Design and Analysis of Linear Equality Constrained Dynamic Systems

Zhansheng Duan  
Center for Information Engineering Science Research  
Xi’an Jiaotong University  
Xi’an, Shaanxi 710049, China  
Email: zduan@uno.edu

X. Rong Li  
Department of Electrical Engineering  
University of New Orleans  
New Orleans, LA 70148, USA  
Email: xli@uno.edu

Jifeng Ru  
ARCON Corporation  
Waltham, MA 02451, USA  
Email: jifeng@arcon.com

Abstract—The state of many dynamic systems evolves subject to some equality constraints. Under the assumption that the equality constrained dynamic systems are already available, most existing work focuses on developing state estimation algorithms for these systems. However, how to design and analyze such a system is rarely addressed, even though it is critically important for performance evaluation and application. In this paper, unlike the existing system conversion-based design techniques, we propose a new systematic way to design and analyze linear equality constrained linear dynamic systems through a direct elimination technique, where the desired model class is given and only the distributions of the initial state and driving process noise need to be determined. Comparatively, it is much easier. It is also found that the existing formulations of linear equality constrained linear dynamic systems only cover a small part of the whole class. Numerical examples are provided to illustrate the effectiveness of the proposed way of design and analysis.

Keywords: Unconstrained system, constrained system, constrained estimation, system modeling.

I. INTRODUCTION

The state of many dynamic systems evolves subject to some equality constraints. For example, in ground target tracking [1]–[3], if we treat roads as curves without width, the road networks can then be described by equality constraints. In an airport, an aircraft moves on the runways or taxiways. In the quaternion-based attitude estimation problem, the attitude vector must have a unit norm [4]. In a compartmental model with zero net inflow [5], mass is conserved. In undamped mechanical systems, such as one with Hamiltonian dynamics, the energy conservation law holds. Likewise, in circuit analysis, Kirchhoff’s current and voltage laws hold.

There are way more types of research problems concerning equality constrained dynamic systems. Two of them more concerned with estimation are as follows. In the first type, the goal is to develop state estimation algorithms for the constrained systems assumed to be given. This is the focus of most existing work. In the second type, the goal is to design and analyze equality constrained dynamic systems, that is, to study how to specify the building blocks of the system so that it can be guaranteed that the system state satisfies the required equality constraints. However, this type of work is scarce—only [6] and [7] are known to us.

For linear equality constrained (LEC) state estimation, numerous results are available [6], [8]–[13]. For example, the dimensionality reduction method equivalently converts a constrained state estimation problem to a reduced dimensional unconstrained one. The equivalence can be achieved by reparameterizing the system model [8] making use of the deterministic relationships, imposed by the linear equality constraint, among components of the state vector. It can also be achieved through null space decomposition as in [14]. With this method, the complexity of the dynamic system to be estimated can be reduced. However, this reduction is not significant because the computational load of the state estimator is mainly determined by the dimension of the measurement, which is unaltered in the dimensionality reduction method. Also, it is hard to extend this method to the nonlinear equality constrained case. Another popular method, the projection method [6], [8], [11], projects the unconstrained estimate onto the constraint subspace by applying classical constrained optimization techniques. Specifically in [8], after the unconstrained estimate has been obtained, the problem is formulated as one of weighted least-squares estimation, in which the unconstrained estimate is treated as data and the inverse of its error covariance matrix is used as the weighting matrix. The pseudo measurement method has also been applied to equality constrained state estimation. Its key idea is to treat the equality constraints as pseudo measurements. Thus the LEC state estimation problem is converted into a regular filtering problem with two types of measurements. However, since the pseudo measurements are noise free, the augmented measurement noise will have a singular covariance. Then numerical problems may occur when the Kalman filter is applied. Moreover, the increase in the dimension of the augmented measurement will increase the computational complexity of the state estimator. That is probably why the pseudo measurement method is not popular in LEC state estimation. To avoid possible numerical problems caused by the singular covariance of the measurement noise [15] if the matrix inverse is used, the Moore-Penrose pseudoinverse was used in [16]. Also, to gain insight and analyze this type of estimation problem and to mitigate the MP inverse of a higher dimension with a demanding computational load.
in batch form, two equivalent sequential forms were obtained in [16] by following the recursibility of LMMSE estimation of [17]. It was found that under certain conditions, although equality constraints are indispensable for the evolution of the state, updating by them is redundant for filtering. If there exists model mismatch, however, updating by them is necessary and helpful. Some other methods claimed to be optimal in the model mismatch, however, updating by them is necessary and helpful. Some other methods claimed to be optimal in the model mismatch, however, updating by them is necessary and helpful. Some other methods claimed to be optimal in the model mismatch, however, updating by them is necessary and helpful. Some other methods claimed to be optimal in the model mismatch, however, updating by them is necessary and helpful.

For nonlinear equality constrained state estimation, by the Taylor series expansion (TSE) based linearization, the result of [9] was claimed to be approximately GML-optimal, and [8] extended their LEC state estimation results to the case with nonlinear equality constraints. The second-order TSE was utilized in [3], [13] to obtain better estimation results. [18] imposed constraints on the moments of the distribution of the estimate and proposed a two-step projection method.

It is important to know how to design LEC dynamic systems and analyze their properties. For example, when evaluating performance of state estimation algorithms for LEC dynamic systems, how can we generate the ground truth for an LEC dynamic system which meets the assumption in the estimator development? When applying developed LEC state estimators, how to model the LEC dynamic system is critically important as well.

The key idea of the design and analysis technique for an LEC linear dynamic system in [6] and [7] is to convert an unconstrained dynamic system to a constrained one. This will be called conversion-based design in this paper. For example, an unconstrained dynamic system is premultiplied by an orthogonal projection matrix to make the conversion work in [6]. The conversion-based design has the pros that once we are given an unconstrained dynamic system, we can find a dynamic system which satisfies the required equality constraint. And this resultant LEC dynamic system can serve the performance evaluation purpose. The disadvantage is that the resultant LEC dynamic system is not necessarily the one we expected for the applications we are dealing with. For conversion-based design, to achieve an expected LEC dynamic system, to know what kind of unconstrained system should be designed is not easy in general. In this paper, inspired by the direct elimination method in linear least squares parameter estimation with a linear equality constraint, we propose a new systematic way to design and analyze LEC linear dynamic systems. The difference is that before design and analysis, we are given the desired model class. That is, the state transition matrix, the deterministic input transition matrix and the deterministic input are given. What we need to design is just the distribution of the initial state and process noise and possibly their crosscorrelation. This is more in accordance with the needs in practical applications.

This paper is organized as follows. Sec. II formulates the problem. Sec. III presents the proposed systematic way to design and analyze LEC linear dynamic system driven by desired model class. Sec. IV provides four numerical examples to verify the effectiveness of the proposed design and analysis technique. Sec. V gives conclusions.

II. Problem Formulation

Consider the following typical form of a linear stochastic dynamic system

\[ x_{k+1} = F_k x_k + G_k u_k + w_k \]

(1)

where \( x_k \in \mathbb{R}^n, u_k \in \mathbb{R}^{n_u} \). It is known in advance that the state of this system satisfies the following linear equality constraint

\[ C_k x_k = d_k \]

(2)

where \( C_k \in \mathbb{R}^{m \times n} \) and \( d_k \in \mathbb{R}^m \) are both known deterministically, \( C_k \) is of full row rank, and \( m < n \).

Remark 1: If \( C_k \) is not of full row rank, it just means that there are redundant constraints on the system state in \( \{2\} \). To get rid of this redundancy, we can simply remove the linearly dependant rows of \( C_k \) and keep its maximal linearly independent rows. Then \( C_k \) will be of full row rank. The case in which \( m = n \) is trivial and is not considered since \( x_k \) can be uniquely determined.

In applications, we are given the constraint \( \{2\} \) first, e.g., a road network in ground target tracking and runways in an airport. Then we are asked to design a linear system \( \{1\} \) which evolves subject to the constraint \( \{2\} \). This is meant to specify the system matrices \( F_k \) and \( G_k \), the governing rule for the change of deterministic input \( u_k \), the distribution of \( x_0, \{w_k\} \) and possibly their crosscorrelation. For a typical unconstrained dynamic system, we usually assume that the process noise \( \{w_k\} \) and the initial state \( x_0 \) are independent. Under this assumption, the design of \( F_k, G_k, u_k \), and the distributions of \( x_0 \) and \( \{w_k\} \) can all be done separately without relying on the specification of the others at all. Also, they can take any value or distribution.

However, such a design for LEC dynamic systems is not easy because the constraint \( \{2\} \) requires that at any time \( k \), all realizations of \( x_k \) satisfy \( \{2\} \). This requirement is stringent and imposes significant difficulty on the design of LEC dynamic systems. For example, \( x_0 \) and \( \{w_k\} \) may not have arbitrary distributions and their moments may have some special structure required by constraint \( \{2\} \). In this paper, we try to discover all these differences from the design of typical unconstrained dynamic systems and provide a systematic way to design LEC dynamic systems. Meanwhile, we provide some analysis of LEC dynamic systems.

III. Design and Analysis of LEC Dynamic Systems in Desired Model Class

Unlike the conversion-based design, in class-directed design, we are given the desired model class for system evolution, represented by \( F_k, G_k, u_k \), and we need to design the distributions of \( x_0 \) and \( \{w_k\} \) and possibly their crosscorrelation. For example, in an airport, we know that the aircraft of interest is moving at a nearly constant velocity or acceleration or taking a coordinated turn. So \( F_k, G_k, \) and \( u_k \) can be
determined easily. However, how to design the distributions of \( x_0 \) and \( \{w_k\} \) and possibly their crosscorrelation is not easy, as shown below.

First, let us see what the linear equality constraint (2) really means. Without loss of generality, assume that the components of \( x_k \) have been reshuffled such that

\[
C_k = \begin{bmatrix} C^1_k & C^2_k \end{bmatrix}
\]

where \( C^1_k \in \mathbb{R}^{m \times m} \) is nonsingular and \( C^2_k \in \mathbb{R}^{m \times (n-m)} \). This can always be guaranteed since \( C_k \) is of full row rank and \( m < n \). Constraint (2) can now be rewritten as

\[
\begin{bmatrix} C^1_k & C^2_k \end{bmatrix} \begin{bmatrix} x^1_k \\ x^2_k \end{bmatrix} = d_k
\]

where \( \begin{bmatrix} (x^1_k)' & (x^2_k)' \end{bmatrix}' = x_k \) and \( x^1_k \in \mathbb{R}^m \), \( x^2_k \in \mathbb{R}^{(n-m)} \).

As a result, it can be easily seen that

\[
x^1_k = (C^1_k)^{-1} d_k - A_k x^2_k
\]

(3)

where \( A_k = (C^1_k)^{-1} C^2_k \). That is, constraint (2) means that some components of the state vector \( x_k \) can be determined deterministically by the rest. This differs from the unconstrained dynamic systems, where no components of the state vector \( x_k \) can be determined deterministically by the others.

Rewrite the state transition equation as a partitioned form accordingly,

\[
\begin{bmatrix} x^1_{k+1} \\ x^2_{k+1} \end{bmatrix} = \begin{bmatrix} F^{11}_k & F^{12}_k \\ F^{21}_k & F^{22}_k \end{bmatrix} \begin{bmatrix} x^1_k \\ x^2_k \end{bmatrix} + \begin{bmatrix} G^1_k \\ G^2_k \end{bmatrix} u_k + \begin{bmatrix} w^1_k \\ w^2_k \end{bmatrix}
\]

where \( F^{11}_k, F^{22}_k, G^1_k, G^2_k, u^1_k, \) and \( u^2_k \) have appropriate dimensions. Then,

\[
\begin{align*}
x^1_{k+1} &= F^{11}_k x^1_k + F^{12}_k x^2_k + G^1_k u_k + w^1_k \\
x^2_{k+1} &= F^{21}_k x^1_k + F^{22}_k x^2_k + G^2_k u_k + w^2_k
\end{align*}
\]

(4)

Substituting (3) into (5), we have

\[
x^2_{k+1} = (F^{22}_k - F^{21}_k A_k) x^2_k + F^{21}_k (C^1_k)^{-1} d_k + G^2_k u_k + w^2_k
\]

(6)

This is a regular unconstrained dynamic system.

As made clear above about the class-directed design, \( F_k, G_k, u_k, C_k \) and \( d_k \) are all given in advance. What we need to design in (6) is just the distributions of \( x^2_0 \) and \( \{w^2_k\} \) and possibly their crosscorrelation. Since (6) is a regular unconstrained dynamic system, \( x^0_0 \) and \( \{w^0_k\} \) can have any arbitrary distributions. For example, we can simply assume they have no crosscorrelation as usual. However, \( x^0_0 \) cannot have an arbitrary distribution since

\[
x^0_0 = (C^0_0)^{-1} d_0 - A_0 x^2_0
\]

(7)

as can be seen from (3), which indicates clearly that the distribution of \( x^0_0 \) is completely determined by the distribution of \( x^2_0 \). This makes the design of LEC dynamic systems different from the design of regular unconstrained linear dynamic systems.

Substituting (3) into (4), we have

\[
(C^1_{k+1})^{-1} d_{k+1} - A_{k+1} x^2_{k+1} = F^{11}_k (C^1_k)^{-1} d_k + (F^{12}_k - F^{11}_k A_k) x^2_k + G^1_k u_k + w^1_k
\]

(8)

Substituting (6) into (8), we have

\[
w^1_k = (C^1_{k+1})^{-1} d_{k+1} - (F^{11}_k (C^1_k)^{-1} + A_{k+1} F^{21}_k (C^1_k)^{-1}) d_k - (G^1_k + A_{k+1} G^2_k) u_k - B_k x^2_k - A_{k+1} w^2_k
\]

where

\[
B_k = A_{k+1} (F^{22}_k - F^{21}_k A_k) + F^{12}_k - F^{11}_k A_k
\]

It can be clearly seen that the distribution of \( w^1_k \) depends on those of \( x^2_k \) and \( w^2_k \) in general. Because of the dependence of \( w^1_k \) on \( w^2_k \), the moments of \( w^1_k \) may have a special structure, as will be seen in Sec. IV later. Because of the dependence of \( w^1_k \) on \( x^2_k \),

\[
E[w^1_k (x^2_k)'] = -B_k E[x^2_k (x^2_k)']
\]

we also have

\[
E[w^1_k (x^2_k)'] = B_k E[x^2_k (x^2_k)'] A_k'
\]

That is, \( w^1_k \) is dependent on the system state \( x_k \). We can also say that \( w_k \) depends on \( x_k \) since \( w_k \) is a subvector of \( w_k \).

Moreover, due to the evolution of the linear dynamic system (1), we can easily see that \( \{w^1_k\} \) will not be a white noise sequence. This is totally against the formulations of LEC dynamic systems in the existing work, where it is usually assumed that \( \{w_k\} \) is a white noise sequence and uncorrelated with the initial state \( x_0 \). As a result of this assumption, \( w_k \) will be independent of \( x_k \). As analyzed above, however, for the LEC dynamic system (2), \( w_k \) usually depends on \( x_k \) and \( \{w_k\} \) is not white. This also makes the design of LEC dynamic systems significantly different from the design of regular unconstrained linear dynamic systems.

Only if

\[
B_k = 0
\]

will the existing formulations be valid. With this condition, it can be guaranteed that \( w_k \) and \( x_k \) have no crosscorrelation and \( \{w_k\} \) is white. That is, the existing formulations only cover a small class of LEC dynamic systems. Correspondingly, the applicability of the state estimation algorithms developed is limited.

With the above analysis, the generation of one run of the linear dynamic system (1) subject to linear equality constraint (2) can be summarized as follows.

**Initialization:** Sample \( x^2_0 \) and \( w^2_0 \) according to the prior distributions

\[
x^2_0 \sim p(x^2_0), \quad w^2_0 \sim p(w^2_0)
\]

where \( p(x^2_0) \) and \( p(w^2_0) \) can be specified as any valid distributions. For simplicity, assume \( \{w^2_k\} \) is white. Then generate \( x^0_0 \) and \( w^0_0 \) according to

\[
\begin{align*}
x^0_1 &= (C^0_0)^{-1} d_0 - A_0 x^2_0 \\
w^0_1 &= (C^1_0)^{-1} d_1 - (F^{11}_0 (C^1_0)^{-1} + A_1 F^{21}_0 (C^1_0)^{-1}) d_0 - (G^1_0 + A_1 G^2_0) u_0 - B_0 x^2_0 - A_1 w^2_0
\end{align*}
\]

**Recursive evolution:** For \( k = 1, 2, \ldots \), repeat the following four steps.
Step 1: Generate $x_k^2$ according to the unconstrained sub-
dynamic system model
\[
x_k^2 = (F_{k-1}^{22} - F_{k-1}^{21}A_{k-1})x_{k-1}^2 + F_{k-1}^{21}(C_{k-1})^{-1}d_{k-1}
+ G_{k-1}^{-1}uk_{k-1} + w_{k-1}^2
\]

Step 2: Generate $w_k^2$ according to the prior distribution
\[
w_k^2 \sim p(w_k^2)
\]
where $p(w_k^2)$ can be specified as any valid distribution.

Step 3: Generate $x_k^1$ according to
\[
x_k^1 = (C_{k-1}^{-1}d_{k-1} - A_kx_k^2
\]

Step 4: Generate $w_k^1$ according to
\[
w_k^1 = (C_{k-1}^{-1}d_{k-1} - (F_k^{11}(C_k^{-1})^{-1} + A_{k+1}F_k^{21}(C_k^{-1})^{-1})d_k
+ (G_k^1 + A_kG_k^2)uk_k - B_kx_k^2 - A_{k+1}w_{k-1}^2
\]

With this procedure, it is guaranteed that at any time $k$, all realizations of $x_k$ satisfy (2).

Next, consider the commonly used Gaussian case. For the linear dynamic system (1) subject to linear equality constraint (2), suppose that
\[
x_0^2 \sim \mathcal{N}(\bar{x}_0^2, P_0^2), \quad w_k^2 \sim \mathcal{N}(0, Q_k^2)
\]
and \{$w_k^2\}$ is white and independent of $x_0^2$. Then, clearly
\[
x_0^1 \sim \mathcal{N}(\bar{x}_0^1, P_0^1)
\]
\[
w_0^1 \sim \mathcal{N}(\bar{w}_0^1, Q_0^1)
\]
and
\[
\text{cov}(x_0^1, w_0^1) = A_0P_0^2B_0'
\]
\[
\text{cov}(x_0^2, w_0^1) = -P_0^2B_0'
\]
\[
\text{cov}(x_0, w_0^2) = 0
\]
where \[
\bar{x}_0^1 = (C_0^1)^{-1}d_0 - A_0\bar{x}_0^2
\]
\[
P_0^1 = A_0P_0^2A_0'
\]
\[
\bar{w}_0^1 = (C_0^{-1})^{-1}d_0 - (F_0^{11}(C_0^{-1})^{-1} + A_1F_0^{21}(C_0^{-1})^{-1})d_0
+(G_0^1 + A_1G_0^2)u_0 - B_0\bar{x}_0^2
\]
\[
Q_0^1 = B_0P_0^2B_0' + A_1Q_0^1A_1'
\]
Note that although $w_0^2$ is zero-mean, $w_0^1$ is not zero-mean in general.

Recursively, for $k = 1, 2, \ldots$, we have
\[
x_k^2 \sim \mathcal{N}(\bar{x}_k^2, P_k^2)
\]
\[
x_k^1 \sim \mathcal{N}(\bar{x}_k^1, P_k^1)
\]
\[
w_k^1 \sim \mathcal{N}(\bar{w}_k^1, Q_k^1)
\]
and
\[
\text{cov}(x_k^1, w_k^1) = A_kP_k^2B_k'
\]
\[
\text{cov}(x_k^2, w_k^1) = -P_k^2B_k'
\]
\[
\text{cov}(x_k, w_k^2) = 0
\]
where
\[
\bar{x}_k^2 = (F_{k-1}^{22} - F_{k-1}^{21}A_{k-1})\bar{x}_{k-1}^2 + F_{k-1}^{21}(C_{k-1}^{-1})^{-1}d_{k-1}
+ G_{k-1}^{-1}uk_{k-1}
\]
\[
P_k^2 = (F_{k-1}^{22} - F_{k-1}^{21}A_{k-1})P_{k-1}^2(F_{k-1}^{22} - F_{k-1}^{21}A_{k-1})' + Q_{k-1}^2
\]
\[
\bar{x}_k^1 = (C_{k-1}^{-1})^{-1}d_k - A_k\bar{x}_k^2
\]
\[
P_k^1 = A_kP_k^2A_k'
\]
\[
\bar{w}_k^1 = (C_{k-1}^{-1})^{-1}d_k - (F_k^{11}(C_k^{-1})^{-1} + A_{k+1}F_k^{21}(C_k^{-1})^{-1})d_k
+(G_k^1 + A_kG_k^2)uk_k - B_kx_k^2 - A_{k+1}w_{k-1}^2
\]
\[
Q_k^1 = B_kP_k^2B_k' + A_kQ_k^1A_k'
\]
Also note that $w_k^1$ is not zero-mean in general.

As shown above, for LEC dynamic systems, the process noise $w_k$ is usually state dependent and not white. This invalidates or at least discredits the state estimation algorithms developed based on the traditional linear Gaussian assumption, which is widely used in the existing work. So we need to develop new state estimation algorithms.

Suppose that a linear measurement model of the system state $x_k$ is given as
\[
z_k = H_kx_k + v_k
\]
where $z_k \in \mathbb{R}^{n_z}$, and \{\$v_k\}$ is a white noise sequence with $v_k \sim \mathcal{N}(0, R_k)$ and is independent of $x_0^2$ and \{$w_k^2\}$.

As was meant by (3), $x_k^1$ can be determined by $x_k^2$ deterministically. So (13) can be rewritten as
\[
z_k = \begin{bmatrix} H_k^1 & H_k^2 \end{bmatrix} \begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix} + v_k
\]
\[
= (H_k^1 - H_k^0A_k)x_k^2 + H_k^1(C_k^0)^{-1}d_k + v_k
\]
where $H_k^1 \in \mathbb{R}^{n_z \times m}$ and $H_k^2 \in \mathbb{R}^{n_z \times (n_m - m)}$.

By taking advantage of the property that the sub-system (6) is just a regular unconstrained dynamic system, the MMSE-optimal estimation for the linear dynamic system (1) subject to the linear equality constraint (2) in the above Gaussian case can be summarized as follows.

Initialization:
\[
\hat{x}_{0|0}^2 = \bar{x}_0^2, \quad P_{0|0}^2 = P_0^2
\]
\[
\hat{x}_{0|0}^1 = (C_0^1)^{-1}d_0 - A_0\hat{x}_{0|0}^2, \quad P_{0|0}^1 = A_0P_{0|0}^2A_0'
\]
For $k = 1, 2, \ldots$,

Prediction:
\[
\hat{x}_{k|k-1}^2 = (F_{k-1}^{22} - F_{k-1}^{21}A_{k-1})\hat{x}_{k-1|k-1}^2 + F_{k-1}^{21}(C_{k-1}^{-1})^{-1}d_{k-1}
+ G_{k-1}^{-1}uk_{k-1}
\]
\[
P_{k|k-1}^2 = (F_{k-1}^{22} - F_{k-1}^{21}A_{k-1})P_{k-1|k-1}^2(F_{k-1}^{22} - F_{k-1}^{21}A_{k-1})' + Q_{k-1}^2
\]
\[
\hat{x}_{k|k-1}^1 = (C_{k-1}^{-1})^{-1}d_k - A_k\hat{x}_{k|k-1}^2
\]
\[
P_{k|k-1}^1 = A_kP_{k|k-1}^2A_k'
\]
the vehicle should be \( \dot{x} \). This means that the vehicle is moving at a constant velocity.

Consider the following commonly used linear equality constraint.

\[
\text{where } P_{k|k-1} = P_{k|k-1}^{-1} - P_{k|k-1}^{-1} (H_k^2 - H_k A_k) P_{k|k-1}^{-1} (H_k^2 - H_k A_k)' + R_k^{-1}
\]

\[
\dot{\hat{x}}_{k|k} = \hat{x}_{k|k-1} + P_{k|k-1}^2 (H_k^2 - H_k A_k)' \cdot (H_k^2 - H_k A_k) P_{k|k-1}^{-1} (H_k^2 - H_k A_k) \hat{x}_{k|k-1} + R_k^{-1}
\]

Update:

\[
\hat{x}_{k|k} = \hat{x}_{k|k-1} + P_{k|k-1}^2 (H_k^2 - H_k A_k)' \cdot (H_k^2 - H_k A_k) P_{k|k-1}^{-1} (H_k^2 - H_k A_k) \hat{x}_{k|k-1} + R_k^{-1}
\]

\[
P_{k|k} = P_{k|k-1} - P_{k|k-1}^2 (H_k^2 - H_k A_k)' \cdot (H_k^2 - H_k A_k) P_{k|k-1}^{-1}
\]

\[
\hat{x}_{k|k} = (C_k^{-1})^{-1} d_k - A_k \hat{x}_{k|k-1}
\]

Partition the linear dynamic system and linear equality constraint as

\[
C_k = \begin{bmatrix} 1 & -\tan \theta & 0 & 0 \end{bmatrix}
\]

\[
F_k = \begin{bmatrix} 1 & 0 & 0 & T \sin \theta \\ 0 & 1 & 0 & T \cos \theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[
G_k = \begin{bmatrix} T \sin \theta \\ T \cos \theta \\ 0 \\ 0 \end{bmatrix}
\]

\[
Q_k = \begin{bmatrix} 10 & 10 \sqrt{3} & 0 & 0 \\ 10 \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 400 & 0 \\ 0 & 0 & 0 & 400 \end{bmatrix}
\]

\[
P_0 = \begin{bmatrix} 15 & 5 \sqrt{3} & 0 & 0 \\ 5 \sqrt{3} & 5 & 0 & 0 \\ 0 & 0 & 400 & 0 \\ 0 & 0 & 0 & 400 \end{bmatrix}
\]

A. Example 1

By using the following well selected setting (Setting 1) of the remaining parameters

\[
\hat{x}_0 = \begin{bmatrix} 10 \sqrt{3} \\ 10 \\ 0 \\ 0 \end{bmatrix}
\]

\[
P_0 = \begin{bmatrix} 15 & 5 \sqrt{3} & 0 & 0 \\ 5 \sqrt{3} & 5 & 0 & 0 \\ 0 & 0 & 400 & 0 \\ 0 & 0 & 0 & 400 \end{bmatrix}
\]

\[
Q_k = \begin{bmatrix} 10 & 10 \sqrt{3} & 0 & 0 \\ 10 \sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 30 & 10 \sqrt{3} \\ 0 & 0 & 10 \sqrt{3} & 10 \end{bmatrix}
\]

Fig. 1 shows one run of the velocity of the vehicle. It can be clearly seen that the heading is really \( \theta \). That is, the linear equality constraint (14) is strictly satisfied.

With the above setting of parameters, the angle between the \( y \) axis and the trajectory of the vehicle is also guaranteed to be \( \theta \), which can be seen from Fig. 2.

For this example, it follows that

\[
A_k = \begin{bmatrix} -\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

So from (9), (10), (11) and (12), the assumption that \( x_0 \) is independent of \( \{w_k\} \) and \( \{w_k\} \) is a white noise sequence is clearly valid.
Furthermore, we have
\[
\begin{cases}
G_1^1 u_k = \sqrt{3} T, \ G_2^1 u_k = [T \ 0 \ 0]', \text{if } k \text{ is odd} \\
G_1^2 u_k = -\sqrt{3} T, \ G_2^2 u_k = [-T \ 0 \ 0]', \text{if } k \text{ is even}
\end{cases}
\]

Then whether \(k\) is odd or even, we always have
\[
-(G_1^1 + A_{k+1} G_2^2) u_k = 0
\]

Also, since \(d_k = 0\), it turns out that
\[
w_1^k = -A_{k+1} u_0 = [\sqrt{3} \ 0 \ 0] w_2^k
\]

That is why \(w_k\) is zero-mean and \(Q_k\) takes the special structure of (17).

Similarly, from (7), we have
\[
x_1^0 = -A_0 x_0^2 = [\sqrt{3} \ 0 \ 0] x_0^2
\]

That is also why \(\bar{x}_0\) and \(P_0\) take the special structures of (15) and (16), respectively.

**B. Example 2**

As analyzed above, the design of a linear dynamic system subject to a linear equality constraint is very sensitive to the setting of parameters. This is well illustrated by the following example, where we have used the following setting (Setting 2) of the remaining parameters

\[
Q_k = \begin{bmatrix}
13.25 & 0.1443 & 0 & 0 \\
0.1443 & 13.0833 & 0 & 0 \\
0 & 0 & 30 & 10\sqrt{3} \\
0 & 0 & 10\sqrt{3} & 10
\end{bmatrix}
\]

\(\bar{x}_0\) and \(P_0\) are still the same as in Setting 1.

One run of the system is shown in Figs. 3 and 4. From the velocity plot, it can be clearly seen that the linear equality constraint (14) is satisfied. However, the angle between the \(y\) axis and the vehicle trajectory is not \(\theta\) any more.

**C. Example 3**

In this example, we use the following setting (Setting 3) of the remaining parameters

\[
\bar{x}_0 = [10\sqrt{3} \ 10 \ 2 \ 3]'
\]

\[
P_0 = \begin{bmatrix}
\frac{15}{\sqrt{3}} & \frac{5\sqrt{3}}{2} & 0 & 0 \\
\frac{5\sqrt{3}}{2} & \frac{5}{2} & 0 & 0 \\
0 & 0 & 410 & 0 \\
0 & 0 & 0 & 405
\end{bmatrix}
\]

\[
Q_k = \begin{bmatrix}
10 & 10\sqrt{3} & 0 & 0 \\
10\sqrt{3} & 10 & 0 & 0 \\
0 & 0 & 30 & 0 \\
0 & 0 & 0 & 10
\end{bmatrix}
\]

One run of the system is shown in Figs. 5 and 6. From the velocity plot, it can be clearly seen that the linear equality constraint (14) is satisfied. However, the angle between the \(y\) axis and the vehicle trajectory is not \(\theta\) any more.

**D. Example 4**

In this example, the following changes are made to the system:

\[
F_{12}^k = [0 \ 1 \ 0], \ u_k = 0
\]

\(G_k \in \mathbb{R}^4\) is arbitrary.
And it is still required that the linear equality constraint (14) be satisfied.

Now it can be easily seen that
\[ B_k = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \]

So the process noise \( w_k \) will be dependent on the system state \( x_k \) and \( \{ w_k \} \) will not be a white sequence, as analyzed before. Given
\[ x_0^2 \sim \mathcal{N}(\bar{x}_0^2, P_0^2), \; w_k^2 \sim \mathcal{N}(0, Q_k^2) \]
where \( \{ w_k^2 \} \) is a white noise sequence and independent of \( x_0^2 \), let (Setting 4)
\[
\bar{x}_0^2 = \begin{bmatrix} 10 & 2 & 3 \end{bmatrix}^T
\]
\[
P_0^2 = \begin{bmatrix}
\frac{5}{2} & 0 & 0 \\
0 & 410 & 0 \\
0 & 0 & 405
\end{bmatrix}
\]
\[
Q_k^2 = \begin{bmatrix}
10 & 0 & 0 \\
0 & 30 & 0 \\
0 & 0 & 10
\end{bmatrix}
\]

By following the general procedure, one run of the system is shown in Figs. 7 and 8. From the velocity plot, it can be clearly seen that the linear equality constraint (14) is satisfied. However, the angle between the y axis and the vehicle trajectory is not \( \theta \).

It seems that the generated system state is similar to what is in Example 3. However, there are significant differences. Fig. 9 shows the corresponding \( w_1^k \) in one run. It can be clearly seen that \( \{ w_1^k \} \) is not a white noise sequence any more. This is in accordance with our analysis above. For the linear dynamic system in this example, the existing state estimation algorithms based on the traditional linear Gaussian assumption are not applicable. However, our MMSE-optimal state estimator still works.
V. CONCLUSIONS

State estimation algorithm development and system design and analysis are two equally important research problems concerning equality constrained dynamic systems. However, most existing work on equality constrained dynamic systems focus only on developing state estimation algorithms by assuming that the constrained systems are already available. And not much work has been done on how to design and analyze equality constrained dynamic systems. The existing conversion-based design techniques provide a way to design equality constrained dynamic systems. But designing an unconstrained dynamic system which can then be converted to the desired unconstrained dynamic system is not easy. In this paper, we propose a new systematic and relatively simple way to design and analyze LEC linear dynamic systems through a direct elimination technique, where the desired model class is given and only the distributions of the initial state and process noise need to be determined. It is also found that the existing formulations only cover a small class of LEC linear dynamic systems. Numerical examples show that the proposed way of design and analysis is effective.

REFERENCES


