Phi^n_{k\,n-k}(f) = \varphi^n_{k\,n-k} f \varphi^{2k}_{n-k\,k} \quad \forall\, f \in \mathfrak{C}(\mathbb{G}^n_k)$$

is an isomorphism between the groups $\mathfrak{C}(\mathbb{G}^n_k)$ and $\mathfrak{C}(\mathbb{G}^n_{n-k})$.

It is easy to see that any linear map of $\mathbb{G}^n_k$ onto $\mathbb{G}^n_k$ is a Chow map and

$$\mathfrak{L}(\mathbb{G}^n_k) \subset \mathfrak{C}(\mathbb{G}^n_k)$$

if $1 < k < n-1$. The inverse inclusion holds and we have the following statement.

Theorem 5.2.1. [2] (see also [3]) *If $1 < k < n-1$ and $n \neq 2k$ then*

$$\mathfrak{C}(\mathbb{G}^n_k) = \mathfrak{L}(\mathbb{G}^n_k) \ .$$

The group $\mathfrak{C}(\mathbb{G}^{2k}_k)$ $(k > 1)$ is generated by the group $\mathfrak{L}(\mathbb{G}^{2k}_k)$ and the homeomorphism $\varphi^{2k}_{k\,k}$.

Theorem 5.2.1 will be proved in the next section. Our proof is a modifecation of the proof from the Chow's paper [2]. In Section 5.4 we show that

$$\mathfrak{R}(\mathbb{G}^n_k) = \mathfrak{C}(\mathbb{G}^n_k)$$

if $1 < k < n-1$. This implies that in this case Theorem 5.3.1 is a consequence of the Chow Theorem.

5.3 Proof of the Chow Theorem

5.3.1. Chow sets. Idea of the proof. Now we consider one special class of subsets of $\mathbb{G}^n_k$. In the next subsections we exploit it's properties to prove the Chow Theorem.

We say that a set $V \subset \mathbb{G}^n_k$ is a *Chow set* ($V \in \mathfrak{C}^n_k$) if any two planes belonging to V are adjacent. A Chow set V is called *maximal* ($V \in \mathfrak{M}\mathfrak{C}^n_k$) if any Chow set W containing V coincides with it.

Proposition 5.3.1. *For any Chow set $V \subset \mathbb{G}^n_k$ there exists a maximal Chow set $W \subset \mathbb{G}^n_k$ containing V.*

Proof. The proof of this statement is similar to the proof of Proposition 2.2.1. ■

Let $s \in \mathbb{G}^n_m$ and $m = k-1$ or $k+1$. Then $\mathbb{G}^n_k(s)$ is a Chow set. It is not difficult to see that a Chow set $V \subset \mathbb{G}^n_k$ is maximal if and only if for any plane l belonging to $\mathbb{G}^n_k \setminus V$ there exists a plane $l' \in V$ such that $d(l, l') > 1$. This implies that the Chow set $\mathbb{G}^n_k(s)$ is maximal. Now we show that there are not other maximal Chow sets.

Proposition 5.3.2. *For any maximal Chow set $V \subset \mathbb{G}^n_k$ there exists a plane $s \in \mathbb{G}^n_m$ ($m = k-1$ or $k+1$) such that $V = \mathbb{G}^n_k(s)$.*

Proof. Consider two planes l_1 and l_2 belonging to the maximal Chow set V. Define $s_1 = l_1 \cap l_2$ and denote by s_2 the plane generated by l_1 and l_2. Since the planes l_1 and l_2 are adjacent we have

$$\dim s_1 = k-1 \text{ and } \dim s_2 = k+1 \, .$$

Assume that $V \neq \mathbb{G}^n_k(s_2)$ and prove that in this case

$$V \setminus \mathbb{G}^n_k(s_2) \subset \mathbb{G}^n_k(s_1) \, . \tag{5.3.1}$$

Consider a plane

$$l_3 \in V \setminus \mathbb{G}^n_k(s_2) \, .$$

This condition shows that

$$\dim s_2 \cap l_3 \leq k-1$$

(otherwise we have $l_3 \in \mathbb{G}^n_k(s_2)$) Recall that the plane s_2 is generated by the planes l_1 and l_2. This implies that the plane $s_2 \cap l_3$ contains the planes $l_1 \cap l_3$ and $l_2 \cap l_3$. However, the equality

$$d(l_1, l_3) = d(l_2, l_3) = 1$$

guarantees that

$$\dim l_1 \cap l_3 = \dim l_1 \cap l_3 = k-1 \, .$$

Therefore the planes

$$l_1 \cap l_3, l_2 \cap l_3, s_2 \cap l_3 \text{ and } s_1 = l_1 \cap l_2$$

coincide. Then and $l \in \mathbb{G}^n_k(s_1)$ and we have proved the inclusion (5.3.1).

Now consider a plane

$$l \in \mathbb{G}^n_k(s_2) \setminus \mathbb{G}^n_k(s_1)$$

The plane l_3 is generated by the plane s_1 and some line which is not contained in s_2. Our plane l does not contains s_2. Therefore

$$\dim l_3 \cap l < k-1$$

and these planes are not adjacent. In other words we have proved that

$$V \cap (\mathbb{G}_k^n(s_2) \setminus \mathbb{G}_k^n(s_1)) = \emptyset \, .$$

Then equation (5.3.1) implies the inclusion $V \subset \mathbb{G}_k^n(s_1)$. The Chow set V is maximal and $\mathbb{G}_k^n(s_1)$ is a Chow set. Therefore the inverse inclusion holds true. ∎

It is trivial that any Chow map of $\mathbb{G}_k^n$ onto $\mathbb{G}_k^n$ preserves the classes $\mathfrak{C}_k^n$ and $\mathfrak{MC}_k^n$. This fact and Proposition 5.3.2 will be used in the next subsections to prove the Chow Theorem.

5.3.2. Let f be a Chow map of $\mathbb{G}_k^n$ onto $\mathbb{G}_k^n$. Proposition 5.3.2 guarantees that for any $s \in \mathbb{G}_{k-1}^n$ there exists a plane $f_{k-1}(s) \in \mathbb{G}_i^n$ (where $i = k-1$ or $i = k+1$) such that

$$f(\mathbb{G}_k^n(s)) = \mathbb{G}_k^n(f_{k-1}(s)) \, .$$

Define

$$V_{k-1}^n(f) = \{ \, s \in \mathbb{G}_{k-1}^n \mid f_{k-1}(s) \in \mathbb{G}_{k-1}^n \, \} \, ,$$
$$W_{k-1}^n(f) = \{ \, s \in \mathbb{G}_{k-1}^n \mid f_{k-1}(s) \in \mathbb{G}_{k+1}^n \, \}$$

and prove the next lemma.

Lemma 5.3.1. *For each Chow map f of $\mathbb{G}_k^n$ onto $\mathbb{G}_k^n$ one of the following equalities*

$$V_{k-1}^n(f) = \mathbb{G}_{k-1}^n \text{ or } W_{k-1}^n(f) = \mathbb{G}_{k-1}^n$$

holds true.

Proof. Consider the case when the set $V_{k-1}^n(V)$ is not empty and show that it coincides with $\mathbb{G}_{k-1}^n$. Let s be a plane belonging to $V_{k-1}^n(V)$. We want to prove that $V_{k-1}^n(f)$ contains any plane $s' \in \mathbb{G}_{k-1}^n$. Recall that there exist planes

$$s_0 = s, s_1, ..., s_m = s' \ (m = d(s,s'))$$

belonging to $\mathbb{G}_{k-1}^n$ and such that the planes s_i, s_{i+1} are adjacent for any $i = 0, ..., m-1$. This implies that we can restrict ourselfs only to the case when s and s' are adjacent planes.

Assume that $s' \in W^n_{k-1}(f)$ and $f_{k-1}(s') \in \mathbb{G}^n_{k+1}$. The equality $d(s,s')=1$ guarantees that the set

$$\mathbb{G}^n_k(s) \cap \mathbb{G}^n_k(s')$$

contains only unique plane generated by s and s'. Then the set

$$f(\mathbb{G}^n_k(s) \cap \mathbb{G}^n_k(s')) = \mathbb{G}^n_k(f_{k-1}(s)) \cap \mathbb{G}^n_k(f_{k-1}(s')) \tag{5.3.2}$$

contains only one plane. This set is not empty if and only if

$$f_{k-1}(s) \subset f_{k-1}(s') \ .$$

The last inclusion implies that the set (5.3.2) contains the infinite number of elements. Therefore our hypothesis fails and $s \in V^n_{k-1}(f)$. ■

Proposition 5.3.2 implies that for any $s \in \mathbb{G}^n_{k+1}$ there exists a plane $f_{k+1}(s) \in \mathbb{G}^n_i$ (where $i = k-1$ or $i = k+1$) such that

$$f(\mathbb{G}^n_k(s)) = \mathbb{G}^n_k(f_{k+1}(s)) \ .$$

Define

$$V^n_{k+1}(f) = \{\ s \in \mathbb{G}^n_{k+1} \mid f_{k+1}(s) \in \mathbb{G}^n_{k+1}\ \} \ ,$$
$$W^n_{k+1}(f) = \{\ s \in \mathbb{G}^n_{k+1} \mid f_{k+1}(s) \in \mathbb{G}^n_{k-1}\ \} \ .$$

Then we have the following lemma.

Lemma 5.3.2. *For each Chow map f of $\mathbb{G}^n_k$ onto $\mathbb{G}^n_k$ one of the following equalities*

$$V^n_{k+1}(f) = \mathbb{G}^n_{k+1} \text{ or } W^n_{k+1}(f) = \mathbb{G}^n_{k+1}$$

holds true.

Proof. Consider the Chow map

$$f' = \Phi^n_{k\,n-k}(f) : \mathbb{G}^n_{n-k} \to \mathbb{G}^n_k \ .$$

An immediate verification shows that

$$V^n_{n-k-1}(f') = \varphi^n_{k+1}(V^n_{k+1}(f))$$

and

$$W^n_{n-k-1}(f') = \varphi^n_{k+1}(W^n_{k+1}(f)) \ .$$

This implies that Lemma 5.3.2 is a consequence of Lemma 5.3.1. ■

5.3.3 For each Chow map f of $\mathbb{G}^n_k$ onto $\mathbb{G}^n_k$ define

$$i(f) = \begin{cases} k-1 & \text{if } V^n_{k-1}(f) = \mathbb{G}^n_{k-1} \\ k+1 & \text{if } W^n_{k-1}(f) = \mathbb{G}^n_{k-1} \end{cases}$$

and

$$j(f) = \begin{cases} k+1 & \text{if } V^n_{k+1}(f) = \mathbb{G}^n_{k+1} \\ k-1 & \text{if } W^n_{k+1}(f) = \mathbb{G}^n_{k+1} \end{cases}$$

Lemmas 5.3.1, 5.3.2 show that the Chow map f induces the map

$$C^n_{k\,k-1}(f) : \mathbb{G}^n_{k-1} \to \mathbb{G}^n_{i(f)}$$

$$C^n_{k\,k-1}(f)(s) = f_{k-1}(s) \quad \forall\, s \in \mathbb{G}^n_{k-1}$$

and the map

$$C^n_{k\,k+1}(f) : \mathbb{G}^n_{k+1} \to \mathbb{G}^n_{j(f)}$$

$$C^n_{k\,k+1}(f)(s) = f_{k+1}(s) \quad \forall\, s \in \mathbb{G}^n_{k-1}\ .$$

Any Chow map f is a bijection of $\mathbb{G}^n_k$ onto $\mathbb{G}^n_k$. Therefore $C^n_{k\,k-1}(f)$ is a bijection of $\mathbb{G}^n_{k-1}$ onto $\mathbb{G}^n_{i(f)}$ and $C^n_{k\,k+1}(f)$ is a bijection of $\mathbb{G}^n_{k+1}$ onto $\mathbb{G}^n_{j(f)}$. These arguments elso imply that $i(f) = k-1$ if and only if $j(f) = k+1$. Therefore each Chow map f of $\mathbb{G}^n_k$ onto $\mathbb{G}^n_k$ setisfies one of the following conditions:

(i) $i(f) = k-1$ and $j(f) = k+1$;

(ii) $i(f) = k+1$ and $j(f) = k-1$.

Lemma 5.3.3. *The maps $C^n_{k\,k-1}(f)$ and $C^n_{k\,k+1}(f)$ preserve the distance between planes.*

Proof. It is not difficult to see that planes s_1 and s_2 belonging to $\mathbb{G}^n_m$ ($m = k-1$ or $k+1$) are adjacent if and only if the set

$$\mathbb{G}^n_k(s_1) \cap \mathbb{G}^n_k(s_2) \tag{5.3.3}$$

contains only one plane. Let $s_1 \in \mathbb{G}^n_{k-1}$ and $s_2 \in \mathbb{G}^n_{k-1}$. The map f is bijective and the set (5.3.2) contains only one plane if and only if the set (5.3.3) contains only one plane. Therefore the planes $f_{k-1}(s_1)$ and $f_{k-1}(s_2)$

are adjacent if and only if s_1 and s_2 are adjacent planes. These arguments show that the similar statement holds for the map $C^n_{k\,k+1}(f)$. ■

5.3.4. Denote by $\mathfrak{C}'(\mathbb{G}^n_k)$ the class of all Chow maps f of $\mathbb{G}^n_k$ onto $\mathbb{G}^n_k$ setisfying the condition (i). Lemma 5.3.3 and the condition (i) impy that if $k > 2$ then for any map f belonging to $\mathfrak{C}'(\mathbb{G}^n_k)$ the map $C^n_{k\,k-1}(f)$ is a Chow map of $\mathbb{G}^n_{k-1}$ onto $\mathbb{G}^n_{k-1}$; and if $k < n-2$ then $C^n_{k\,k-1}(f)$ is a Chow map of $\mathbb{G}^n_{k+1}$ onto $\mathbb{G}^n_{k+1}$. An immediate verification shows that

$$C^n_{k-1\,k}(C^n_{k\,k-1}(f)) = f$$

and

$$C^n_{k+1\,k}(C^n_{k\,k+1}(f)) = f\ .$$

Therefore $C^n_{k\,k-1}(f)$ and $C^n_{k\,k+1}(f)$ are elements of $\mathfrak{C}'(\mathbb{G}^n_{k-1})$ and $\mathfrak{C}'(\mathbb{G}^n_{k+1})$ respectively.

Define

$$\mathfrak{C}'(\mathbb{G}^n_1) = C^n_{2,1}(\mathfrak{C}'(\mathbb{G}^n_2))$$

and

$$\mathfrak{C}'(\mathbb{G}^n_{n-1}) = C^n_{n-2,n-1}(\mathfrak{C}'(\mathbb{G}^n_{n-2}))\ .$$

Then for any $1 < k < n-1$ the maps

$$C^n_{k\,k-1} : \mathfrak{C}'(\mathbb{G}^n_k) \to \mathfrak{C}'(\mathbb{G}^n_{k-1})$$

and

$$C^n_{k\,k+1} : \mathfrak{C}'(\mathbb{G}^n_k) \to \mathfrak{C}'(\mathbb{G}^n_{k+1})$$

are bijections of $\mathfrak{C}'(\mathbb{G}^n_k)$ onto $\mathbb{G}^n_{k-1}$ and a bijection of $\mathfrak{C}'(\mathbb{G}^n_k)$ onto $\mathbb{G}^n_{k+1}$ respectively.

For any

$$1 \geq k \leq n-1$$

consider the map

$$C^n_{k\,m} : \mathfrak{C}'(\mathbb{G}^n_k) \to \mathfrak{C}'(\mathbb{G}^n_m)$$

defined by the following condition

$$C^n_{k\,m} = \begin{cases} C^n_{m+1\,m} ... C^n_{k-1\,k-2} C^n_{k\,k-1} & \text{if } m < k\ , \\ C^n_{m-1\,m} ... C^n_{k+1\,k+2} C^n_{k\,k+1} & \text{if } m > k\ . \end{cases}$$

It is not difficult to see that this map is a bijection of $\mathfrak{C}'(\mathbb{G}^n_k)$ onto $\mathfrak{C}'(\mathbb{G}^n_m)$. Moreover, for any map $f \in \mathfrak{C}'(\mathbb{G}^n_k)$ we have

$$f(\mathbb{G}^n_k(s)) = \mathbb{G}^n_k(C^n_{k\,m}(f)(s)) \quad \forall\ s \in \mathbb{G}^n_m\ . \tag{5.3.4}$$

In the cases $m = k-1, k+1$ this equality is trivial. For the general case it can be proved by the induction by m.

The Fundamental Theorem of Projective Geometry, Lemma 3.1.1 and the equation (5.3.4) for $k = 1$ show that

$$\mathfrak{C}'(\mathbb{G}_1^n) \subset \mathfrak{L}(\mathbb{G}_1^n) .$$

Then the equations (3.2.4) and (5.3.4) imply that

$$L^n_{1,k}(\mathfrak{C}'(\mathbb{G}_1^n)) = \mathfrak{C}'(\mathbb{G}_k^n) \subset \mathfrak{L}(\mathbb{G}_k^n) .$$

The inverse inclusion is trivial and we have

$$\mathfrak{L}(\mathbb{G}_k^n) = \mathfrak{C}'(\mathbb{G}_k^n) \tag{5.3.5}$$

for any $1 \geq k \leq n-1$.

5.3.5. Proof of the Chow Theorem for the case $n \neq 2k$. An immediate verification shows us that in this case

$$\alpha^n_{k-1} \neq \alpha^n_{k+1} .$$

Then Lemma 5.3.3 implies that the condition (ii) does not holds for any Chow map f of $\mathbb{G}_k^n$ onto $\mathbb{G}_k^n$ and

$$\mathfrak{C}'(\mathbb{G}_k^n) = \mathfrak{C}(\mathbb{G}_k^n) .$$

Then the equality (5.3.5) implies that the Chow Theorem holds for the case $n \neq 2k$. ■

5.3.6. Proof of the Chow Theorem for the case $n = 2k$. Consider a Chow map f of $\mathbb{G}_k^{2k}$ onto $\mathbb{G}_k^{2k}$ setisfying the condition (ii). Define $f' = \varphi^{2k}_{k\,k} f$. Lemma 5.2.1 implies that f' is a Chow map of $\mathbb{G}_k^{2k}$ onto $\mathbb{G}_k^{2k}$. An immediate verification shows that

$$C^{2k}_{k\,k-1}(f') = \varphi^{2k}_{k+1} C^{2k}_{k\,k-1}(f)$$

and

$$C^{2k}_{k\,k+1}(f') = \varphi^{2k}_{k-1} C^{2k}_{k\,k+1}(f) .$$

Therefore the Chow map f' setisfies the condition (i). We have proved that the group $\mathfrak{C}(\mathbb{G}_k^{2k})$ is generated by the class $\mathfrak{C}'(\mathbb{G}_k^{2k})$ and the Chow map $\varphi^{2k}_{k\,k}$. Then the Chow Theorem is a consequence of the equality (5.3.5). ■

5.4 Proof of Theorem 3.3.1 for the case $1 < k < n-1$

5.4.1. Now we exploit Theorem 4.2.1 and prove that

$$\mathfrak{R}(\mathbb{G}_k^n) \subset \mathfrak{C}(\mathbb{G}_k^n)$$

for any $1 < k < n-1$. The Chow Theorem shows that the inverse inclusion is trivial.

Let f be a regular map of $\mathbb{G}_k^n$ onto $\mathbb{G}_k^n$ and $R \subset \mathbb{G}_k^n$ be a maximal regular set. Then $R' = f(R)$ is a maximal regular set. Consider the coordinate systems $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ associated with the maximl regular sets R and R' respectively.

Lemma 5.4.1. *Let s be an $(n-1)$-dimensional coordinate plane for the system $\{x_i\}_{i=1}^n$. Then the following statements hold true:*

(i) *if $n \neq 2k$ then there exists an $(n-1)$-dimensional coordinate plane s' for the system $\{y_i\}_{i=1}^n$ such that*

$$f(R(s)) = R'(s') \; ; \tag{5.4.1}$$

(ii) *if $n = 2k$ then there exists an m-dimensional coordinate plane s' for the system $\{y_i\}_{i=1}^n$ (where $m = 1$ or $m = n-1$) setisfying the condition (5.4.1).*

Proof. If $k \leq n-k$ then our statement is a trivial consequence of Theorem 4.2.1 and Proposition 4.1.1.

In the case $k > n-k$ consider the axis x_i transverse to the plane s. Theorem 3.2.1 and Proposition 4.4.1. guarantee the existence of an axis y_j such that

$$f(R(x_i)) = R'(y_j) \; .$$

Denote by s' the $(n-1)$-dimensional coordinate plane for the system $\{y_i\}_{i=1}^n$ transverse to the axis y_j. Then the equalities

$$R(s) = R \setminus R(x_i)$$

and

$$R'(s') = R' \setminus R'(y_j)$$

imply (5.4.1). ■

In the next subsection Lemma 5.4.1 will be used to prove the following statement.

Lemma 5.4.2. *Let s be a $(k+1)$-dimensional coordinate plane for the system $\{x_i\}_{i=1}^n$. Then the following statements hold true:*

(i) *if $n \neq 2k$ then there exists a $(k+1)$-dimensional coordinate plane s' for the system $\{y_i\}_{i=1}^n$ setisfying the condition* (5.4.1)*;*

(ii) *if $n = 2k$ then there exists an m-dimensional coordinate plane s' for the system $\{y_i\}_{i=1}^n$ (where $m = k-1$ or $m = k+1$) setisfying the condition* (5.4.1)*.*

For any two planes belonging to $\mathbb{G}_k^n$ there exists a maximal regular subset of $\mathbb{G}_k^n$ containing them. Therefore Lemma 5.4.2 shows that f is a Chow map and we have proved the required inclusion.

5.4.2. Proof of Lemma 5.4.2. Denote by s_i and s'_i the $(n-1)$-dimensional coordinate planes in the systems $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ transverse to the axis x_i and y_i respectively. Lemma 5.4.1 implies that in the case $n \neq 2k$ for any $i = 1, ..., n$ there exists a number j_i such that

$$f(R(s_i)) = R_f(s'_{j_i}) \ .$$

Prove our statement for the plane s generated by the axes $x_1, ..., x_{k+1}$. The equality

$$R(s) = \bigcap_{i=k+2}^{n} R(s_i)$$

implies that

$$f(R(s)) = \bigcap_{i=k+2}^{n} R_f(s'_{j_i}) = R_f(s') \ ;$$

where s' is the plane generated by the axes $y_{j_1}, ..., y_{j_{k+1}}$.

In the case $n = 2k$ Lemma 5.4.1 shows that the map f setisfies one of the following two conditions:

(a) there exists a number j_1 such that equation (5.4.1) holds for $i = 1$;

(b) there exists a number j_1 such that

$$f(R(s_1)) = R_f(y_{j_1}) \ .$$

First of all prove that in the case (a) for any $i = 2, ..., n$ there exists a number j_i setisfying the condition (5.4.1). Then in this case the proof of the statement (ii) is similar to the proof of the statement (i). Assume that there exist numbers i and j_i such that

$$f(R(s_i)) = R_f(y_{j_i}) .$$

Define $\hat{s} = s_1 \cap s_i$. Then

$$R(\hat{s}) = R(s_1) \cap R(s_i)$$

and

$$f(R(\hat{s})) = R_f(s'_{j_1}) \cap R_f(y_{j_i}) .$$

The homeomorphism $\varphi_{s'_{j_1}}$ transfers

$$R_f(s'_{j_1}) \cap R_f(y_{j_i})$$

to a regular set $R' \subset \mathbb{G}_k^{2k-1}$ such that any plane belonging to R' contains some fixed line. This implies that

$$|f(R(\hat{s}))| = c_{k-1}^{2k-1} .$$

Hovever

$$|R(\hat{s})| = c_k^{2k-2} .$$

An immediate verification shows that

$$c_{k-1}^{2k-1} = \frac{(2k-1)!}{k!(k-1)!} ,$$

$$c_k^{2k-2} = \frac{(2k-2)!}{k!(k-2)!} = \frac{(2k-2)!(k-1)}{k!(k-1)!} .$$

Therefore

$$c_k^{2k-2} \neq c_{k-1}^{2k-2}$$

and our hypothesis fails.

In the case (b) the regular map $\varphi_{k\,k}^{2k} f$ satisfies the conditions of the case (a). Therefore there exists $s'' \in \mathbb{G}_{k+1}^{2k}$ such that

$$(\varphi_k^{2k} f)(R(s)) = R''(s'') ,$$

where $R'' = \varphi_k^{2k}(R')$. This implies that in this case the equation (5.4.1) holds if $s' = \varphi_{k+1\,k-1}^{2k}(s'')$. ■

Chapter 6

Number characteristics of irregular sets

In the chapter we introduce two dual number characteristics of subsets of the Grassman manifolds and exploit them to study the structure of irregular sets. The results of this chapter will be used in the next chapter to prove Theorem 3.4.1.

6.1 Definitions

6.1.1. Let V be a subset of $\mathbb{G}^n_k$. Consider the set

$$N_1(V) = \{ \, t \in \mathbb{G}^n_1 \mid \mathbb{G}^n_k(t) \subset V \, \} \, .$$

If this set is not empty then the lines belonging to it generate some plane $s_1(V)$. Define

$$n_1(V) = \begin{cases} \dim s_1(V) & \text{if } N_1(V) \neq \emptyset \\ 0 & \text{if } N_1(V) = \emptyset \, . \end{cases}$$

Consider the dual set

$$N_{n-1}(V) = \{ \, t \in \mathbb{G}^n_{n-1} \mid \mathbb{G}^n_k(t) \subset V \, \} \, .$$

If this set is not empty then the intersection of all planes belonging to it is a plane passing through of the point O (we suppose that O is a zero-dimensional plane). This plane will be denoted by $s_{n-1}(V)$. Define

$$n_{n-1}(V) = \begin{cases} \dim s_{n-1}(V) & \text{if } N_{n-1}(V) \neq \emptyset \\ n & \text{if } N_{n-1}(V) = \emptyset \, . \end{cases}$$

Proposition 6.1.1. *For any set $V \subset \mathbb{G}^n_k$ we have*

$$n_{n-1}(V) = n - n_1(\varphi^n_{k\,n-k}(V)) .$$

Proof. The equation (2.3.1) shows that the plane $s_{n-1}(V)$ is the orthogonal complement to the plane $s_1(\varphi^n_{k\,n-k}(V))$ and the set $N_1(\varphi^n_{k\,n-k}(V))$ is empty if and only if the set $N_{n-1}(V)$ is empty. ■

Proposition 6.1.2. *If a set $V \subset \mathbb{G}^n_k$ is irregular then it setisfies the following two conditions*

$$n_1(V) \le n - k \;\; \textit{and} \;\; n_{n-1}(V) \ge n - k .$$

Proof. Let $t_1, ..., t_{n_1(V)}$ be lines belonging to the set $N_1(V)$ and generating the plane $s_1(V)$. Consider lines $t_{n_1(V)+1}, ..., t_n$ such that $T = \{t_i\}^n_{i=1}$ is a maximal regular subset of $\mathbb{G}^n_1$. In the case when $n_1(V) > n - k$ each plane belonging to the maximal regular set $R = r^n_{1\,k}(T)$ contains at least one of the lines $t_1, ..., t_{n_1(V)}$. This implies the inclusion $R \subset V$ and the set V is not irregular.

The set $\varphi^n_{k\,n-k}(V)$ is an irregular subset of $\mathbb{G}^n_{n-k}$. We have proved that

$$n_1(\varphi^n_{k\,n-k}(V)) \le k .$$

Therefore second inequality is a consequence of Proposition 6.1.1. ■

6.1.2. Examples. Let $s \in \mathbb{G}^n_m$. Then

$$n_1(X^n_k(s)) = m \text{ if } m \le n - k$$

and

$$n_{n-1}(Y^n_k(s)) = m \text{ if } m \ge n - k$$

In the case $m < n - k$ any plane $t \in \mathbb{G}^n_{n-1}$ contains a k-dimensional plane intersecting s only in the point O. Therefore the inclusion

$$\mathbb{G}^n_k(t) \subset X^n_k(s)$$

does not holds for any $t \in \mathbb{G}^n_{n-1}$. This implies that

$$n_{n-1}(X^n_k(s)) = 0 \text{ if } m < n - k .$$

Proposition 6.1.1 and the equation (2.4.2) show that

$$n_1(Y^n_k(s)) = 0 \text{ if } m > n - k .$$

In the case $m = n - k$ the set $X^n_k(s)$ coincides with $Y^n_k(s)$ and

$$n_1(X^n_k(s)) = n_{n-1}(X^n_k(s)) = n - k\ .$$

In Chapter 7 we construct a maximal irregular set $V \subset \mathbb{G}^n_k$ satisfying the inequality $n(V) < n - k$.

The inclusion $V \subset W$ implies the inequalities

$$n_1(V) \leq n_1(W) \text{ and } n_{n-1}(V) \geq n_{n-1}(W)\ .$$

Therefore for any set $V \subset \mathbb{G}^n_k$ containing one of the sets $X^n_k(s)$ or $Y^n_k(s)$ we have

$$n_1(V) \geq \dim s \text{ or } n_{n-1}(V) \leq \dim s$$

respectively. In the next section we show that for maximal irregular sets the inverse statement holds true.

6.1.3. In the case $n \neq 2k$ the equalities

$$n_1(V) = n_1(W) \text{ and } n_{n-1}(V) = n_{n-1}(W)$$

holds for any similar subsets V and W of $\mathbb{G}^n_k$. For the case $n = 2k$ this statement fails. Consider, for example, the similar sets

$$V = X^{2k}_k(s) \quad s \in \mathbb{G}^n_{k-1}$$

and

$$W = Y^{2k}_k(s^\perp) = \varphi^{2k}_{k\,k}(X^{2k}_k(s))\ .$$

It is trivial that in this case we have $n_1(V) = k - 1$ and $n_1(W) = 0$.

6.2 Number characteristics and the structure of irregular sets. I

6.2.1. The section is devoted to prove the following statement.

Theorem 6.2.1. *For any maximal irregular set $V \subset \mathbb{G}^n_k$ the following two inclusions*

$$X^n_k(s_1(V)) \subset V$$

and

$$Y^n_k(s_{n-1}(V)) \subset V$$

hold true.

Proof. Consider first inclusion. In the case $n_1(V) = 0$ the set $X_k^n(s_1(V))$ is empty and our inclusion is trivial.

Let $n_1(V) \geq 1$. Assume that the inclusion fails and there exists a plane l belonging to the set

$$X_k^n(s_1(V)) \setminus V \ .$$

Let

$$t_1, ..., t_m \ , \ m = n_1(V)$$

be lines belonging to the set $N(V)$ and generating the plane $s_1(V)$. Lemma 2.2.1 implies the existence of a set $R' \subset V$ such that $R = R' \cup \{l\}$ is a maximal regular set. Lemma 2.4.1 shows that in the coordinate system associated with R there exist $n - m$ coordinate axes which are not conteined in the plane $s_1(V)$. Denote them by $t_{m+1}, ..., t_n$. Then $T = \{t_i\}_{i=1}^n$ is a maximal regular subset of $\mathbb{G}_1^n$. Now we prove the inclusion

$$R'' = r_{1,k}^n(T) \subset V$$

showing that the set V is not irregular.

The maximal regular set R'' can be represented as the union

$$R'' = R_1'' \cup R_2'' \ ,$$

where R_1'' is the set of all planes belonging to R'' and containing at least one of the lines $t_1, ..., t_m$ and $R_2'' = R'' \setminus R_1''$ is the set of all k-dimensional planes generated by the lines

$$t_{i_1}, ..., t_{i_k} \ , \quad m + 1 \leq i_1 < ... < i_k \leq n$$

(Proposition 6.1.1 shows that $m \leq n - k$). The set V contains R_1'' (since $t_1, ..., t_m$ are lines belonging to $N_1(V)$). It is not difficult to see that

$$R_2'' \subset R \setminus \{l\} = R' \subset V$$

and we have the required inclusion.

Consider the set $\varphi_{k\,n-k}^n(V)$. It is a maximal irregular subset of $\mathbb{G}_{n-k}^n$. We have proved above that

$$X_{n-k}^n(s_1(\varphi_{k\,n-k}^n(V))) \subset \varphi_{k\,n-k}^n(V) \ . \qquad (6.2.1)$$

It is easy to see that $s_1(\varphi_{k\,n-k}^n(V))$ is the orthogonal complement to $s_{n-1}(V)$ (see the proof of Proposition 6.1.1). Therefore second inclusion is a consequence of the equations (2.4.2) and (6.2.1). ■

6.2.2. Example. Now we show that for irregular subsets which are not maximal Theorem 3.2.1 fails. Consider planes $l \in \mathbb{G}^n_k$ and $s \in \mathbb{G}^n_{n-k}$ setisfying the condition

$$0 < \dim l \cap s < n - k \ . \tag{6.2.2}$$

Then the irregular set

$$V = X^n_k(s) \setminus \{l\}$$

is not maximal and first inclusion of Theorem 3.2.1 does not hold. We want to prove that $s_1(V) = s$.

The equation (6.2.2) implies the existence of $n-k$ linearly independend lines $l_1, ..., l_{n-k}$ belonging to $\mathbb{G}^n_1$ and setisfying the following conditions:

(i) the plane s is generated by $l_1, ..., l_{n-k}$;

(ii) each l_i $(i = 1, ..., n-k)$ does not coincide with the plane $l \cap s$.

The condition (ii) shows that $l \notin \mathbb{G}^n_k(l_i)$ and

$$\mathbb{G}^n_k(l_i) \subset V \quad \forall\, i = 1, ..., n-k \ .$$

Then the condition (i) implies that the plane s and $s_1(V)$ coincide.

6.2.3. Now we consider a few corollaries of Theorem 6.2.1.

Corollary 6.2.1. *If an irregular set $V \subset \mathbb{G}^n_k$ setisfies the condition*

$$n_1(V) = n - k$$

then $X^n_k(s_1(V))$ is unique maximal irregular set containing V. Therefore if the irregular set V is maximal then it coincides with $X^n_k(s_1(V))$.

Proof. Let W be a maximal irregular subset of $\mathbb{G}^n_k$ containing V. Then

$$s_1(V) \subset s_1(W) \text{ and } n_1(V) \leq n_1(W) \ .$$

Proposition 6.1.2 shows that $n_1(W) = n - k$ and the planes $s_1(V)$, $s_1(W)$ coincide. Theorem 6.2.1 implies that

$$X^n_k(s_1(V)) = X^n_k(s_1(W)) \subset W \ .$$

The statement (v) of Proposition 2.4.2 shows that the irregular set $X^n_k(s_1(W))$ is maximal and the inverse inclusion holds true. ■

Corollary 6.2.2. *If an irregular set $V \subset \mathbb{G}^n_k$ setisfies the conditions*

$$n_{n-1}(V) = n - k$$

then $Y^n_k(s_{n-1}(V))$ is unique maximal irregular set containing V. Therefore if the irregular set V is maximal then it coincides with $Y^n_k(s_{n-1}(V))$.

Proof. Let W be a maximal irregular subset of $\mathbb{G}^n_k$ containing V. Then $\varphi^n_{k\,n-k}(W)$ is a maximal irregular subset of $\mathbb{G}^n_{n-k}$ containing the irregular set $\varphi^n_{k\,n-k}(V)$. The plane $s_1(\varphi^n_{k\,n-k}(V))$ is the orthogonal complement to the plane $s_{n-1}(V)$ and

$$n_1(\varphi^n_{k\,n-k}(V)) = k \ .$$

Then Corollary 6.2.1 implies that

$$X^n_{n-k}(s_1(\varphi^n_{k\,n-k}(V))) = \varphi^n_{k\,n-k}(W)$$

and the equality (2.4.2) guarantees that

$$Y^n_k(s_{n-1}(V)) = W \ .$$

■

Corollaries 6.2.1, 6.2.2 and the statement (v) of Proposition 2.4.2 show that for any maximal regular set $V \subset \mathbb{G}^n_k$ the conditions

$$n_1(V) = n - k \text{ and } n_{n-1}(V) = n - k$$

are equivalent.

6.3 Number characteristics and the structure of irregular sets. II

Let V be an irregular subset of $\mathbb{G}^n_k$ and $t \in \mathbb{G}^n_m$. Define

$$V(t) = \varphi_t(V \cap \mathbb{G}^n_k(t)) \ .$$

In this section we study the set $V(t)$ and prove the following statement.

Theorem 6.3.2. *Let $s \in \mathbb{G}^n_m$. Then the next two statements holds true:*

(i) *if* $s \in \mathbb{G}^n_m$ *is a plane setisfing the condition* $s \subset s_1(V)$ *then for any plane* $t \in \mathbb{G}^n_{n-m}$ *transverse to* s *the set* $V(t)$ *is an irregular subset of* $\mathbb{G}^{n-m}_k$ *or a regular subset of* $\mathbb{G}^{n-m}_k$ *which is not maximal;*

(ii) *if* $s \in \mathbb{G}^n_m$ *is a plane setisfing the condition* $s_{n-1}(V) \subset s$ *then for any plane* $t \in \mathbb{G}^n_{n-m}$ *transverse to* s *the set* $V(t)$ *is an irregular subset of* $\mathbb{G}^m_{k-n+m}$ *or a regular subset of* $\mathbb{G}^m_{k-n+m}$ *which is not maximal.*

Proof. First of all prove the statement (i) for the case when the irregular set V is maximal. Then Theorem 6.2.1 implies the inclusion $X^n_k(s) \subset V$. Assume that our statement fails. Then there exists a plane $t \in \mathbb{G}^n_{n-m}$ transverse to s and such that the set $V(t)$ contains a maximal regular set R. Let

$$\{c_i\}_{i=1}^{n-m} = \varphi_t^{-1}(r^{n-m}_{k\,1}(R))$$

and $c_{n-m+1}, ..., c_n$ be a collection of m lines belonging to $\mathbb{G}^n_1$ and genereting the plane s. Then $C = \{c_i\}_{i=1}^n$ is a maximal regular subset of $\mathbb{G}^n_1$. Show that V contains the maximal regular set $R' = r^n_{1\,k}(C)$. This implies that the set V is not irregular.

Represent R' as the union

$$R' = R'_1 \cup R'_2 \ ,$$

where R'_1 is the set of all planes belonging to R' and containing at least one of the lines $c_{n-m+1}, ..., c_n$ and $R'_2 = R' \setminus R'_1$ is the set of all planes generated by the lines

$$c_{i_1}, ..., c_{i_k} \ , \quad 1 \le i_1 < ... < i_k \le n-m$$

(Proposition 6.1.2 shows that $m < n-k$). Then

$$R'_1 \subset X^n_k(s) \subset V$$

and

$$R'_2 = \varphi_t^{-1}(R) \subset V \ .$$

This implies that $R' \subset V$.

If the irregular set V is not maximal then consider a maximal irregular set $W \subset \mathbb{G}^n_k$ containing it. For this set our statement is proved. Then the inclusions

$$s \subset s_1(V) \subset s_1(W)$$

and $V \subset W$ show that the required statement holds for the set V.

In the case when $s_{n-1}(V) \subset s$ consider the irregular set $\varphi^n_{k\,n-k}(V) \subset \mathbb{G}^n_{n-k}$ and the plane $s^\perp \in \mathbb{G}^n_{n-m}$. The plane $s_1(\varphi^n_{k\,n-k}(V))$ is the orthogonal complement to the plane $s_{n-1}(V)$. Therefore

$$s^\perp \subset s_1(\varphi^n_k(V)) .$$

For any plane $t \in \mathbb{G}^n_{n-m}$ transverse to s the plane $t^\perp \in\in \mathbb{G}^n_m$ is transverse to $s^\perp$. Proposition 6.3.1 shows that

$$\varphi_{t^\perp}(\varphi^n_{k\,n-k}(V) \cap \mathbb{G}^n_{n-k}(t^\perp)) = \varphi_{t^\perp}\varphi^n_{k\,n-k}(V \cap \mathbb{G}^n_k(t))$$

is an irregular subset of $\mathbb{G}^{n-m}_{n-k}$ or a regular subset of $\mathbb{G}^{n-m}_{n-k}$ which is not maximal. Recall that in this case

$$\varphi_t = \varphi^{n-m}_{n-k\,k-m}\varphi_{t^\perp}\varphi^n_{k\,n-k}$$

(see Subsection 2.3.2). Therefore the set $V(t)$ setisfies the required conditions. ■

Corollary 6.3.1. *Let V be an irregular subset of $\mathbb{G}^n_k$ and there exist two planes s_1 and s_2 belonging to $\mathbb{G}^n_{m_1}$ and $\mathbb{G}^n_{m_2}$ ($m_1 \le n-k$, $m_2 \ge n-k$) respectively and such that*

$$X^n_k(s_1) \subset V\ ,\ Y^n_k(s_2) \subset V\ .$$

Then $s_1 \subset s_2$.

Proof. Consider a line $p \in \mathbb{G}^n_1$ which is not contained in the plane s_2. There exists a plane $t \in \mathbb{G}^n_{n-1}$ transverse to p and containing s_2. Then $\mathbb{G}^n_k(t) \subset V$ and the set $V(t)$ coincides with $\mathbb{G}^{n-1}_k$. Therefore the plane s_1 does not contain p and we have the required inclusion (otherwise Theorem 6.3.1 shows that the set $V(t)$ does not coincide with $\mathbb{G}^{n-1}_k$). ■

6.4 One class of irregular sets which elements are sets of first category

6.4.1. Here we exploit the number characteristics introduced in Section 6.1 to obtain one class of irregular sets which elements are sets of first category.

Recall that a subset A of some topological space X is called a set of *first category* if it can be represented as a countable uniom of nowhere dense subsets of X. Otherwise we say that A is a set of *second category*.

Proposition 6.4.1. *An irregular set $V \subset \mathbb{G}_k^n$ satisfying one of the following condition*

$$n_1(V) \geq n - k - 1$$

or

$$n_{n-1}(V) \leq n - k + 1$$

is a set of first category in $\mathbb{G}_k^n$.

6.4.2. Proof of Proposition 6.4.1. Show that Proposition 6.4.1 is a consequence of Theorem 6.3.1, Proposition 2.4.1 and the following lemma.

Lemma 6.4.1. *Let $V \subset \mathbb{G}_k^n$ and there exists a line $p \in \mathbb{G}_1^n$ such that for any plane $t \in \mathbb{G}_{n-1}^n$ transverse to p the set $V(t)$ is a set of first category in $\mathbb{G}_k^{n-1}$. Then V is a set of first category in $\mathbb{G}_k^n$.*

Remark 6.4.1. The map $\varphi_t : \mathbb{G}_k^n(t) \to \mathbb{G}_k^{n-1}$ is a homeomorphism. Therefore $V(t)$ is a set of first category in $\mathbb{G}_k^{n-1}$ if and only if $\mathbb{G}_k^n(t) \cap V$ is a set of first category in $\mathbb{G}_k^n(t)$.

Consider the case $n_1(V) \geq n - k - 1$. Define $m = n - k - 1$. If $m = 1$ then we have $n = k + 2$ and Proposition 6.3.1 implies that for any plane $t \in \mathbb{G}_{n-1}^n$ transverse to $s_1(V) \in \mathbb{G}_1^n$ the set $V(t)$ is an irregular subset of $\mathbb{G}_k^{k+1}$. Then Proposition 2.4.1 shows that this set is nowhere dense in $\mathbb{G}_k^{k+1}$ and our statement is a consequence of Lemma 6.4.1.

Assume that the statement holds for any number m such that $m < m_0$ ($m_0 > 1$) and consider the case $m = m_0$. Fix a line $p \in \mathbb{G}_1^n$ belonging to the set $N_1(V)$ (in other words p is a line lying in the plane $s_1(V)$) and for any $t \in \mathbb{G}_{n-1}^n$ transverse to p define

$$s(t) = \varphi_t(s \cap t) \ .$$

It is easy to see that $s(t) \in \mathbb{G}_{m_0-1}^{n-1}$. Theorem 6.2.1 shows that $X_k^n(s_1(V)) \subset V$. Therefore

$$X_k^{n-1}(s(t)) \subset V(s(t))$$

for each $t \in \mathbb{G}_{n-1}^n$ transverse to the line p. Theorem 6.3.1 implies that $V(t)$ is an irregular subset of $\mathbb{G}_k^{n-1}$. Then the inductive hypothesis and Lemma 6.4.1 guarantee the fulfilment of the required statement for the case $m = m_0$.

In the case $n_{n-1}(V) \le n-k+1$ consider the set irregular $\varphi^n_{k\,n-k}(V) \subset \mathbb{G}^n_{n-k}$. Proposition 6.1.1 shows that it satisfies the condition

$$n_1(\varphi^n_{k\,n-k}(V)) \ge k-1 = n-(n-k)-1 \ .$$

Therefore $\varphi^n_{k\,n-k}(V)$ is a set of first category in $\mathbb{G}^n_{n-k}$. The map $\varphi^n_{k\,n-k}$ is a homeomorphism and V is a set of first category in $\mathbb{G}^n_k$. ■

6.4.3. Proof of Lemma 6.4.1. We prove Lemma 6.4.1 in the following convenient form.

Lemma 6.4.2. *Let $V \subset \mathbb{G}^n_k$ and there exists plane $s \in \mathbb{G}^n_{n-1}$ such that for any line $t \in \mathbb{G}^n_1$ transverse to s the set $V(t)$ is a set of first category in $\mathbb{G}^{n-1}_{k-1}$. Then V is a set of first category in $\mathbb{G}^n_k$.*

It is not difficult to see that Lemma 6.4.1 is a consequence of Lemma 6.4.2.

Proof of Lemma 6.4.2. Let $\{x_i\}^n_{i=1}$ be a coordinate system for $\mathbb{R}^n$ such that s is the coordinate plane generated by the axes $x_2, ..., x_n$. Denote by s' the $(n-k)$-dimensional coordinate plane generated by the axes $x_{k+1}, ..., x_n$. Recall that any plane l belonging to the open everywhere dense set

$$U^n_k(s') = \mathbb{G}^n_k \setminus X^n_k(s')$$

can be represented as a linear map of $\mathbb{R}^k$ into $\mathbb{R}^{n-k}$. Moreover, the map

$$A^n_k(s') : U^n_k(s') \to \mathbb{R}^{k(n-k)}$$

comparing to $l \in U^n_k(s')$ the matrix

$$A^n_k(s')(l) = \{x_{i\,j}\}^{n-k\,k}_{i=1\;j=1}$$

of this linear map in the coordinates $\{x_i\}^k_{i=1}$, $\{x_i\}^n_{i=k+1}$ is a homeomorphism (see Chapter 1).

Represente our set V as the union

$$V = V_1 \cup V_2 \ ,$$

where

$$V_1 = V \cap X^n_k(s')$$

and

$$V_2 = V \cap U^n_k(s') = V \setminus X^n_k(s') \ .$$

It is trivial that V_1 is nowhere dense subset of $\mathbb{G}_k^n$. Therefore V is a set of first ctegory if and only if V_2 is a set of first ctegory. In what follows we show that

$$V' = A_k^n(s')(V_2)$$

is a set of first category in $\mathbb{R}^{k(n-k)}$. This implies that V_2 and V are sets of first category in $\mathbb{G}_k^n$.

For any

$$a = (a_1, ..., a_{n-k}) \in \mathbb{R}^{n-k}$$

denote by $t(a)$ the line belonging to $\mathbb{G}_1^n$ and containing the vector

$$(1, 0, ..., 0, a_1, ..., a_{n-k}) \ .$$

Then for any $l \in \mathbb{G}_k^n(t(a))$ the linear map $A_k^n(s')(l)$ transfers the vector

$$(1, 0, ..., 0) \in \mathbb{R}^k$$

to a. Therefore

$$A_k^n(s')(\mathbb{G}_k^n(t(a)))$$

is the $(k-1)(n-k)$-dimensional plane in $\mathbb{R}^{k(n-k)}$ defined by the conditions:

$$x_{1\,1} = a_1, ..., x_{n-k\,1} = a_{n-k} \ .$$

Denote this plane by $L(a)$.

For any $a \in \mathbb{R}^{n-k}$ the line $t(a)$ is not contained in s' and

$$\mathbb{G}_k^n(t(a)) \cap V_2 \subset \mathbb{G}_k^n(t(a)) \cap V$$

is a set of first category in $\mathbb{G}_k^n(t(a))$. Then

$$V' \cap L(a) = A_k^n(s')(\mathbb{G}_k^n(t(a)) \cap V_2)$$

is a set of first category in

$$L(a) \approx \mathbb{R}^{(k-1)(n-k)}$$

and the following lemma guarantees that V' is a set of first category in $\mathbb{R}^{k(n-k)}$.

Lemma 6.4.3. *Let X be a set of first category in $\mathbb{R}^n$. Then $X \times \mathbb{R}^m$ is a set of first categry in $\mathbb{R}^{n+m}$.*

Remark 6.4.2. There is a more complicated statement which is called the Kuratovski – Ulam Theorem (see [11]). Lemma 6.4.3 is a trivial consequence of it.

Proof. Let

$$X \subset \bigcup_{i=1}^{\infty} X_i \ ,$$

where each X_i is a closed nowhere dense subset of $\mathbb{R}^n$. Then

$$X \times \mathbb{R}^m \subset \bigcup_{i=1}^{\infty} X_i \times \mathbb{R}^m$$

and each $X_i \times \mathbb{R}^m$ is a closed nowhere dense subset of $\mathbb{R}^{n+m}$. ■

Remark 6.4.3. The similar arguments show that the next statement holds true: if $V \subset \mathbb{G}^n_k$ and there exists plane $s \in \mathbb{G}^n_m$ $(m \neq k)$ such that for any plane $t \in \mathbb{G}^n_{n-m}$ transverse to s the set $V(t)$ is a set of first category in $\mathbb{G}^{n-m}_k$ $(k > m)$ or $\mathbb{G}^{n-m}_{k-m}$ $(k < m)$ then V is a set of first category in $\mathbb{G}^n_k$.

Chapter 7

Proof of Theorem 3.4.1

In the chapter we construct a maximal irregular subset of $\mathbb{G}_k^n$ $(1 < k < n-1)$ which is not similar to maximal irregular subsets of $\mathbb{G}_k^n$ considered in Section 2.4.

7.1 One special class of irregular subsets of $\mathbb{G}_{n-2}^n$

7.1.1. Let us consider a set $V \subset \mathbb{G}_{n-2}^n$ setisfies the condition

($\mathcal{S}$) for any plane $s \in \mathbb{G}_{n-1}^n$ the set $V(s)$ is a maximal irregular subset of $\mathbb{G}_{n-2}^{n-1}$

(see Section 6.3).

Lemma 7.1.1. *Any subset of $\mathbb{G}_{n-2}^n$ satisfying the condition ($\mathcal{S}$) is irregular. However, this irregular set is not maximal.*

Proof. Let V be a subset of $\mathbb{G}_{n-2}^n$ satisfying the condition ($\mathcal{S}$). Assume that the set V is not irregular. Then it contains a maximal regular set R. It is not difficult to see that for any plane s belonging to $r_{n-2,n-1}^n(R)$ the set $V(s)$ contains the maximal regular set $\varphi_s(R(s))$. This implies that $V(s)$ is not irregular and V does not satisfy the condition ($\mathcal{S}$).

Consider a maximal irregular set $V \subset \mathbb{G}_{n-2}^n$ and some plane l belonging to $\mathbb{G}_{n-2}^n \setminus V$. Lemma 2.2.1 guarantees the existence of a set $R' \subset V$ such that $R = R' \cup \{l\}$ is maximal regular set. For any plane $s \in r_{n-2,n-1}^n(R)$ which does not contain l the set $V(s)$ is not irregular (it contains the maximal regular set $\varphi_s(R(s))$). Therefore the set V does not satisfy the condition $(\mathcal{S})$. ■

Proposition 2.4.1 shows that for any $s \in \mathbb{G}^n_{n-1}$ there exists a line $t(s) \in \mathbb{G}^n_1$ contained in s and such that

$$V(s) = \mathbb{G}^{n-1}_{n-2}(\varphi_s(t(s))) \ .$$

Consider the map

$$F_V : \mathbb{G}^n_{n-1} \to \mathbb{G}^n_1 \ ,$$
$$F_V(s) = t(s) \quad \forall s \in \mathbb{G}^n_{n-1} \ .$$

Lemma 7.1.2. *For any set $V \subset \mathbb{G}^n_{n-2}$ setisfying the condition $(\mathcal{S})$ the map F_V satisfies the following two conditions:*

(S'_1) *for any plane $s \in \mathbb{G}^n_{n-1}$ the line $F_V(s)$ are contained in s;*

(S'_2) *for any two planes s_1 and s_2 belonging to $\mathbb{G}^n_{n-1}$ the line $F_V(s_1)$ is contained in s_2 if and only if $F_V(s_2)$ is a line contained in s_1.*

Proof. First condition is trivial. Show that second condition holds true. Consider the plane $l = s_1 \cap s_2$ belonging to $\mathbb{G}^n_{n-2}$. In the case when $l \in V$ the sets $V(s_1)$ and $V(s_2)$ contains the planes $\varphi_{s_1}(l)$ and $\varphi_{s_2}(l)$ respectively. Therefore l contains the lines $F_V(s_1)$ and $F_V(s_2)$. If $l \notin V$ then $V(s_1)$ does not contain the planes $\varphi_{s_1}(l)$ and $V(s_2)$ does not contain the planes $\varphi_{s_2}(l)$. In other words l does not contain the lines $F_V(s_1)$ and $F_V(s_2)$. This implies the required condition. ■

Now define

$$f_V = F_V \varphi^n_{n-1\,1} \ .$$

Lemma 7.1.2 implies that the map $f_V : \mathbb{G}^n_1 \to \mathbb{G}^n_1$ satisfies the conditions:

(S_1) $f_V(t) \perp t$ (the lines t and $f_V(t)$ are orthogonal) for any $t \in \mathbb{G}^n_1$;

(S_2) for any two lines t_1 and t_2 belonging to $\mathbb{G}^n_1$ we have $f_V(t_1) \perp t_2$ if and only if $f_V(t_2) \perp t_1$.

The conditions (S_1) and (S_2) are trivial consequences of the conditions (S'_1) and (S'_2) respectively.

It is not difficult to see that each map $f : \mathbb{G}^n_1 \to \mathbb{G}^n_1$ setisfying the conditions (S_1) and (S_2) defines a set $V(f) \subset \mathbb{G}^n_{n-2}$ setisfying the condition

$(\mathcal{S})$. Moreover sets V and W setisfying the condition $(\mathcal{S})$ coinside if and only if $f_V = f_W$. Therefore there is an one-to-one correspondence between the class of subsets of $\mathbb{G}^n_{n-2}$ satisfying the condition $(\mathcal{S})^k_{n-2}$ and the class of maps of $\mathbb{G}^n_1$ into $\mathbb{G}^n_1$ satisfying the conditions (S'_1) and (S'_2).

Now we show that in the case $n = 2k$ the class of all subset of $\mathbb{G}^n_{n-2}$ setisfying the condition $(\mathcal{S})$ is not empty. Consider the linear map of $\mathbb{R}^{2k}$ onto $\mathbb{R}^{2k}$ defined by the matrix

$$\begin{pmatrix} 0 & -1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -1 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

This map induces the linear map $f_0 : \mathbb{G}^{2k}_1 \to \mathbb{G}^{2k}_1$ satisfying the conditions (S_1) and (S_2). The map f_0 defines a set $V_0 \subset \mathbb{G}^{2k}_{2k-2}$ satisfying the condition $(\mathcal{S})$.

7.1.2. Remarks. Now we announce one result wich will be not proved here.

In the case $n \neq 2k$ there are not subsets of $\mathbb{G}^n_{n-2}$ setisfying the condition $(\mathcal{S})^n_{n-2}$. If $n = 2k$ then the class of subsets of $\mathbb{G}^n_{n-2}$ setisfying the condition $(\mathcal{S})$ is not empty and any two sets belonging to it are similar.

This result is a consequence of the following statement. In the case $n \neq 2k$ there are not maps of $\mathbb{G}^n_1$ into $\mathbb{G}^n_1$ satisfying the conditions (S_1) and (S_2). If $n = 2k$ then for any map $f : \mathbb{G}^n_1 \to \mathbb{G}^n_1$ satisfying the conditions (S_1) and (S_1) there exists an orthogonal linear map g of $\mathbb{R}^n$ onto $\mathbb{R}^n$ such that

$$f = L^n_1(g^{-1}) f_0 L^n_1(g)\ .$$

7.1.2. Proof of Theorem 3.4.1 for the case when $n = 2i$ and $k = 2, n-2$. Presuppose that $n = 2k$ and consider a maximal irregular subset of $\mathbb{G}^n_{n-2}$ containing a set setisfying the condition $(\mathcal{S})$.

Lemma 7.1.3. *For any $s \in \mathbb{G}^n_2$ the set $X^n_{n-2}(s)$ does not contain V_0* (V_0 is the set introduced in Subsection 7.1.1).

Proof. Assume that the inclusion $V_0 \subset X^n_{n-2}(s)$ holds for some $s \in \mathbb{G}^n_2$. Then for any $t \in \mathbb{G}^n_{n-1}$ the line $f_0(t)$ is contained in the plane s. However, an immediate verification shows that $f(\mathbb{G}_1^n) = \mathbb{G}_1^n$. Therefore our hypothesis fails. ■

Lemma 7.1.3 and Remark 3.4.1 guarantee that each maximal irregular subset of $\mathbb{G}_{n-2}^n$ containing V_0 is not similar to $X_{n-2}^n(s)$ ($s \in \mathbb{G}_2^n$). Then Lemma 3.4.1 show that any maximal irregular subset of $\mathbb{G}_2^n$ containing $\varphi_{n-2\,2}^n(V_0)$ is not similar to $X_{n-2}^n(s')$ ($s' \in \mathbb{G}_{n-2}^n$). ■

7.2 Proof of Theorem 3.4.1 for the general case

7.2.1. In the next subsection we show that Theorem 3.4.1 is a consequence of the following lemma.

Lemma 7.2.1. *For any maximal irregular set $V' \subset \mathbb{G}_k^m$ and any plane $t \in \mathbb{G}_m^n$ there exists a maximal irregular set $V \subset \mathbb{G}_k^n$ such that $V(t) = V'$.*

Proof. Fix some plane $s \in \mathbb{G}_{n-m}^n$ transverse to t and define

$$V'' = X_k^n(s) \cup \varphi_t^{-1}(V) \ .$$

It is not difficult to see that V'' is an irregular subset of $\mathbb{G}_k^n$. Consider a maximal irregular subset V of $\mathbb{G}_k^n$ containing V''. Theorem 6.3.1 implies that $V(t)$ is an irregular subset of $\mathbb{G}_k^m$ containing V'. The irregular set V' is maximal. Therefore $V(t)$ coincides with V'. ■

7.2.2. In the previous section Theorem 3.4.1 was proved for the case when $n = 2i$ and $k = 2, n-2$. Now we consider the case $n = 2i+1$. Fix a plane $t \in \mathbb{G}_{n-1}^n$ and a maximal irregular set $V' \subset \mathbb{G}_2^{n-1}$ which is not similar to $X_2^{n-1}(s)$ ($s \in \mathbb{G}_{n-3}^{n-1}$). Let V be a maximal irregular subset of $\mathbb{G}_2^n$ setisfying the condition $V(t) = V'$. Then the following simple lemma implies that for any $s \in \mathbb{G}_{n-2}^n$ the set $X_2^n(s)$ does not coincide with V.

Lemma 7.2.2. *If $1 < k < n-1$ and $V = X_k^n(s)$ ($s \in \mathbb{G}_{n-k}^n$) then for any $t \in \mathbb{G}_{n-1}^n$ one of the equalities*

$$V(t) = \mathbb{G}_k^{n-1}$$

or

$$V(t) = X_k^{n-1}(s') \quad (s \in \mathbb{G}_{n-k-1}^n)$$

holds true.

Proof. Proposition 2.4.2 shows that first equality holds if the plane t contains the plane s. Otherwise, we have second equality. ■

Remark 3.4.1 guarantees that for any $s \in \mathbb{G}^n_{n-2}$ the sets $X^n_2(s)$ and V are not similar.

We have proved that in the case $k = 2$ there exists a maximal irregular subset of $\mathbb{G}^n_k$ which is not similar to $X^n_k(s)$ ($s \in \mathbb{G}^n_{n-k}$). Lemma 3.4.1 implies that this statement holds for the case $k = n - 2$.

For the gneral case the proof is analogous. Define $i = n - k$. For the case $i = 2$ our statement is proved. Assume that it holds for any i such that $i \leq i_0$ ($i_0 > 2$). Then Lemmas 7.2.1, 7.2.2 and the inductive hypothesis imply the required statement. ■

Remark 7.2.1. We have proved the existence of a maximal irregular set $V \subset \mathbb{G}^n_k$ which does not coincide with $X^n_k(s)$ ($s \in \mathbb{G}^n_{n-k}$). Theorem 6.2.1 shows that this irregular set setisfies the condition $n_1(V) \leq n - k$.

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