On Super and Restricted Connectivity of Some Interconnection Networks*

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Abstract

The super (resp., edge-) connectivity of a connected graph is the minimum cardinality of a vertex-cut (resp., an edge-cut) whose removal does not isolate a vertex. In this paper, we consider the two parameters for a special class of graphs $G(G_0, G_1; M)$, proposed by Chen et al [Applied Math. and Computation, 140 (2003), 245-254], obtained from two $k$-regular $k$-connected graphs $G_0$ and $G_1$ with the same order by adding a perfect matching between their vertices. Our results improve ones of Chen et al. As applications, the super connectivity and the super edge-connectivity of the $n$-dimensional hypercube, twisted cube, cross cube, Möbius cube and locally twisted cube are all $2^n - 2$.

Keywords: Connectivity; super connectivity; restricted connectivity; hypercubes; twisted cubes; cross cubes; Möbius cubes; locally twisted cubes

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1 Introduction

We follow [19] for graph-theoretical terminology and notation not defined here. Throughout this paper, a graph $G = (V, E)$ always means a simple graph (without loops and multiple edges), where $V = V(G)$ is the vertex-set and $E = E(G)$ is the edge-set. The symbols $K_{1,n-1}$ and $K_n$ denote a star graph and a complete graph with order $n$, respectively. For a subset $X \subset V(G)$, the symbol $\partial_G(X)$ the set of edges incident with some vertex

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in $X$. $\xi(G) = \min\{d_G(x) + d_G(y) : e = xy \in E(G)\} - 2$ is the minimum edge-degree of $G$.

It is well known that when the underlying topology of an interconnection network is modelled by a connected graph $G$, the connectivity $\kappa(G)$ and the edge-connectivity $\lambda(G)$ are two important measurements for fault-tolerance of the network [18]. The two parameters, however, have a obvious deficiency, that is to tacitly assume that all elements in any subset of $G$ can potentially fail at the same time. To compensate for this shortcoming, Bauer et al [4] suggested the concept of the super connectedness. A connected graph is said to be super vertex-connected (resp., super edge-connected), if every minimum vertex-cut (resp., edge-cut) isolates a vertex. Many super connected graphs have been found in the literature (see, for example, [1, 2, 3, 4, 5, 6, 7, 9, 10, 13, 15, 16]). A quite natural problem is that if a connected graph $G$ is super vertex-connected or super edge-connected then how many vertices or edges must be removed to disconnect $G$ such that every component of the resulting graph contains no isolated vertices. This problem results in the concept of the super connectivity, introduced in [13] (see also [3, 15]).

A subset $F \subset V(G)$ is said to be nontrivial if it contains no $N_G(x)$ as its subset for some vertex $x \in V(G) \setminus F$, and a subset $B \subset E(G)$ is said to be nontrivial if it contains no $\partial_G(x)$ as its subset for some vertex $x \in V(G)$. A nontrivial vertex-set (resp., edge-set) $S$ is called a nontrivial vertex-cut (resp., edge-cut) if $G - S$ disconnected. The super vertex-connectivity $\kappa_s(G)$ (resp., edge-connectivity $\lambda_s(G)$) of a connected graph $G$ is defined as the minimum cardinality of a nontrivial vertex-cut (resp. edge-cut) if $G$ has a nontrivial vertex-cut (resp., a nontrivial edge-cut), and does not exist otherwise, denoted by $\infty$.

Esfahanian and Hakimi [11, 12] generalized the notion of connectivity by introducing the concept of the restricted connectivity in point of view of network applications. A set $S \subset V(G)$ (resp., $S \subset E(G)$) is called a restricted vertex-set (resp., edge-set) if it contains no $N_G(x)$ (resp., $\partial_G(x)$) as its subset for any vertex $x \in V(G)$. A restricted vertex-set (resp., edge-set) $S$ is called a restricted vertex-cut (resp., edge-cut) if $G - S$ is disconnected. The restricted vertex-connectivity (resp., edge-connectivity) of a connected graph $G$, denoted by $\kappa_r(G)$ (resp., $\lambda_r(G)$), is defined as the minimum cardinality of a restricted vertex-cut (resp., edge-cut) if $G$ has a restricted vertex-cut (resp., edge-cut), and does not exist otherwise.

The four parameters $\kappa_s, \kappa_r, \lambda_s$ and $\lambda_r$ in conjunction with $\kappa$ and $\lambda$ can provide more accurate measurements for fault tolerance of a large-scale interconnection network. What relationships exist between $\kappa_s$ and $\kappa_r$, $\lambda_s$ and $\lambda_r$?

From definitions, there is no difference between two concepts of nontrivial edge-cuts and restricted edge-cuts, and so $\lambda_s(G) = \lambda_r(G)$ for any
graph $G$ provided they exist. There are a number of journal papers on $\lambda_s(G)$ or $\lambda_r(G)$ (see, for example, [2, 3, 11, 12, 14, 17, 20]), due to the fact Esfahanian and Hakimi [12] solved the existence of $\lambda_r(G)$ for a graph $G$ by proving the following proposition.

**Proposition 1** If $G$ is neither $K_{1,n}$ nor $K_3$, then $\lambda(G) \leq \lambda_r(G) \leq \xi(G)$.

![Diagram](image.png)

Figure 1: $\kappa_r(G)$ does not exist, while $\kappa_s(G) = 3$

However, there is a slightly difference between two concepts of nontrivial vertex-cuts and restricted vertex-cuts. For example, consider the graph $G$ shown in Figure 1. The graph $G$ has a unique nontrivial vertex-cut $S = \{x_2, x_5, x_7\}$, and no restricted vertex-cut, and so $\kappa_s(G) = 3$. A unique possible restricted vertex-cut is also $S$, however, it contains $N_G(x_7)$ as a subset. Thus, $\kappa_r(G)$ does not exist. Up to now, a few results on $\kappa_s(G)$ and $\kappa_r(G)$ for a graph $G$ have been known. Indeed, the existence of $\kappa_s(G)$ and $\kappa_r(G)$ has not been yet solved for a general graph $G$. However, for a graph $G$ if $\kappa_r(G)$ exists then $\kappa_s(G)$ exists and $\kappa_s(G) \leq \kappa_r(G)$ since any restricted vertex-cut is certainly a nontrivial vertex-cut. Conversely, if $\kappa_s(G)$ does not exist then $\kappa_r(G)$ does not exist. The following proposition holds obviously, which shows relationships between the super connectivity and the restricted connectivity.

**Proposition 2** Let $G$ be a connected graph, neither $K_{1,n}$ nor $K_3$. Then

1. $\kappa_r(G) \geq \kappa_s(G) \geq \kappa(G)$, and if $\kappa_s(G) > \kappa(G) = \delta(G)$ then $G$ is super-connected.
2. $\lambda_r(G) = \lambda_s(G) \geq \lambda(G)$, and if $\lambda_s(G) > \lambda(G) = \delta(G)$ then $G$ is super edge-connected.

In this paper, we consider a special class of graphs, proposed by Chen et al [10]. Let $G_0$ and $G_1$ be two $k$-regular $k$-connected graphs with $n$ vertices, and $M$ be an arbitrary perfect matching between the vertices of $G_0$ and $G_1$. The graph $G(G_0, G_1; M)$ is defined as a graph $G$ with the vertex-set...
\( V(G) = V(G_0) \cup V(G_1), \) and the edge-set \( E(G) = E(G_0) \cup E(G_1) \cup M. \) We will call the edges in \( M \) cross-edges. The well-known \( n \)-dimensional hypercube \( Q_n \), the twisted cube \( TQ_n \), the cross cube \( CQ_n \), the Möbius cube \( MQ_n \) and the locally twisted cube \( LTQ_n \), each of them can be viewed as a special \( G(G_0, G_1; M) \) for some two graphs \( G_0, G_1 \) and some perfect matching \( M \). Chen et al.\cite{Ref10} have shown that \( G(G_0, G_1; M) \) is super vertex-connected if and only if either \( n > k + 1 \) or \( n = k + 1 \) with \( k = 2 \), and is super edge-connected if and only if \( n > k + 1 \). Applying these results, they proved \( Q_n, TQ_n, CQ_n \) and \( MQ_n \) all are super vertex-connected and super edge-connected.

We, in the present paper, study the super connectivity and the restricted connectivity of the graph \( G(G_0, G_1; M) \). As applications, we determine these parameters for \( Q_n, TQ_n, CQ_n, MQ_n \) and \( LTQ_n \) all being \( 2n - 2 \). The above-mentioned Chen et al.’s results will be referred to as direct consequences of our results.

\section{Main Results}

The following theorem holds obviously, the proof is omitted here.

\textbf{Theorem 1} \( \kappa(G(G_0, G_1; M)) = \lambda(G(G_0, G_1; M)) = k + 1 \) if and only if \( n \geq k + 1 \) for any \( k \geq 1 \).

\textbf{Theorem 2} Let \( G = G(G_0, G_1; M) \) and \( k \geq 2 \). Then
\begin{enumerate}
  \item \( \kappa_s(G) = \kappa_r(G) = k + 1 \) if and only if \( n = k + 1 \) with \( k \geq 3 \);
  \item \( k + 1 < \kappa_s(G) \leq \kappa_r(G) \leq 2k \) if and only if either \( n \geq k + 2 \); and
  \item \( \kappa_s(G) = \kappa_r(G) = 2k \) if each of \( G_0 \) and \( G_1 \) contains no triangles.
\end{enumerate}

\textbf{Proof} (1) Assume \( \kappa_s(G) = \kappa_r(G) = k + 1 \). Then \( n \geq k + 1 \) with \( k \geq 2 \) and there is a vertex-cut \( S \) with \( |S| = k + 1 \) such that every component of \( G - S \) contains no isolated vertices. Clearly, no component of \( G - S \) is included in both \( G_0 \) and \( G_1 \) since \( \kappa(G_i) = k \) for \( i = 1, 2 \) and \( 2k > k + 1 \) for \( k \geq 2 \). Moreover, \( G - S \) contains exactly two components, the one in \( G_0 \) and the other in \( G_1 \). Let \( H \) be the only component of \( G - S \) that is included in \( G_0 \). Then \( N_G(H) = S \). Because every vertex in \( H \) is matched by \( M \) with a vertex in \( G_1 \), we have \( |V(H)| = |N_G(H) \cap V(G_1)| \). It follows that
\begin{align*}
  k + 1 & \leq n = |V(G_0)| = |V(H)| + |S \cap V(G_0)| \\
  & = |N_G(H) \cap V(G_1)| + |S \cap V(G_0)| \\
  & \leq |S| = k + 1,
\end{align*}
which gives \( n = k + 1 \). Thus, both \( G_0 \) and \( G_1 \) are isomorphic to \( K_{k+1} \) since they are \( k \)-regular. Since \( G - S \) has at least two components and every component has at least two vertices, thus, \( 2k + 2 = 2n \geq (k + 1) + 4 = k + 5 \), which yields \( k \geq 3 \).
Conversely, we clearly have $k + 1 = \kappa(G) \leq \kappa_s(G) \leq \kappa_r(G)$ by Theorem 1 and Proposition 2. We want to show $\kappa_r(G) \leq k + 1$ if $n = k + 1$ with $k \geq 3$. Let $u$ and $v$ be any two adjacent vertices in $G_0$ and $S = N_G(u, v)$. Then $|S| = k + 1$ since both $G_0$ and $G_1$ both are isomorphic to a complete graph $K_{k+1}$. Moreover, $G - S$ is disconnected since $2n - (k + 1) - 2 \geq 2$ for $k \geq 3$. It is easy to verify that $S$ is a restricted vertex-cut of $G$. Thus, $\kappa_r(G) \leq |S| = k + 1$.

(2) We note that $k \geq 2$ is required only by $n \geq k + 2$ and to ensure $2k > k$. By the conclusion in (1), we only need to show that $\kappa_r(G) \leq |S| \leq 2k$ for $n \geq k + 2$. To this aim, let us arbitrarily choose two adjacent vertices $u$ and $v$ in $G_0$ and $S = N_G(u, v)$. Then, $|S| \leq 2k$. Clearly, $G - S$ is disconnected since $2n - (2k) - 2 \geq 2$ for $n \geq k + 2$. Since for every vertex in $G$, at least one of its neighbors is not in $S$, $S$ is a restricted vertex-cut of $G$. This shows that $\kappa_r(G)$ exists and $\kappa_r(G) \leq |S| \leq 2k$.

(3) Clearly, the hypothesis that $G_i$ contains no triangles and $k \geq 2$ implies $n \geq k + 2$. By the conclusion in (2), which also implies the existence of $\kappa_s(G)$, we only need to prove $\kappa_s(G) \geq 2k$ if each of $G_0$ and $G_1$ contains no triangles. To the end, we only need to show that for any nontrivial vertex-set $F$ in $G$, if $|F| \leq 2k - 1$ then then $G - F$ is connected.

Let $F_0 = F \cap V(G_0)$, and $F_1 = F \cap V(G_1)$. Obviously, $F_0 \cap F_1 = \emptyset$. Thus, either $|F_0| \leq k - 1$ or $|F_1| \leq k - 1$. We can, without loss of generality, suppose that $|F_1| \leq k - 1$. Then $G_1 - F_1$ is connected since $\kappa(G_1) = k$.

We show that any vertex $u_0$ in $G_0 - F_0$ can be connected to the connected graph $G_1 - F_1$. Let $u_0u_1$ be a cross-edge, where $u_1 \in V(G_1)$. If $u_1 \notin F_1$, then we are done. So we assume that $u_1 \in F_1$. Since $F$ is a nontrivial vertex-set, there exists a vertex $v_0$ adjacent to $u_0$ in $G_0 - F_0$. Consider $N = N_G(u_0, v_0)$. Then $|N| = 2k > 2k - 1$ since $G$ contains no triangles. Thus, there is a vertex $x_0 \in N \cap V(G_0)$ such that the cross-edge $x_0x_1$, where $x_1 \in V(G_1)$, is not incident with any vertex in $F$. This implies that $u_0$ in $G_0 - F_0$ can be connected to $G_1 - F_1$ via the cross-edge $x_0x_1$.

Thus, we show that $|S| \geq 2k$ for any nontrivial vertex-cut $S$ in $G$, that is, $\kappa_s(G) \geq 2k$ if each of $G_0$ and $G_1$ contains no triangles. The theorem follows.

By Theorem 2 and Proposition 2, we obtain the following corollary immediately.

**Corollary** (Chen et al [10]) $G(G_0, G_1; M)$ is $(k + 1)$-regular super connected if and only if either $n > k + 1$ or $n = k + 1$ with $k = 1, 2$.

**Theorem 3** Let $G = G(G_0, G_1; M)$ and $k \geq 1$. Then

1. $\lambda_s(G) = \lambda_r(G) = k + 1$ if and only if $n = k + 1$;
2. $k + 1 < \lambda_s(G) = \lambda_r(G) \leq 2k$ if and only if $n \geq k + 2$; and


(3) \( \lambda_s(G) = \lambda_r(G) = 2k \) if each of \( G_0 \) and \( G_1 \) contains no triangles.

**Proof** (1) Clearly, \( n \geq k + 1 \) is necessary for \( k \geq 1 \). Assume \( \lambda_s(G) = k + 1 \), then there is a nontrivial edge-cut \( S \) with \( |S| = k + 1 \) such that \( G - S \) contains no isolated vertices. Let \( H \) be a component of \( G - S \) and let \( h = V(H) \). Then \( \partial_G(H) = S \). We can easily verify that \( H \) is certainly included in one of \( G_0 \) and \( G_1 \). Without loss of generality, assume that \( H \) is included in \( G_0 \). Consider the degree-sum of vertices in \( H \). Since \( G \) is \((k+1)\)-regular, we have

\[
h(h - 1) \geq \sum_{x \in V(H)} d_H(x) = \sum_{x \in V(H)} d_G(x) - |S| = h(k + 1) - (k + 1) = (h - 1)(k + 1),
\]

from which we have \( h \geq k + 1 \). On the other hand, noting that \( \partial_G(H) = S \) and every vertex in \( H \) is matched by \( M \) with a vertex in \( G_1 \), we have \( |S \cap E(G_0)| = |S| - h \leq 0 \), which implies \( H = G_0 \), \( \partial_G(H) = S = M \), and so \( h \leq |M| = |S| = k + 1 \). Thus, we have \( n = h = k + 1 \).

Conversely, if \( n = k + 1 \) with \( k \geq 1 \), then \( G_0 \) and \( G_1 \) are isomorphic to \( K_n \). Clearly, the perfect matching \( M \) is a nontrivial edge-cut. Thus, \( k + 1 = \lambda(G) \leq \lambda_s(G) \leq |M| = n = k + 1 \), which yields \( \lambda_r(G) = k + 1 \).

(2) By the conclusion in (1) and Proposition 1, the assertion holds clearly.

(3) By Proposition 2 and the conclusion in (2), we need to prove that \( \lambda_s(G) \geq 2k \) if each of \( G_0 \) and \( G_1 \) contains no triangles.

Assume that \( F \) is a nontrivial edge-set of \( G \). We need to prove that if \( |F| \leq 2k - 1 \) then \( G - F \) is connected. Since \( \lambda(G_0) = \lambda(G_1) = k \), at least one of \( G_0 - F \) and \( G_1 - F \) is connected. We can, without loss of generality, suppose that \( G_0 - F \) is connected. In order to prove that \( G - F \) is connected, we only need to show that any vertex \( x_1 \) in \( G_1 \) can be connected to some vertex in \( G_0 - F \).

If the cross edge \( x_0x_1 \) is not in \( F \), then there is nothing to do. Suppose that \( x_0x_1 \in F \). Since \( F \) is a nontrivial edge-set, which does not isolate a vertex, there exists an edge \( x_1y_1 \) in \( G_1 \) such that \( x_1y_1 \notin F \). Since \( G \) contains no triangles, then \( |N_G(x_1, y_1)| = 2k > 2k - 1 \). Thus, there exists at least one \( u_1 \in N_G(x_1, y_1) \) such that the cross-edge \( u_0u_1 \) is not in \( F \). Thus, \( x_1 \) can be connected to \( G_0 - F \) via the cross-edge \( u_0u_1 \).

Thus, we show that \( |S| \geq 2k \) for any nontrivial edge-cut \( S \) in \( G \), that is, \( \lambda_r(G) \geq 2k \).

The theorem follows. \( \square \)

By Theorem 3 and Proposition 1, we obtain the following corollary immediately.
Corollary (Chen et al [10]) \( G(G_0, G_1; M) \) is \((k + 1)\)-regular super edge-connected if and only if \( n > k + 1 \) for any \( k \geq 1 \).

3 Applications

Topologies of many interconnection networks can be viewed as \( G(G_0, G_1; M) \) for some \( k \)-regular graphs \( G_0 \) and \( G_1 \), such as the hypercube \( Q_n \), the twisted cube \( TQ_n \), the cross cube \( CQ_n \), the Möbius cube \( MQ_n \) and the locally twisted cube \( LTQ_n \). Chen et al [10] have proved that each of these networks is super connected and super edge-connected. Applying our results, we immediately obtain that their super connectivity, super edge-connectivity, restricted connectivity and restricted edge-connectivity all are \( 2n - 2 \) for \( n \geq 3 \) and, thus, are super connected and super edge-connected. The proofs are omitted here.

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References


