Fundamental groups of moduli stacks of smooth Weierstrass fibrations

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Abstract

We give finite presentations for the fundamental group of moduli stacks of smooth Weierstrass curves over $\mathbf{P}^n$ which extend the classical result for elliptic curves to positive dimensional base. We thus get natural generalisations of $\mathrm{SL}_2\mathbb{Z}$ presented in terms of $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right)$ and pave the way to understanding the fundamental group of moduli stacks of elliptic surfaces in general.

Our approach exploits the natural $\mathbb{Z}_2$-action on Weierstrass curves and the identification of $\mathbb{Z}_2$-fixed loci with smooth hypersurfaces in an appropriate linear system on a projective line bundle over $\mathbf{P}^n$. The fundamental group of the corresponding discriminant complement can be presented in terms of finitely many generators and relations using methods in the Zariski tradition, which were successfully elaborated in [Lö3].

1 Introduction

Our primary objects are hypersurfaces of the ruled manifold $X_{n,d} = \mathbf{P}\left(\mathcal{O}_{\mathbf{P}^n}(d) \oplus \mathcal{O}_{\mathbf{P}^n}\right)$ in the linear system $|3\sigma_0|$, where σ_0 denotes the divisor on $X_{n,d}$ defined by the zero section of $\mathcal{O}_{\mathbf{P}^n}(d)$. They are assembled in the universal hypersurface $\mathcal{H}_{n,d}$ which is a hypersurface in $X_{n,d} \times \mathbf{P}V_{n,d}$, $V_{n,d} = \Gamma(X_{n,d}, \mathcal{O}_{X_{n,d}}(3\sigma_0))$.

Upon a choice of homogeneous coordinates y, y_0 on a fibre and $x_0, ..., x_n$ on the base, $V_{n,d}$ is identified with the polynomials of $\mathbb{C}[y_0, y, x_0, ..., x_n]$ which have degree 3 in the variables y_0, y and weighted degree $3d$ in the variables y, x_i of weights d and 1 respectively.

On $V_{n,d}$ we introduce coordinates u_ν with respect to the monomial basis. Since each monomial in $V_{n,d}$ is uniquely determined by the exponents of the x_i, each coordinate u_ν is unambiguously specified by a multiindex $\nu \in \{(\nu_1, ...\nu_n) \,|\, \nu_i \geq 0, |\nu| = \sum_i \nu_i \leq 3d\}$. The equation of $\mathcal{H}_{n,d}$ then reads in multiindex notation, $x^\nu = x_0^{\nu_0} \cdots x_n^{\nu_n}$,

$$u_{\mathbf{0}} y^3 + \sum_{|\nu|=d} u_\nu y_0 y^2 x^\nu + \sum_{|\nu|=2d} u_\nu y_0^2 y x^\nu + \sum_{|\nu|=3d} u_\nu y_0^3 x^\nu \qquad (1)$$

Its projection to the factor $\mathbf{P}V_{n,d}$ has singular values precisely along the discriminant

$$\mathcal{D}_{n,d} \quad = \quad \{u \in \mathbf{P}V_{n,d} \,|\, \mathcal{H}_u \text{ is singular}\}$$

which is the union of the hyperplane $\{u_{\mathbf{0}} = 0\}$ and the projective dual of weighted projective space $\mathbf{P}^{n+1}_{d,1,1,...,1}$ given as the image of $X_{n,d}$ under the projective morphism defined by the base point free linear system $|3\sigma_0|$.

The problem we want to address in the first stage is to give a geometrically distinguished finite presentation of the fundamental group of the complement $\mathcal{U}_{n,d}$ of $\mathcal{D}_{n,d}$.

It may be viewed as a special instance of the vastly open problem posed by Dolgachev and Libgober, [DL], to determine the fundamental group of the discriminant complement of any (complete) linear system.

They handle the case of linear systems of elliptic curves on $\mathbf{P}^2$ and $\mathbf{P}^1 \times \mathbf{P}^1$ as well as linear systems on curves, but actually the first result of that kind is due to Zariski who considered the complete linear systems on $\mathbf{P}^1$, which in degree $l = 3$ may be viewed as the $n = 0$ analogue of our set up.

Theorem 1 (Zariski [Za] and Fadell, van Buskirk [FvB])
The fundamental group $\pi_1(\mathcal{U}_{\mathbf{P}^1,l})$ of the discriminant complement associated to the complete linear system of degree l on $\mathbf{P}^1$ is finitely presented by generators $\sigma_1, ..., \sigma_{l-1}$ and relations

i) $\sigma_i\sigma_j = \sigma_j\sigma_i$, *if* $|i-j| \geq 2$, $1 \leq i, j < l$,

ii) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, *if* $1 \leq i < l-1$,

iii) $\sigma_1 \cdots \sigma_{l-2}\sigma_{l-1}\sigma_{l-1}\sigma_{l-2} \cdots \sigma_1 = 1$.

We have previously extended these results to all complete linear systems on projective spaces, cf. [Lö5], and we provide now a series of examples of linear systems on ruled manifolds (The claim will be made more explicit later in the paper, cf. section 7.):

Theorem 2 *There is a graph $\Gamma_{n,d} = (\mathcal{V}_{n,d}, E_{n,d} \subset {\mathcal{V}_{n,d}}^2)$ with vertex set $\mathcal{V}_{n,d}$ linearly ordered by $\prec$, such that $\pi_1(\mathcal{U}_{n,d})$ is generated by elements*

$$t_{\boldsymbol{i}}, \quad \boldsymbol{i} \in \mathcal{V}_{n,d}$$

and a complete set of relations is provided by:

i) $t_{\boldsymbol{i}}t_{\boldsymbol{j}} = t_{\boldsymbol{j}}t_{\boldsymbol{i}}$ *for all* $(\boldsymbol{i}, \boldsymbol{j}) \notin E_{n,d}$,

ii) $t_{\boldsymbol{i}}t_{\boldsymbol{j}}t_{\boldsymbol{i}} = t_{\boldsymbol{j}}t_{\boldsymbol{i}}t_{\boldsymbol{j}}$ *for all* $(\boldsymbol{i}, \boldsymbol{j}) \in E_{n,d}$,

iii) $t_{\boldsymbol{i}}t_{\boldsymbol{j}}t_{\boldsymbol{k}}t_{\boldsymbol{i}} = t_{\boldsymbol{j}}t_{\boldsymbol{k}}t_{\boldsymbol{i}}t_{\boldsymbol{j}}$ *for all* $\boldsymbol{i} \prec \boldsymbol{j} \prec \boldsymbol{k}$ *such that* $(\boldsymbol{i}, \boldsymbol{j}), (\boldsymbol{i}, \boldsymbol{k}), (\boldsymbol{j}, \boldsymbol{k}) \in E_{n,d}$,

iv) *for all* $\boldsymbol{i} \in \mathcal{V}_{n,d}$

$$t_{\boldsymbol{i}} \left(t_{\boldsymbol{i}}^{-1} \prod_{\boldsymbol{i} \in \mathcal{V}_{n,d}^{\prec}} t_{\boldsymbol{i}} \right)^{3d-1} = \left(t_{\boldsymbol{i}}^{-1} \prod_{\boldsymbol{i} \in \mathcal{V}_{n,d}^{\prec}} t_{\boldsymbol{i}} \right)^{3d-1} t_{\boldsymbol{i}}$$

In the second stage we are interested in hypersurfaces given by a Weierstrass equation. On the $\mathbf{P}^2$-bundles $Y_{n,d} = \mathbf{P}\left(\mathcal{O}_{\mathbf{P}^n} \oplus \mathcal{O}_{\mathbf{P}^n}(2d) \oplus \mathcal{O}_{\mathbf{P}^n}(3d)\right)$ we introduce homogeneous fibre coordinates y_0, y_1, y_2 and homogeneous base coordinates $x_0, ..., x_n$.

A Weierstrass fibration is then defined to be given by an equation of the form

$$y_0 y_2^2 \quad = \quad y^3 + \sum_{|\nu|=2d} u_\nu y_0^2 y x^\nu + \sum_{|\nu|=3d} u_\nu y_0^3 x^\nu \tag{2}$$

with d necessarily even, where the coefficients u_ν are coordinates of a vector subspace $V'_{n,d}$ of $V_{n,d}$ generated by the elements $x^\nu, |\nu| \geq 2d$.

The same equation also defines the associated tautological Weierstrass hypersurface in $Y_{n,d} \times V'_{n,d}$. Its projection to $V'_{n,d}$ has singular values along the discriminant

$$\mathcal{D}'_{n,d} \quad = \quad \mathcal{D}_{n,d} \cap V'_{n,d}$$

where $V'_{n,d}$ is embedded into $\mathbf{P}V_{n,d}$ as the affine part (w.r.t. the hyperplane $u_{\mathbf{0}} = 0$) of the projective subspace generated by $V'_{n,d}$ and $x^{\mathbf{0}}$, the polynomial with coefficient $u_{\mathbf{0}}$. That property is easily checked on equations and reflects the following fact. A Weierstrass fibration is a double cover of a smooth surface and therefore smooth if the $\mathbb{Z}_2$-fixed locus is. Its fixed part off the hypersurface $y_2 = 0$ is always smooth, hence smoothness is equivalent to smoothness of the restriction to $y_2 = 0$ which yield precisely the smoothness condition considered in the first part.

The complement $\mathcal{U}'_{n,d}$ of $\mathcal{D}'_{n,d}$ in $V'_{n,d}$ is the base of a versal family of smooth Weierstrass fibrations. The moduli stack $\mathcal{M}_{n,d}$ is in naturally obtained as the quotient of of $V'_{n,d}$ by the group of automorphisms, which is given as the direct product of the group of linear projective transformations of the base and $\mathbb{C}^*$ acting on the coordinates u_ν with weight $|\nu|/d$. We will show $\pi_1(\mathcal{U}_{n,d}) \cong \pi_1(\mathcal{U}'_{n,d})$ and derive the topological fundamental group of $\mathcal{M}_{n,d}$ from a homotopy exact sequence.

In this way we are able to generalise the old result giving the fundamental group of the moduli stack of elliptic curves, i.e. the $n = 0$ case.

Theorem 3 *The (orbifold) fundamental group* $\mathrm{SL}_2\mathbb{Z}$ *of* $\mathcal{M}_0$ *is finitely presented as*

$$\langle \quad \sigma_1, \sigma_2 \quad | \quad \sigma_1\sigma_2\sigma_1 \quad = \quad \sigma_2\sigma_1\sigma_2, \quad (\sigma_1\sigma_2)^6 \quad = \quad 1 \rangle$$

Of course the natural generalisation relies heavily on our theorem 2:

Theorem 4 *There is a graph* $\Gamma_{n,d} = (\mathcal{V}_{n,d}, E_{n,d})$ *with linearly ordered vertex set* $\mathcal{V}_{n,d}$ *and bijective maps for* $\kappa \in \{0, ..., n\}$*,*

$$\boldsymbol{i}_\kappa : \{1, ..., 2(3d-1)^n\} \to \mathcal{V}_{n,d}$$

such that the (orbifold) fundamental group $\pi_1(\mathcal{M}_{n,d})$ *is generated by elements* $t_{\boldsymbol{i}}$*,* $\boldsymbol{i} \in \mathcal{V}_{n,d}$ *and a complete set of relations is given by* $i) - iv)$ *above and two additional relations*

v)

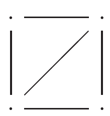
$$\prod_{\kappa=0}^{n} \left(\prod_{m=1}^{2(3d-1)^n} t_{\boldsymbol{i}_\kappa(m)} \right)^6 \quad = \quad 1 \quad = \quad \left(\prod_{m=1}^{2(3d-1)^n} t_{\boldsymbol{i}_0(m)} \right)^d$$

Theorem 3 is recovered with Γ the connected graph on 2 vertices and the map $\boldsymbol{i}_0$ enumerating its vertices. In that case the sets $ii), iii)$ and $iv)$ of relations are void.

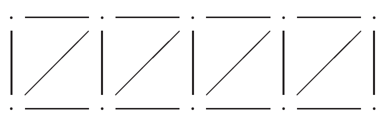
We want to stress the fact that the relations $i) - iii)$ of our presentations have a distinctive flavour since they stem from a different setting: If we consider the Brieskorn-Pham polynomial in the variables $y, x_1, .., x_n$,

$$f \quad = \quad y^3 + x_1^{3d} + \cdots + x_n^{3d},$$

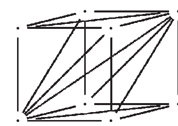
we are naturally led to consider a versal unfolding of the isolated hypersurface singularity it defines. In fact the complement of the discriminant in the unfolding base was shown to have fundamental group generated as in the theorem but with relations $i) - iii)$ only, [Lö3], in terms of a distinguished Dynkin graph $\Gamma_{n,d}$ associated to the singularity, eg. the following and higher dimensional analoga:



$\Gamma_{1,1}$



$\Gamma_{1,2}$



$\Gamma_{2,1}$

We will explain in detail in section 4 how this result can be used in the present paper.

Relations $iv)$ on the contrary are due to degenerations along the hypersurface $x_0 = 0$, while those in $v)$ originate in the action of the automorphism group.

The present paper should be viewed as a further contribution in our ongoing project to understand families of smooth elliptic surfaces and their monodromies, for which we have given an outline in the introduction of [Lö4]. In fact [Lö1] and [Lö2] may be seen as a starting point, since there we have determined the images of homological monodromy.

Moduli stacks enter the stage, since they provide the appropriate means to study all families of a specified kind at once. In particular all their monodromy maps should assemble into a monodromy homomorphism defined on the topological fundamental group of the stack.

A particular nice example – which motivated our research – is provided by the families of elliptic curves, where the homological monodromies assemble into an isomorphism from the orbifold fundamental group of the stack $\mathbb{H}/\operatorname{SL}_2\mathbb{Z}$ to the automorphism group $\operatorname{SL}_2\mathbb{Z}$ of the first homology of a curve, cf. theorem 3. Our aim is to investigate possible generalisations to the case of families of elliptic surfaces, which we believe to be tractable and still to exhibit many characteristic features of the surface case in general.

A major difference to the curve case is the existence of – at least – three distinct moduli problems for families of elliptic surfaces which attract our attention:

i) for smooth elliptic surfaces with a section. The coarse moduli space has been constructed by Miranda and Seiler as the moduli space of Weierstrass fibrations with at most rational double points, cf. [Mi, Sei1].

ii) for smooth elliptic surfaces with a section and irreducible fibres only, equivalently for surfaces with a smooth Weierstrass model. That case is an instance of a moduli problem for polarised elliptic surfaces as considered by Seiler [Sei2].

iii) for smooth elliptic surfaces with a section and nodal fibres only, which were considered in [Lö4] for the benefit of allowing a special kind of monodromy, cf. below.

To hope for as nice a result as in the elliptic curve case, we are forced to adjust the choice of monodromy to the choice of moduli problem. An educated guess among some natural monodromies leads to the following tentative list:

i) algebraic or geometric monodromy. It takes values in the automorphism group of integer (co)homology respective the group of isotopy classes of diffeomorphism.

ii) symplectic monodromy. Both the ambient space and the polarisation may be employed to construct a symplectic connection. The monodromy then takes values in the group of symplectic isotopy classes of symplectomorphisms.

iii) bifurcation braid monodromy. We exploit the fact, that families of elliptic surfaces with nodal fibres only, they naturally give rise to continuous families of finite sets in the base. Thus in case of regular surfaces the monodromy takes values in the braid group of the two-sphere, cf. [Lö4].

Since symplectic monodromy remains quite mysterious despite the efforts of Seidel and others to enlighten the structure of symplectomorphism groups we have proposed a replacement of *ii*) of a more topological flavour: (More details and motivation from a comparison with symplectic monodromy can be found in [Lö4].)

ii') braid class monodromy: Obtained from braid monodromy by imposing just as many relations on the image of braid monodromy as to make sure, that it is well defined on the larger moduli stack.

In any case it is desirable to understand the topological fundamental groups of the moduli stacks and the target groups of the monodromies. While our previous contributions were to monodromies in case *i*) and *iii*), the present paper yields the fundamental group in case *ii*).

Our results also prepare the ground to handle the fundamental group in the other cases. To address *iii*) we have to discard some parts of the moduli stack. On the level of discriminant complements this corresponds to taking the bifurcation divisor into account, the set of parameters u, such that the projection of the corresponding hypersurface $\mathcal{H}_u$ to $\mathbf{P}^1$ is non-generic.

For case *i*) on the other hand, we need to glue in some orbifold divisor to account for some families which are allowed in addition. The associate coarse space is naturally the coarse moduli space of elliptic surfaces with a section constructed as a moduli space of Weierstrass fibrations with at most rational double point singularities. To construct the appropriate stack structure over that space, to get the actual moduli stack for families of smooth elliptic surfaces, will be the task of a forthcoming paper.

Of course we can initiate an analogous program in higher dimension. For example our new results have no dimension restriction. Nevertheless we should note a number of potential obstacles:

i) In higher dimension a generalised bifurcation monodromy can be assigned as long as we admit only family of Weierstrass fibrations with generic bifurcation set of their fibrations. However this monodromy maps but to a group detecting the braiding in $\mathbf{P}^n$ of the critical loci, which then are positive dimensional and singular, cf. the interpretation of $\pi_1(\mathcal{U}_{\mathbf{P}^n,d})$ as group of braiding in $\mathbf{P}^n$ ([Lö5]).

ii) Admitting also families of smooth Weierstrass fibrations, the need of a bifurcation class monodromy has to be checked and if necessary relations have to be imposed on the image of bifurcation monodromy.

iii) A suitable relation of smooth elliptic fibrations with section to Weierstrass fibrations with mild singularities is needed for any progress on geometric monodromy.

2 the discriminant polynomial

The aim of this section is to gain a better insight into the geometric properties of $\mathcal{D}_{n,d}$ and $\mathcal{U}_{n,d}$. Upon identification of the complement of the hyperplane $u_{\mathbf{0}} = 0$ in $\mathbf{P}V_{n,d}$ with the affine hyperplane $u_{\mathbf{0}} = 1$ in $V_{n,d}$, $\mathcal{U}_{n,d}$ is the complement of a hypersurface given by a polynomial

$$p_{n,d} \in \mathbb{C}\big[u_\nu \,\big|\, |\nu| \in \{d, 2d, 3d\}\big].$$

We distinguish the parameter $z = u_{3d,0,..,0}$ (and emphasize the distinction by the new notation from now on). With respect to the parameter z we define the discriminant polynomial $q_{n,d} = discr_z p_{n,d}$ of $p_{n,d}$ and its leading coefficient $\ell_{n,d}$, which together with $p_{n,d}$ will be the targets of our ensuing investigations.

For convenience we recall some topological Euler numbers:

$$e_n = n+1, \quad \text{of complex projective space } \mathbf{P}^n,$$

$$e_{n;d} = n+1+\frac{(1-d)^{n+1}-1}{d}, \quad \text{of smooth hypersurfaces of degree } d \text{ in } \mathbf{P}^n,$$

$$e_{n;d,d} = n+1+(n-1)(1-d)^n +2\frac{(1-d)^n-1}{d}, \quad \text{of smooth complete intersections of two hypersurfaces of degree } d \text{ in } \mathbf{P}^n,$$

$$e_{_H} = 3e_n - 2e_{n;3d}, \quad \text{of a smooth } H \text{ in } |\mathcal{O}_{X_{n,d}}(3\sigma_0)|,$$

$$e_{_{H\cap H'}} = 3e_{n;3d} - 2e_{n;3d,3d}, \quad \text{of a smooth intersection of two hypersurfaces in } |\mathcal{O}_{X_{n,d}}(3\sigma_0)|.$$

The last two formulas are immediate from the fact that there is a smooth hypersurface H which is a triple cover over $\mathbf{P}^n$ totally branched over a hypersurface of degree $3d$, resp. an intersection of two such hypersurfaces which is a triple cover over a hypersurface of degree $3d$ totally branched over the intersection with another such hypersurface in $\mathbf{P}^n$.

Lemma 2.1 *The discriminant polynomial $p_{n,d}$ is of degree*

$$\deg p_{n,d} \quad = \quad 2(n+1)(3d-1)^n.$$

Proof: The degree of $p_{n,d}$ is by definition the number intersection points of its zero set with a generic affine line, hence the number of singular hypersurfaces of the corresponding affine pencil.

The hypersurfaces of such a pencil are contained in an open part of $X_{n,d}$ isomorphic to $\mathcal{O}_{\mathbf{P}^n}(d)$. We obtain the degree of $p_{n,d}$ computing the Euler number $n+1 = e_{\mathbb{C}}e_n$ of $\mathcal{O}_{\mathbf{P}^n}(d)$ from a decomposition into constructible strata with respect to a generic affine pencil:

First the set of points, which belong to fibres intersecting the base locus. It is a $\mathbb{C}$-fibre space over a degree $3d$ hypersurface of $\mathbf{P}^n$ and has Euler number $e_{n;3d}$.

Second the set of points not in the first set, which belong to singular hypersurfaces. This set consists of $\deg p_{n,d}$ hypersurfaces, each regular except for a single ordinary double point and deprived of the base locus of the pencil. Its Euler number is therefore $\deg p_{n,d}(3e_n - 2e_{n;3d} - (-1)^n - (3e_{n;3d} - 2e_{n;3d,3d}))$.

Third the set of points not in the first set, which belong to smooth hypersurfaces. It consists of a smooth family of smooth hypersurfaces each deprived of the base locus of the pencil over the affine line punctured at the $\deg p_{n,d}$ parameters of singular hypersurfaces. The Euler number is therefore $(1-\deg p_{n,d})(3e_n - 2e_{n;3d} - (3e_{n;3d} - 2e_{n;3d,3d}))$.

If we equate the sum of their Euler numbers with $n+1$ and use the numerical values provided above, we get by a straightforward calculation

$$\begin{aligned} n+1 &= e_{n;3d} - \deg p_{n,d}(-1)^n + e_{\mathbb{C}}(3e_n - 2e_{n;3d} - (3e_{n;3d} - 2e_{n;3d,3d})) \\ \iff \deg p_{n,d} &= (-1)^n(2e_n - 4e_{n;3d} + 2e_{n;3d,3d}) \\ &= 2(n+1)(3d-1)^n \end{aligned}$$

□

Lemma 2.2 *The discriminant polynomial $p_{n,d}$ as a polynomial in the coefficient z of x_0^{3d} only has degree*

$$\deg_z p_{n,d} \quad = \quad 2(3d-1)^n.$$

Proof: The degree $\deg_z p_{n,d}$ is equal to the number of singular hypersurfaces in a generic affine pencil with varying part zx_0^{3d}.

We obtain the degree computing the Euler number $n+1 = e_{\mathbb{C}}e_n$ of $\mathcal{O}_{\mathbf{P}^n}(d)$ from a decomposition into three constructible strata with respect to that pencil:

First the set of points, which belong to fibres intersecting the base locus. It is a $\mathbb{C}$-fibre space over the hyperplane $x_0 = 0$ of $\mathbf{P}^n$ and has Euler number e_{n-1}.

Second the set of points not in the first set, which belong to singular hypersurfaces. This set consists of $\deg_z p_{n,d}$ hypersurfaces, each regular except for a single ordinary double point and deprived of the base locus of the pencil. Its Euler number is therefore $\deg_z p_{n,d}(3e_n - 2e_{n;3d} - (-1)^n - (3e_{n-1} - 2e_{n-1;3d}))$.

Third the set of points not in the first set, which belong to smooth hypersurfaces. It consists of a smooth family of smooth hypersurfaces each deprived of the base locus of the pencil over the affine line punctured at the $\deg_z p_{n,d}$ parameters of singular hypersurfaces. The Euler number is thus $(1-\deg_z p_{n,d})(3e_n - 2e_{n;3d} - (3e_{n-1} - 2e_{n-1;3d}))$.

If we equate the sum of their Euler numbers with $n+1$ and use the numerical values provided above, we get by a straightforward calculation

$$\begin{aligned} n+1 &= e_{n-1} - \deg_z p_{n,d}(-1)^n + e_{\mathbb{C}}(3e_n - 2e_{n;3d} - (3e_{n-1} - 2e_{n-1;3d})) \\ \iff \deg_z p_{n,d} &= (-1)^n(2e_n - 2e_{n-1} - 2e_{n;3d} + 2e_{n-1;3d}) \\ &= 2(3d-1)^n \end{aligned}$$

□

To cope with their rôle in the following discussion we introduce the shorthand u'_ν for the parameters of monomials x^ν not containing x_0.

Lemma 2.3 *The discriminant $\mathcal{D}_{n,d}$ defined by the polynomial $p_{n,d}$ is irreducible.*

Proof: In case $n = 0$ the discriminant is the cuspidal cubic, hence irreducible. In case $n > 0$ the fibres of $\mathcal{D}_{n,d}$ under projection to the variables u'_ν are generically irreducible.

If $\mathcal{D}$ were reducible one component thus had to be a union of fibres and therefore had to be defined by a polynomial g in $\mathbb{C}[u'_\nu]$. Of course g must be a factor of each coefficient of $p_{n,d}$ considered as a polynomial in z, in particular of the leading coefficient $\ell_{n,d}$. By the induction hypothesis we conclude that g is a factor of $p_{n-1,d}$. But since there are parameter points on the zero set of $p_{n-1,d}$, which correspond to smooth hypersurfaces, it does not belong to $\mathcal{D}_{n,d}$. Hence there is no fibral component of the discriminant, so we conclude that the discriminant coincides with its irreducible vertical component. □

Lemma 2.4 *The discriminant polynomial $p_{n,d}$ as a polynomial in the coefficient z of x_0^d only has leading coefficient*

$$\ell_{n,d} \quad = \quad p_{n-1,d}^{3d-1}.$$

Proof: For any pencil of hypersurfaces defined by polynomials with varying part zx_0^{3d} we can compute the degree $\deg_z$ of the discriminant polynomial in z by an evaluation of topological Euler numbers again.

We consider the decomposition of $\mathcal{O}_{\mathbf{P}^n}(d)$ into three constructible strata with respect to the given pencil:

First the set of points, which belong to fibres intersecting the base locus. It is the line bundle over the hyperplane $x_0 = 0$ of $\mathbf{P}^n$ and has Euler number e_{n-1}.

Second the set of points not in the first set, which belong to singular hypersurfaces. This set consists of $\deg_z$ hypersurfaces, each regular except for a single ordinary double point and deprived of the base locus Bs of the pencil. Its Euler number is therefore $\deg_z(3e_n - 2e_{n;3d} - (-1)^n - e_{Bs})$.

Third the set of points not in the first set, which belong to smooth hypersurfaces. It consists of a smooth family of smooth hypersurfaces each deprived of the base locus Bs of the pencil over the affine line punctured at the finitely many parameters of singular hypersurfaces. The Euler number is thus $(1 - \deg_z)(3e_n - 2e_{n;3d} - e_{Bs})$.

If we equate the sum of their Euler numbers with $n+1$ and use the numerical values provided above, we get by a straightforward calculation

$$\begin{aligned} n+1 &= e_{n-1} - \deg_z(-1)^n + e_{\mathbb{C}}(3e_n - 2e_{n;3d} - e_{Bs}) \\ \Longleftrightarrow \deg_z &= (-1)^n(2e_n - 2e_{n-1} - 2e_{Bs}) \end{aligned}$$

where the base locus Bs and $\deg_z$ depend on the pencil.

Hence the degree $\deg_z$ drops if and only if the Euler number of Bs differs by a positive multiple of $(-1)^{n-1}$ as compared to the Euler number of the base locus for a generic pencil. That change occurs if and only if Bs is singular, which is a condition on the restriction to the hyperplane $x_0 = 0$ of H_0.

We conclude that the degree $\deg_z$ drops if and only if the discriminant polynomial in the appropriate variables – the parameters u'_ν of monomials not containing x_0 – vanishes. Therefore the reduced equation for the zero set of $\ell_{n,d}$ is $p_{n-1,d}$, which is irreducible by the irreducibility of the discriminant.

We find the multiplicity of $p_{n-1,d}$ in $\ell_{n,d}$ by a comparison of degrees: By lemma 2.1 the polynomial $p_{n,d}$ is homogeneous of degree $2(n+1)(3d-1)^n$ and it has to match the sum of the homogenous degree of $\ell_{n,d}$, which is a multiple of $\deg p_{n-1,d} = 2n(3d-1)^{n-1}$, and $\deg_z p_{n,d}$ which is $2(3d-1)^n$ by lemma 2.2. So we infer our claim. □

Lemma 2.5 *The discriminant polynomial $p_{n,d}$ as a polynomial in the coefficient z of x_0^{3d} only has coprime coefficients.*

Proof: By the preceding lemma the leading coefficient has a unique irreducible factor $p_{n-1,d}$. So the coefficients are not coprime only if $p_{n-1,d}$ is a factor of each.

In that case the zero set of $p_{n-1,d}$ belongs to the discriminant and must be equal to the discriminant since both are irreducible.

This is not true because there are singular hypersurfaces which are regular when restricted to $x_0 = 0$ so contrary to our assumption the coefficients are coprime. □

Lemma 2.6 *The bifurcation polynomial $q_{n,d}$ is homogeneous of degree*

$$(2n+1)2(3d-1)^n\,(2(3d-1)^n-1)\,.$$

Proof: The polynomial $q_{n,d}$ is obtained as the discriminant of $p_{n,d}$ with respect to the variable z and so is homogeneous itself. In fact it can be computed from the Sylvester matrix of $p_{n,d}$ and $\partial_z p_{n,d}$; up to a factor consisting of the leading coefficient polynomial $\ell_{n,d}$ it is the determinant of that matrix.

Hence it is sufficient to add the degrees along the diagonal for any reordering of the matrix. In fact we can arrange on this diagonal $\deg_z p_{n,d}$ times the leading coefficient of degree $\deg p_{n,d} - \deg_z p_{n,d}$ and $\deg_z p_{n,d} - 1$ times the constant coefficient of degree $\deg p_{n,d}$. By the above we have to subtract the degree of the leading coefficient and thus we get

$$\begin{aligned}
\deg q_{n,d} &= \deg_z p_{n,d}(\deg p_{n,d} - \deg_z p_{n,d}) + (\deg_z p_{n,d} - 1)\deg p_{n,d} \\
&\quad -(\deg p_{n,d} - \deg_z p_{n,d}) \\
&= (\deg_z p_{n,d} - 1)(\deg p_{n,d} - \deg_z p_{n,d}) + (\deg_z p_{n,d} - 1)\deg p_{n,d} \\
&= (\deg_z p_{n,d} - 1)(2\deg p_{n,d} - \deg_z p_{n,d}) \\
&= (2(3d-1)^n - 1)\,(4(n+1)(3d-1)^n - 2(3d-1)^n) \\
&= (2n+1)\,(2(3d-1)^n - 1)\,2(3d-1)^n
\end{aligned}$$

□

In the next instances we consider $p_{n,d}$ to be weighted homogeneous. The weight we assign to each u_ν is equal to the exponent of x_0 in the monomial of which it is the parameter:

$$wt(u_\nu) \quad = \quad \nu_0.$$

In particular the weight is zero if x_0 does not occur, ie. in case of parameters u'_ν, and it is $3d$ precisely in the case of the parameter z.

Lemma 2.7 *$p_{n,d}$ is weighted homogeneous of degree*

$$\text{w-deg}\, p_{n,d} \quad = \quad 2\cdot 3d(3d-1)^n.$$

Proof: The leading term of $p_{n,d}$ is of degree $2(3d-1)^n$ in z which is of weight $3d$ with coefficient $\ell_{n,d}$, a polynomial in the u'_ν. Hence $\ell_{n,d}$ is of weight zero and the claim follows. □

Lemma 2.8 *$q_{n,d}$ is weighted homogeneous of degree*

$$\text{w-deg}\, q_{n,d} \quad = \quad 2\cdot 3d(3d-1)^n\,(2(3d-1)^n - 1)\,.$$

Proof: The leading coefficient of $p_{n,d}$ with respect to z is of weighted degree zero, hence $q_{n,d}$ is the determinant of $p_{n,d}$ with respect to z up to a factor of vanishing weighted degree and an argument as in lemma 2.6 yields

$$\text{w-deg}\, p_{n,d} \quad = \quad \text{w-deg}\, p_{n,d}(\deg_z p_{n,d} - 1).$$

With the numerical values given in lemma 2.7 and lemma 2.2 we get the claim. □

We next investigate the properties of $p_{n,d}$ with respect to the parameters u_ν of the monomials $x_0^{3d-1}x_\kappa$ only. They will be called linear coefficients and denoted by v_κ, whenever we want to emphasize their distinguished rôle.

Lemma 2.9 *$q_{n,d}$ as a polynomial in the linear coefficients v_κ is of degree*

$$\deg_v q_{n,d} \quad = \quad 2 \cdot 3d(3d-1)^{n-1}\left(2(3d-1)^n - 1\right).$$

Proof: The linear coefficients are of weight $3d-1$, so we get the upper bound for $\deg_v q_{n,d}$ to be $\text{w-deg}\, q_{n,d}$ divided by $(3d-1)$.

The existence of at least one non-trivial coefficient can be deduced from the special family

$$y^3 - 3yx_0^{2d} + \sum a_\kappa x_\kappa^{3d} + \sum v_\kappa x_\kappa x_0^{3d-1} + zx_0^{3d}.$$

For all $a_\kappa = 1$ and all v_κ positive real of sufficiently distinct magnitude

$$0 < v_1 \ll v_2 \ll \cdots \ll v_n < 1$$

the discriminant polynomial has simple roots only so the bifurcation polynomial for the family is non-zero.

On the other hand the bifurcation polynomial of our special family is weighted homogeneous again with $\text{w-deg} = 6d(3d-1)^n(2(3d-1)^n - 1)$ to which in fact only the v_κ contribute since all a_κ have weight 0. So from the non-triviality above we conclude our claim. □

Lemma 2.10 *Consider $q_{n,d}$ as a polynomial in the parameters v_κ with coefficients in $\mathbb{C}\big[u_\nu \big| \nu_0 < 3d-1\big]$. Given a monomial of degree $\deg_v q_{n,d}$ in the parameters v_κ, its coefficient c in $q_{n,d}$ is a polynomial in the parameters u'_ν of monomials x^ν not containing x_0 and it is either zero or has degree*

$$\deg c \quad = \quad (2n(3d-1)-1)\, 2(3d-1)^{n-1}\left(2(3d-1)^n - 1\right).$$

Proof: The degree is just the difference between $\deg q_{n,d}$ and $\deg_v q_{n,d}$. The other claim follows from the fact that the leading coefficient has weighted degree 0, so must be a polynomial in the weight 0 parameters u'_ν. □

Lemma 2.11 *Consider $q_{n,d}$ as a polynomial with coefficients in $\mathbb{C}\big[u_\nu \big| \nu_0 < 3d-1\big]$. Then the greatest common divisor of all these coefficient polynomials is trivial.*

Proof: By construction a polynomial f belongs to the zero set of $q_{n,d}$ if and only if at least one of the polynomials $f + zx_0^{3d}, z \in \mathbb{C}$ has more than a single ordinary double point singularity or the restriction $f|_{x_0=0}$ has more than a single ordinary double point singularity.

If f is any polynomial then some perturbation $\tilde{f}$ of f by terms $v_\kappa x_\kappa x_0^{3d-1}$ has the property that $\tilde{f}$ has non-degenerate critical points only. Moreover by changing the perturbation ever so slightly we may assume, that $\tilde{f}$ has even no multiple critical values. Hence $\tilde{f}$ belongs to the zero set of $q_{n,d}$ if and only if the restriction $\tilde{f}|_{x_0=0}$ is non-generically singular.

The zero set of a common factor of all coefficients is either a divisor or empty since $q_{n,d}$ is non-trivial.

Moreover by the preceding lemma 2.10 the coefficients of monomials in $\mathbb{C}[v_\kappa]$ of highest degree contain only parameters u' of monomials x^ν not containing x_0, hence the same must be true for a common factor of all coefficients.

Therefore if a polynomial f belongs to the zero set of a common factor then so does every perturbation $\tilde{f}$ as above.

Hence a polynomial can only belong to the zero set if its restriction $f|_{x_0=0}$ is non-generically singular. Since the set of polynomials with non-generically singular restriction to $x_0 = 0$ is of codimension two, the zero set of any common factor is empty and therefore any such common factor is a non-zero constant. □

3 Zariski arguments

The ideas of Zariski as elaborated by Bessis [Be] for the affine set up provide the tool to get hold of a presentation for the fundamental group of $\mathcal{U}_{n,d}$. A key notion concerns distinguished elements:

Definition An element in a fundamental group representable by a path isotopic to the boundary of a small disc transversal to a divisor is called a *geometric element*.

A basis of the fundamental group of a punctured affine line is called a *geometric basis*, if its elements are geometric and simultaneously representable by paths disjoint except for the base point.

We denote now by $\mathbb{C}^N$ the affine parameter space complement to the hypersurface $u_{\mathbf{0}} = 0$ containing divisors $\mathcal{D}, \hat{\mathcal{A}}, \hat{\mathcal{B}}$ given by $p_{n,d}$, $p_{n-1,d}$ and $q_{n,d}$ respectively. Note that for notational convenience we suppress the dependence on integers n, d occasionally.

We project $\mathbb{C}^N$ to $\mathbb{C}^{N-1}$ along the distinguished parameter z and get divisors $\mathcal{A}, \mathcal{B}$ defined by $p_{n-1,d}$ and $q_{n,d}$ again. By construction $\mathcal{A}, \mathcal{B}$ pull back to $\hat{\mathcal{A}}, \hat{\mathcal{B}}$ along the projection and $\mathcal{D}$ is finite over $\mathbb{C}^{N-1}$, branches along $\mathcal{B}$ and has $\hat{\mathcal{A}}$ as its vertical asymptotes.

Lemma 3.1 *Suppose L is a fibre of the projection such that its intersection $\mathcal{D}_L$ with the discriminant $\mathcal{D}$ consists of* $\deg_z p_{n,d}$ *points, then there is a split exact sequence*

$$1 \to \pi_1(L - \mathcal{D}_L) \to \pi_1(\mathbb{C}^N - \hat{\mathcal{A}} - \hat{\mathcal{B}} - \mathcal{D}) \to \pi_1(\mathbb{C}^{N-1} - \mathcal{A} - \mathcal{B}) \to 1.$$

with a splitting map which takes geometric elements associated to $\mathcal{B}$ to geometric elements associated to $\hat{\mathcal{B}}$.

Proof: In fact over the complement of $\mathcal{A} \cup \mathcal{B}$ the discriminant is a finite topological cover and its complement is a locally trivial fibre bundle with fibre the affine line punctured at $\deg_z p_{n,d}$ points. The exact sequence is now obtained from the long exact sequence of that fibre bundle. Exactness on the left follows from the fact that no free group of rank more than 1 admits a normal abelian subgroup.

Since the points on $\mathcal{D}$ vary continuously with the parameters in $\mathbb{C}^{N-1} - \mathcal{A}$ so does a suitably chosen real upper bound on their moduli. This bound defines a topological section inducing an algebraic one.

Moreover it maps boundaries of small discs transversal to $\mathcal{B}$ to boundaries of small discs transversal to $\hat{\mathcal{B}}$ and disjoint to any other divisor. (Note that the last claim does not hold for boundaries of arbitrarily small discs transversal to $\mathcal{A}$.) □

We derive an immediate corollary on the level of presentations:

Lemma 3.2 *Suppose there is a presentation for the fundamental group of the base*

$$\pi_1(\mathbb{C}^{N-1} - \mathcal{A} - \mathcal{B}) \quad \cong \quad \langle r_\alpha | \mathcal{R}_q \rangle.$$

in terms of geometric generators then there is a presentation

$$\pi_1(\mathbb{C}^n - \hat{\mathcal{A}} - \hat{\mathcal{B}} - \mathcal{D}) \quad \cong \quad \langle t_i, \hat{r}_\alpha | \hat{r}_\alpha t_i^{-1} \hat{r}_\alpha^{-1} \phi_\alpha(t_i), \mathcal{R}_q \rangle.$$

with elements t_i of a free geometric basis for a generic fibre $L - \mathcal{D}_L$ and ϕ_α, $\hat{r}_\alpha$ the monodromy automorphism, resp. the lift associated to r_α.

To get hold on $\pi_1(\mathbb{C}^{N-1} - \mathcal{A} - \mathcal{B})$ we will exploit a further projection.

Lemma 3.3 *There is a linear combination v_Σ of the v_κ such that the projection $p_v : \mathbb{C}^{N-1} \to \mathbb{C}^{N-2}$ along v_Σ has the following property:*

> *There exists a divisor $\bar{\mathcal{C}}$ such that no component of its pull-back $\mathcal{C}$ to $\mathbb{C}^{N-1}$ is a component of $\mathcal{B}$ and such that the induced map $p_v| : \mathcal{B} \to \mathbb{C}^{N-2}$ is a topological finite covering over the complement of $\bar{\mathcal{C}}$.*

Proof: For general v_Σ the set of singular values for the induced map $p_v| : \mathcal{B} \to \mathbb{C}^{N-2}$ is a divisor $\bar{\mathcal{C}}$, since we equip $\mathcal{B}$ with its reduced structure. Moreover in its complement $\mathcal{B}$ must be a topological fibration hence a topological covering.

Since a common component of $\mathcal{B}$ and the divisor $\mathcal{C}$ can be detected algebraically as a nontrivial factor of $q_{n,d}$ which is independent of the variable v_Σ, It suffices to show that for a suitable choice of v_Σ there is no such factor.

We decompose the polynomial algebra $\mathbb{C}[u_\nu]$ according to the degree $\deg_v$ of each monomial considered as a monomial in the v_κ only. With respect to the $\deg_v$ decomposition we have the summand q_{max} of $q_{n,d}$ of highest degree. Its coefficients are in $\mathbb{C}[u'_\nu]$ by lemma 2.10.

Therefore q_{max} defines a proper hypersurface in some trivial affine bundle over $\mathbf{P}^{n-1}$. If we replace the projective coordinates $v_1 : v_2 : ... : v_n$ of $\mathbf{P}^{n-1}$ by new ones v'_κ in such a way that a point in the complement has coordinates $(0 : ... : 0 : 1)$ in the factor $\mathbf{P}^{n-1}$, then $q_{n,d}$ is a polynomial of highest possible degree in the variable $v_\Sigma = v'_n$. The leading coefficient is thus in $\mathbb{C}[u'_\nu]$.

With that choice, a non-trivial factor of $q_{n,d}$ independent of v_Σ may only depend on the u'_ν. But we know already from the proof of lemma 2.11, that no divisor defined in terms of the u'_ν only can be a component of $\mathcal{B}$, hence there is no common component of $\mathcal{C}$ and $\mathcal{B}$. $\square$

We suppose from now on a projection $p_v : \mathbb{C}^{N-1} \longrightarrow \mathbb{C}^{N-2}$ as in lemma 3.3 has been fixed with L' a generic fibre and we denote by $\bar{\mathcal{A}}$ the divisor in $\mathbb{C}^{N-2}$ defined by $p_{n-1,d}$.

Lemma 3.4 *Suppose there are geometric elements r_a associated to $\mathcal{A}$ and a geometric basis consisting of elements r_b of $\pi_1(L' - \mathcal{B}_{L'})$ such that the r_a generate $\pi_1(\mathbb{C}^{N-1} - \mathcal{A})$, then the r_a and r_b together generate*

$$\pi_1(\mathbb{C}^{N-1} - \mathcal{A} - \mathcal{B}).$$

Proof: Given the elements r_a which generate $\pi_1(\mathbb{C}^{N-1} - \mathcal{A})$ we may conclude, cf. [Be], that there are geometric elements r_c associated to $\mathcal{C}$ which can be taken in the complement of $\mathcal{B}$ such that together they generate

$$\pi_1(\mathbb{C}^{N-1} - \mathcal{A} - \mathcal{C}) \quad \cong \quad \pi_1(\mathbb{C}^{N-2} - \bar{\mathcal{A}} - \bar{\mathcal{C}}).$$

In fact the situation is similar to that of lemma 3.1 since $\mathbb{C}^{N-1} - \mathcal{A} - \mathcal{B} - \mathcal{C}$ fibres locally trivial over $\mathbb{C}^{N-2} - \bar{\mathcal{A}} - \bar{\mathcal{C}}$ with fibre the appropriately punctured complex line $L' - \mathcal{B}_{L'}$. In the associated short exact sequence

$$\pi_1(L_b - \mathcal{B}_L) \longrightarrow \pi_1(\mathbb{C}^{N-1} - \mathcal{A} - \mathcal{B} - \mathcal{C}) \longrightarrow \pi_1(\mathbb{C}^{N-2} - \bar{\mathcal{A}} - \bar{\mathcal{C}})$$

the elements r_b generate the group on the left hand side and the images of the r_a and r_c generate the group on the right hand side, so together they generate the group in the middle. But then we may deduce, that together they generate

$$\pi_1(\mathbb{C}^{N-1} - \mathcal{A} - \mathcal{B})$$

and that the elements r_c are trivial in that group, so our claim holds. $\square$

Lemma 3.5 *Suppose there is a presentation for the fundamental group of $\mathbb{C}^{N-1} - \mathcal{A}$*

$$\pi_1(\mathbb{C}^{N-1} - \mathcal{A}) \quad \cong \quad \langle r_a | \mathcal{R}_a \rangle.$$

in terms of geometric generators and that $\pi_1(L_b - \mathcal{B}_L)$ is generated by a geometric basis r_b then there is a presentation

$$\pi_1(\mathbb{C}^N - \hat{\mathcal{A}} - \mathcal{D}) \quad \cong \quad \langle t_i, \hat{r}_a | t_i^{-1} \phi_b(t_i), \hat{r}_a t_i^{-1} \hat{r}_a^{-1} \phi_a(t_i), \mathcal{R}_a \rangle.$$

where ϕ_a (ϕ_b) is the automorphism associated to r_a (r_b), t_i is a free geometric basis of $\pi_1(L - \mathcal{D}_L)$ and the $\hat{r}_a$ are lifts of r_a by the topological section.

Proof: In the presentation of lemma 3.2 we have simply to set the geometric generators associated to $\hat{\mathcal{B}}$ to be trivial and to discard them from the set of generators. $\square$

Lemma 3.6 *The fundamental group of $\mathbb{C}^N - \mathcal{D}$ has a presentation*

$$\pi_1 \quad \cong \quad \langle t_i | t_i^{-1} \phi_b(t_i), \rho_a t_i^{-1} \rho_a^{-1} \phi_a(t_i) \rangle.$$

where

i) the t_i form a geometric basis of $\pi_1(L - \mathcal{D}_L)$,

ii) the ρ_a can be expressed in terms of t_i such that $\hat{r}_a \rho_a^{-1}$ is a geometric element associated to $\hat{\mathcal{A}}$ and a lift of r_a,

iii) the ϕ_a, resp. ϕ_b, are the braid monodromies associated to r_a, resp. r_b.

Proof: We deduce this lemma using the presentation of lemma 3.6. Since each $\hat{r}_a$ is transversal to $\hat{\mathcal{A}}$ it must be equal to a geometric element $\hat{r}'_a$ for $\hat{\mathcal{A}}$ up to some $\hat{\rho}_a$ expressible in terms of geometric elements for $\mathcal{D}$. These elements in turn are expressible in terms of t_i since the subgroup generated by the t_i is normal.

In the absence of $\hat{\mathcal{A}}$ the geometric elements $\hat{r}'_a$ are obviously trivial, so from the presentation of lemma 3.6 we discard the generators $\hat{r}_a$ and we replace each $\hat{r}_a$ by $\hat{\rho}_a$ in the relations. The relations $\mathcal{R}_a$ are thus replaced by a set of relations $\mathcal{R}(\hat{\rho}_a)$. But these are in fact relations which we want to show to be superfluous.

So we choose $L'' \subset H'' \subset \mathbb{C}^{N-1}$ such that

i) the preimages $E \subset H \subset \mathbb{C}^N$ contain the line L,

ii) L'' and H'' are generic for $\mathcal{A}$, thus in particular

$$\pi_1(L'' - \mathcal{A}_{L''}) \twoheadrightarrow \pi_1(H'' - \mathcal{A}_{H''}) \cong \pi_1(\mathbb{C}^{N-1} - \mathcal{A}).$$

These choices give rise to commutative diagrams

$$\begin{array}{ccccccc}
\pi_1(L - \mathcal{D}_L) & \longrightarrow & \pi_1(E - \mathcal{D}_E - \hat{\mathcal{A}}_E) & \longrightarrow & \pi_1(L'' - \mathcal{A}_{L''}) & \longrightarrow & 1 \\
\| & & \downarrow & & \downarrow & & \\
\pi_1(L - \mathcal{D}_L) & \longrightarrow & \pi_1(H - \mathcal{D}_H - \hat{\mathcal{A}}_H) & \longrightarrow & \pi_1(H'' - \mathcal{A}_{H''}) & \longrightarrow & 1
\end{array}$$

and

$$\begin{array}{ccccc}
\pi_1(E - \mathcal{D}_E - \hat{\mathcal{A}}_E) & \longrightarrow & \pi_1(E - \mathcal{D}_E) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \\
\pi_1(H - \mathcal{D}_H - \hat{\mathcal{A}}_H) & \longrightarrow & \pi_1(H - \mathcal{D}_H) & \longrightarrow & 1.
\end{array}$$

And we have the following presentations:

$$\begin{aligned}
\pi_1(E - \mathcal{D}_E) &\cong \langle t_i | t_i^{-1} \phi_b(t_i), \rho_a t_i^{-1} \rho_a^{-1} \phi_a(t_i) \rangle, \\
\pi_1(H - \mathcal{D}_H) &\cong \langle t_i | t_i^{-1} \phi_b(t_i), \rho_a t_i^{-1} \rho_a^{-1} \phi_a(t_i), \mathcal{R}(\hat{\rho}_a) \rangle
\end{aligned}$$

Our aim is to show that E is sufficiently generic to imply $\pi_1(E - \mathcal{D}_E) \cong \pi_1(\mathbb{C}^N - \mathcal{D})$.

One way to see this is to proceed as follows: Each relation in $\mathcal{R}(\hat{\rho}_a)$ is represented by a path in $E - \mathcal{D}_E$. By our choice of H for each such path there exists a 2-cell in $H - \mathcal{D}_H$ with boundary freely homotopic in $H - \mathcal{D}_H$ to that path.

We will ultimately show that then each such path must be null-homotopic in $E - \mathcal{D}_E$ already, thus proving our claim.

But first we remark that each relation in $\mathcal{R}_a$ among elements r_a in $\pi_1(H'' - \mathcal{A}_{H''})$ originates in a singularity of the affine plane curve $\mathcal{A}_{H''}$ or one of its asymptotes parallel to L''.

By genericity each singularity is either a cusp or a node and we may the 2-cell we need in an arbitrarily close 3-sphere avoiding $\mathcal{A}_{H''}$ and the line parallel to L''.

At the asymptotes parallel to L'' each relation is imposed by a singularity at infinity which is of A type and the 2-cell can be found in a 3-sphere around that point again avoiding $\mathcal{A}_{H''}$ and the asymptote parallel to L'' but also the line at infinity.

Each path and each 2-cell is lifted to $H - \mathcal{D}_H$ via our topological section, so they correspond precisely to the relations in $\hat{\mathcal{R}}(\hat{\rho}_a)$.

Using projection of H'' along L'' to some affine line $\mathbb{C}$, the total space $H - \mathcal{D}_H$ is mapped to $\mathbb{C}$, such that outside a finite set in the target containing the values of singularities of $\mathcal{A}_{H''}$ and of vertical asymptotes we get a locally trivial fibration with fibre equal to $E - \mathcal{D}_E$.

By construction each 2-cell belongs to the total space of this fibration. So each path in the fibre $E - \mathcal{D}_E$ is null-homotopic in the total space due to the existence of the corresponding 2-cell, hence it must be null-homotopic in the fibre by the homotopy exact sequence taking into account that π_2 of a punctured affine line is trivial. $\square$

The claim of the lemma is of course only an intermediate step on our way to give a presentation of the fundamental group. Obviously we have to make the relations explicit in the sense that every relation is given in terms of the chosen generators only.

Moreover we should try to reduce the number of relations whenever it is sensible to do so.

Remark We are very lax about the base points. They should be chosen in such a way that all maps of topological spaces are in fact maps of pointed spaces. (In particular in the presence of a topological section there is no choice left; in the fibre and in the total space the base point is the intersection of the section with the fibre and its projection to the base yields the base point there.)

4 Brieskorn Pham unfolding

In this section we first construct a distinguished set of generators for $\pi_1(\mathbb{C}^N - \mathcal{D})$. We pick some distinguished fibres L_v of the projection $p_z : \mathbb{C}^N \to \mathbb{C}^{N-1}$ along the variable z where in each case $L_v - \mathcal{D}_L$ can be equipped with a distinguished geometric basis by the method of Hefez and Lazzeri [HL]. For later use in section 5 we establish a relation between different such bases.

Secondly we want to give explicitly an exhaustive set of relations associated to a geometric basis for the complement of $\mathcal{B}$. So we exploit the relation of two natural spaces of perturbations of the Brieskorn-Pham polynomial $f = y^3 + x_1^{3d} + \cdots + x_n^{3d}$. On one hand there is our space $\mathbb{C}^N$ considered as the affine subspace of $V_{n,d}$ of polynomials of weighted degree at most $3d$, which are monic as polynomials in y. On the other hand there is a space germ $\mathbb{C}^\mu, 0$ of dimension $\mu = 2(d-1)^n$, the base of a versal unfolding of f capturing the perturbations of f in the set-up of singularity theory. That space and the fundamental group of its discriminant complement has been under scrutiny in [Lö3] and we will transfer some results to the current situation.

4.1 Hefez Lazzeri path system

First we want to describe a natural geometric basis for some fibres of the projection $p : \mathbb{C}^N \to \mathbb{C}^{N-1}$. Since we follow Hefez and Lazzeri [HL] we will call such bases accordingly. We note first that fibres L_u of the projection correspond to affine pencil of polynomials

$$f_u(x_1, ..., x_n)z$$

and their discriminant points $\mathcal{D}_L$ are exactly the z such that the z-level of f is singular. As in [HL] we restrict our attention to the linearly perturbed Brieskorn-Pham polynomial:

$$f \quad = \quad y^3 - 3v_0 y + \sum_{\kappa=1}^{n} (x_\kappa^{3d} - 3dv_\kappa x_\kappa).$$

In that family the discriminant points for any generic pencil are in bijection to the elements in the multiindex set of cardinality $2(3d-1)^n$:

$$I_{n,d} \quad = \quad \{\, (i_0, i_1, ..., i_n) \,|\, 1 \le i_0 < 3,\ 1 \le i_\nu < 3d-1 \,\}.$$

More precisely we get an expression for the critical values from [HL]:

Lemma 4.1 (Hefez Lazzeri) *The polynomial defining the critical value divisor is given by the expansion of the formal product (η primitive of order $3d-1$)*

$$\prod_{\boldsymbol{i} \in I_{n,d}} \left(-z + 2(-1)^{i_0} v_0^{\frac{3}{2}} + (3d-1) \sum_{\kappa=1}^{n} \eta^{i_\kappa} v_\kappa^{\frac{3d}{3d-1}} \right).$$

We deduce two immediate corollaries, that the discriminant sets are equal for suitably related parameter values and that they can be constructed inductively:

Lemma 4.2 *The discriminant of the linearly perturbed polynomial f is invariant under the multiplication of v_0 by a third root of unity and of any v_κ by a $3d$-th root of unity.*

Proof: From the expansion above we see that the discriminant polynomial is a polynomial in $v_\kappa^{\frac{3d}{3d-1}}$ but of course it is also a polynomial in v_κ, hence it must be a polynomial in v_κ^{3d} since that is the least common power of both. Then it is obviously invariant under multiplying v_κ by a d-th root. The statement for v_0 is proved analogously. □

Lemma 4.3 *The critical values of f are distributed on circles of radius $(3d-1)|v_n|^{\frac{3d}{3d-1}}$ centred around the critical values of the polynomial*

$$f' \quad = \quad y^3 - 3v_0 y + \sum_{\kappa=1}^{n-1} (x_\kappa^{3d} - 3dv_\kappa x_\kappa).$$

Proof: Again we can use lemma 4.1. A formal zero of the discriminant polynomial for f differs by a term $(3d-1)v_n^{\frac{3d}{3d-1}}$ from a zero of the discriminant polynomial of f', and that difference is of the claimed modulus. □

We assume now that all v_κ are positive real and of sufficiently distinct modulus

$$|v_n| \ll \cdots \ll |v_1| \ll |v_0|.$$

In case $n = 1$ we define the Hefez Lazzeri geometric basis as indicated in figure 1 for $3d-1 = 5$, where each geometric generator is depicted as a tail and a loop around a critical value.

Of course the geometric element associated to a loop-tail pair is represented by a closed path based at the free end of the tail which proceeds along the tail, counterclockwise around the loop and back along the tail again.

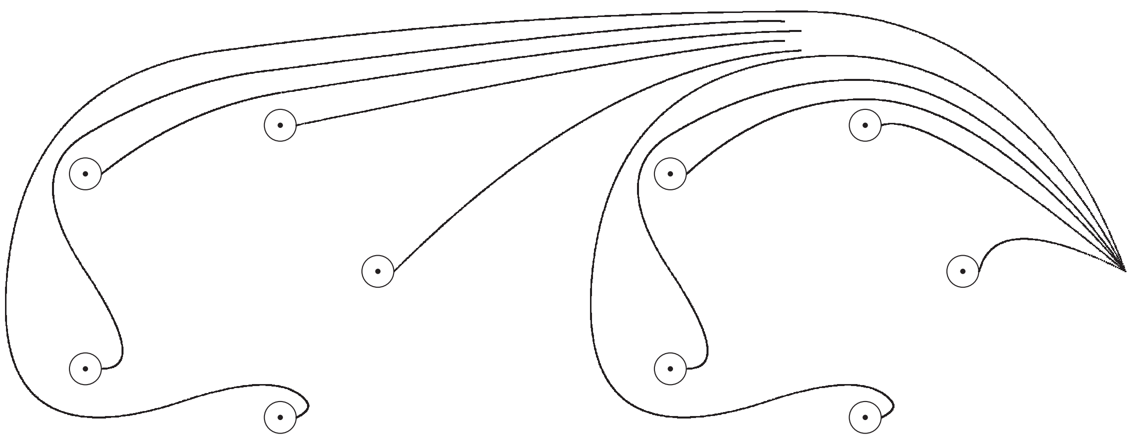


Figure 1: Hefez Lazzeri system in case $n = 1$, $3d-1 = 5$

The $3d-1$ elements of the base associated to punctures on the right are denoted by $t_{1,1}, ..., t_{1,3d-1}$, such that the corresponding critical values are enumerated counterclockwise starting on the positive real line. Similarly the remaining elements are denoted by $t_{2,1}, ..., t_{2,3d-1}$. Each tail on the right is thus labelled unambiguously by its second index in $\{1, ..., 3d-1\}$.

For the inductive step we suppose that the elements of a Hefez Lazzeri base for

$$f' \quad = \quad y^3 - 3v_0 y + \sum_{\kappa=1}^{n-1} (x_\kappa^{3d} - 3d v_\kappa x_\kappa).$$

are given by tail loop pairs each labeled by some multi-index in $I_{n-1,d}$. By assumption v_n is sufficiently small compared to v_{n-1} so we may assume that all critical values of

$$f \quad = \quad y^3 - 3v_0 y + \sum_{\kappa} (x_\kappa^{3d} - 3d v_\kappa x_\kappa).$$

are inside the loops and in fact distributed at distance $(3d-1)|v_\kappa|^{\frac{3d}{3d-1}}$ from their centres.

In the inductive step each loop and its interior are erased and replaced by a scaled copy of the right hand side of the Hefez Lazzeri base in the $n = 1$ case, cf. figure 2. Each tail-loop pair with label $i_n \in \{1, ..., 3d-1\}$ in an inserted disc fits with a tail labeled by some $\boldsymbol{i}' = i_0 i_1 \cdots i_{n-1}$ to form a tail-loop pair representing an element of the new Hefez-Lazzeri base which is labeled by $\boldsymbol{i} = i_0 i_1 \cdots i_{n-1} i_n$.

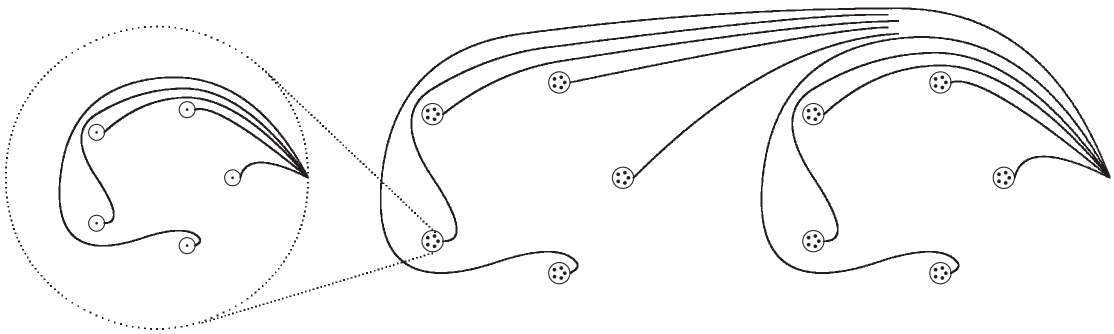


Figure 2: Hefez Lazzeri system in case $n = 2$, $3d - 1 = 5$

The element δ_0 represented by a path enclosing all critical values counter-clockwise deserves our special attention. In the case $n = 1$ we see immediately that as an element in the fundamental group it can be expressed as

$$t_{2,3d-1} t_{2,3d-2} \cdots t_{2,2} t_{2,1} t_{1,3d-1} t_{1,3d-2} \cdots t_{1,2} t_{1,1}.$$

To exploit the inductive construction for the general case we define $t^+_{\boldsymbol{i}'}$, $\boldsymbol{i}' \in I_{n-1,d}$, in the Hefez Lazzeri fibre of

$$f \quad = \quad y^3 - 3v_0 y + \sum_\kappa (x_\kappa^{3d} - 3dv_\kappa x_\kappa)$$

to be represented by the same tail-loop pair as the element $t_{\boldsymbol{i}'}$ in the Hefez Lazzeri fibre of

$$f' \quad = \quad y^3 - 3v_0 y + \sum_{\kappa=1}^{n-1} (x_\kappa^{3d} - 3dv_\kappa x_\kappa).$$

Lemma 4.4 *For any $\boldsymbol{i}' \in I_{n-1,d}$ there is a relation*

$$t^+_{\boldsymbol{i}'} \quad = \quad t_{\boldsymbol{i}'(3d-1)} t_{\boldsymbol{i}'(3d-2)} \cdots t_{\boldsymbol{i}'2} t_{\boldsymbol{i}'1},$$

where $\boldsymbol{i}'(3d-1), \boldsymbol{i}'(3d-2), \ldots, \boldsymbol{i}'2, \boldsymbol{i}'1$ are the obvious elements of $I_{n,d}$.

Proof: The loop of $t^+_{\boldsymbol{i}'}$ is the path which encloses counter-clockwise $3d - 1$ of the critical values which are inserted in the inductive step. So it is homotopic to the product with descending indices of the loop-tail pairs inserted into that loop.

This relation is preserved under appending the tail of $t_{\boldsymbol{i}'}$ and so the claim follows. □

To formulate the general claim we define a linear order $\prec_0$ on $I_{n,d}$ to be the lexicographical order with respect to the linear order $>$ (!) on each component. Accordingly we introduce an order preserving enumeration function

$$\boldsymbol{i}_0 : (\{1, ..., 2(3d-1)^n\}, <) \longrightarrow (I_{n,d}, \prec_0).$$

Lemma 4.5 *Suppose the element δ_0 is represented by a path which encloses counterclockwise all critical values of a Hefez Lazzeri fibre, then*

$$\delta_0 \quad = \quad \prod_{k=1}^{2(3d-1)^n} t_{\boldsymbol{i}_0(k)}.$$

Proof: In case $n = 1$ we have given above an expression for δ_0 which is the expression claimed here, as we have taken care that the order given by $\prec_0$ on $I_{1,d}$ is

$$2, 3d-1 \prec_0 2, 3d-2 \prec_0 \cdots \prec_0 2, 1 \prec_0 1, 3d-1 \prec_0 1, 3d-2 \prec_0 \cdots \prec_0 1, 1.$$

Inductively we then get an expression for δ_0 from that of δ_0' using the definition of $t^+_{\boldsymbol{i}'}$:

$$\delta_0' \quad = \quad \prod_{k=1}^{2(3d-1)^{n-1}} t_{\boldsymbol{i}_o'(k)}. \qquad \Longrightarrow \qquad \delta_0 \quad = \quad \prod_{k=1}^{2(3d-1)^{n-1}} t^+_{\boldsymbol{i}_o'(k)}$$

Finally it suffices to replace each $t^+_{\boldsymbol{i}'}$ using lemma 4.4 above to get our claim. □

As we noticed in lemma 4.2 the set of singular values remains unchanged upon multiplication of any of the real v_κ by a $3d$-th root of unity ξ, resp. v_0 by a third root. The corresponding fibres are thus equipped with the same Hefez Lazzeri systems of paths.

We denote by $t_{\boldsymbol{i}}(\boldsymbol{j})$, $\boldsymbol{j} \in I_{n,d}$, the elements of the Hefez-Lazzeri basis in the fibre at $v(\boldsymbol{j}) = (v_0\xi^{d(j_0-1)}v_1\xi^{j_1-1}, ..., v_n\xi^{j_n-1})$ and by z_0 the fibre coordinate of the Hefez Lazzeri base point, which may be assumed to belong to a topological section as in the proof of lemma 3.1.

We can now compare the fundamental groups $\pi_1(L_{v(\boldsymbol{j})} - \mathcal{D}_{L_{\boldsymbol{j}}}, (v(\boldsymbol{j}), z_0))$ along paths

$$\omega_{\boldsymbol{j}}: \quad s \mapsto (v_0\xi^{sd(j_0-1)}v_1\xi^{s(j_1-1)}, ..., v_n\xi^{s(j_n-1)}, z_0).$$

Lemma 4.6 *Conjugation by a path $\omega_{\boldsymbol{j}}$ induces an isomorphism*

$$\omega^*_{\boldsymbol{j}}: \pi_1(L_{v(\boldsymbol{j})} - \mathcal{D}_{L_{\boldsymbol{j}}}, (v(\boldsymbol{j}), z_0)) \to \pi_1(L_v - \mathcal{D}_L, (v, z_0))$$

such that for $\boldsymbol{i}_0(1) = 11\cdots1 \in I_{n,d}$

$$\omega^*_{\boldsymbol{j}}(t_{\boldsymbol{j}}(\boldsymbol{j})) \quad = \quad t_{\boldsymbol{i}_0(1)}$$

Proof: We consider the case $j_0 = ... = j_{n-1} = 0$ first. Then along $\omega_{\boldsymbol{j}}$ all punctures move counterclockwise in the innermost inserted discs covering an angle of $(j_n - 1)\vartheta$, $\vartheta = \frac{2\pi}{3d-1}$.

Accordingly the final part of each tail has to be adjusted alongside, but all other parts may just be kept fixed. In particular the tail segment with label j_n is moved to the segment with label 1.

Similarly for any $j_\kappa \neq 1$, the κ-th segment of each tail is affected. While precursory segments are unaffected in this more general case successive segments are moved as well, but they are moved to segments having the same label.

We may conclude that in the general case the segments labeled by the components of $\boldsymbol{j}$ are moved to segments labeled by 1 and so our claim holds. □

4.2 Brieskorn-Pham monodromy

The next aim is to determine the set of relations imposed on Hefez-Lazzeri generators by the geometric generators associated to $\mathcal{B}$.

Our strategy is to use the relations imposed by the geometric elements associated to the bifurcation set $\mathcal{B}_f$ in a truncated versal unfolding of the Brieskorn-Pham polynomial

$$f = y^3 + x_1^{3d} + x_2^{3d} + \cdots + x_n^{3d},$$

with base the affine space germ $\mathbb{C}^{\mu-1}, 0$, where μ is the Milnor number of f.

We can do so by means of the *truncated subdiagonal unfolding*, the unfolding of f by non-constant monomials of degree less than d. Its affine base A with bifurcation set $\mathcal{B}_A$ is naturally a subspace of $(\mathbb{C}^{N-1}, \mathcal{B})$, the germ at 0 a subgerm of $(\mathbb{C}^{\mu-1}, \mathcal{B}_f, 0)$.

In fact our aim is to produce a projection $p_v : \mathbb{C}^{N-1} \to \mathbb{C}^{N-2}$ as in lemma 3.4 and a suitable fibre L_v of it which belongs to A.

For the following discussion we observe that a fibre L_v defines a pencil of polynomials, which are linearly related in the sense, that their differences are scalar multiples of a linear polynomial. Moreover a critical point with value z_0 of any such polynomial corresponds to a critical point on the hypersurface defined as its z_0-level.

We will require two conditions on L_v which are specified in the following results:

Lemma 4.7 *There is a pencil of linearly related polynomials in A, each of which has only non-degenerate critical points, except for a finite number of polynomials with degenerate critical points of type A_2 only.*

Proof: First we consider the case $n = 1$. In this case we can perturb x_1^{3d} to a polynomial f_1 of degree $3d$ in x_1 such that the pencil $y^3 + v_0 y + f_1 + v_1 x_1$ of linear perturbations has the required property.

For the general case we just take the sum of such perturbations and thus get a family depending on n parameters v_k, which are the coefficients of the linear monomials:

$$y^3 + y + f_1 + v_1 x_1 + \cdots + f_n + v_n x_n.$$

Since the critical points of the sum are the points such that each coordinate is a critical point of the corresponding summand and the Hessian is diagonal, we deduce, that to get a degenerate critical point at least in one coordinate the corresponding critical point must be degenerate.

In fact it is non-generically degenerate if and only if either the critical point is degenerate in two coordinates or it is non-generically degenerate in one coordinate.

But with our choice this happens only in a codimension two set. Hence we can find a pencil as claimed in the n-dimensional family obtained from the one-dimensional perturbation in each coordinate. □

We remark that if a polynomial in x_1 has one degenerate critical point, then its sum with a polynomial in x_2 has several degenerate critical points in bijection to the critical points of the second summand.

Lemma 4.8 *There is a pencil of linearly related polynomials in A, such that all polynomials have critical points with distinct values, except for a finite set of polynomials, for which there is one multiple value. It must belong to a single pair of non-degenerate critical points.*

Proof: We consider the case $n = 1$. If in the variable x_1 we pick a pencil $f_1 + v_1 x_1$ of polynomials as in the proof above, there are at most two coinciding critical values for each polynomial. Hence there is a lower bound $\varepsilon > 0$ such that at most one pair of critical values has distance less than ε.

In the second variable we pick a generic pencil $f_2 + v_2 x_2$ of linearly related polynomials with a polynomial f_2 of which all critical values and their difference are distinct and within the bound ε of each other.

Their sum yields a two-dimensional family of linearly related polynomials

$$f_1 + v_1 x_1 + f_2 + v_2 x_2.$$

Its critical values are sums of the critical values of both summands.

We deduce that the subfamily $f_1 + v_1 x_1 + f_2$ has only polynomials with at most two coinciding critical values.

Suppose now one of the critical points involved were degenerate. Then we can choose a sufficiently close polynomial such that three critical values are arbitrarily close in contradiction to our construction.

So the corresponding line – and sufficiently close parallel lines – in the parameter plane v_1, v_2 do not meet the locus of polynomials with non-generically conflicting values. Thus this locus must be contained in codimension two, i.e. points, or in parallel lines.

Suppose there is such a parallel line corresponding to a family $f_1 + v_1 x_1 + f_2'$. Then there is a constant a, such that the constant summand and all polynomials of the variable summand have critical values of difference a. Now the set of critical values of the family $f_1 + v_1 x_1$ is described a monic discriminant polynomial $d = d(v_1, z)$ which is irreducible for a generic choice. The condition on the distance of values translates into the condition

$$\mathrm{res}_z(d(v_1, z), d(v_1, z - a)) = 0.$$

But then the two polynomials have a common factor and are thus equal by their irreducibility, which is possible only for $a = 0$. So such a line does not exists and we have shown that polynomials with bad critical value properties occur only in codimension 2.

In the general case $n \geq 2$ we argue as before, but start with a generic pencil in all but the last variable. □

The full genericity property we need comprises the two properties above and the property, that projection along L' is generic in the sense of lemma 3.4.

Lemma 4.9 *There is a projection as in lemma* 3.4 *such that a fibre* L_v *corresponds to pencils as in lemma* 4.7 *and* 4.8.

Proof: By lemma 3.4 we know that there is a projection $p_v : \mathbb{C}^{N-1} \to \mathbb{C}^{N-2}$ such that a generic fibre is generic for $\mathcal{B}$. Notice that we have an open condition, hence a Zariski open set of projections has that property.

Similarly the properties of the two preceding lemmas 4.7 and 4.8 put open conditions on the choice of the projection. Hence we conclude, that a projection p_v and a fibre L_v can be found as claimed. □

We remark that it is not true, that L_v necessarily is a generic fibre of p_v but still it is sufficiently good. We use L_v and an arbitrarily close generic parallel L' to compare the braid monodromy relations of [Lö3] with the relations imposed by geometric elements associated to $\mathcal{B}$ in $\pi_1(\mathbb{C}^N - \mathcal{D})$. To facilitate the use of citations from [Lö3] let us recall:

> Given a linear projection $\mathbb{C}^2 \to \mathbb{C}$ which induces a finite map of a plane curve $C \subset \mathbb{C}^2$ to $\mathbb{C}$, any closed path in the image avoiding the bifurcation set of C gives rise to a braid and the associated isotopy class of diffeomorphisms of the punctured fibre.
>
> The braids thus obtained form the *braid monodromy group*. It acts naturally on the free group generated by a geometric basis in the punctured fibre.
>
> Generalisations to divisors in affine spaces or to the analogue situation for germs are immediate.

To pursue our argument further let t_i be a Hefez Lazzeri geometric base for the fibre L of the projection $p : \mathbb{C}^N \to \mathbb{C}^{N-1}$ over a point corresponding to a linear perturbation of the polynomial f.

We may assume that this polynomial belongs to the pencil determined by L_v.

Proposition 4.10 *Suppose $\pi_1(L' - \mathcal{B})$ is generated by a geometric basis r_b with base point close to point to which L projects. Suppose further that the braid monodromy group Γ of the discriminant in the base of the semi-universal unfolding of f is generated by elements $\{\beta_s\}$. Then there is an identity of normally generated normal subgroups*

$$\langle\langle t_i^{-1} \phi_b(t_i) \rangle\rangle \quad = \quad \langle\langle t_i^{-1} \beta_s(t_i) \rangle\rangle$$

Proof: Arbitrarily close to L_v there is a parallel line L'_f in the base $\mathbb{C}^{\mu-1}$ of the truncated unfolding of f, which is generic with respect to $\mathcal{B}_f$. We denote now by E', E_v and E_f the complex affine planes which project to the lines L', L_v and L'_f respectively. By intersection with the respective discriminants they contain discriminant curves $\mathcal{D}_{E'}, \mathcal{D}_{E_v}$ and $\mathcal{D}_{E_f}$.

By the result of [Lö3] the normal subgroup on the right hand side is the kernel of the natural map from the free group generated by the t_i to $\pi_1(E_f - \mathcal{D}_{E_f}) \cong \pi_1(\mathbb{C}^\mu - \mathcal{D}_f)$. By the genericity of L' the left hand side is the kernel of the map to $\pi_1(E' - \mathcal{D}_{E'})$.

Hence both sides are equal, if we show, that the curves $\mathcal{D}_{E_f}$ and $\mathcal{D}_{E'}$ are isotopic through a path of plane curves.

For the isotopy claim we first observe that both curve naturally degenerate to $\mathcal{D}_{E_v}$. By lemma 4.7 and 4.8 above $\mathcal{D}_{E_v}$ has only double points. Since parallels to L_v are transversal to $\mathcal{B}_f$ in $\mathbb{C}^{\mu-1}$ the degeneration from $\mathcal{D}_{E_f}$ to $\mathcal{D}_{E_v}$ is equisingular and so they are isotopic (though not via an isotopy which preserves the projection). Hence $\mathcal{D}_{E_v}$ is – like $\mathcal{D}_{E_f}$ – a plane curve with only ordinary cusps and ordinary nodes. Therefore the degeneration from $\mathcal{D}_{E'}$ to $\mathcal{D}_{E_v}$ is also equisingular, so $\mathcal{D}_{E'}$ and $\mathcal{D}_{E_f}$ are isotopic. □

The explicit description of the normal subgroup in term of a finite set of relations will be taken from [Lö3] and used in the proof of the main theorem in section 7.

5 Asymptotes

Having fixed a preferred choice of generators for the discriminant complement in the previous section, it is now time to get the explicit relations imposed by the degenerations along the divisor $\mathcal{A}$.

The first and easier task is to understand a local model, that is the restriction of the projection along z to the preimage of a small disc transversal to $\mathcal{A}$. In fact we will get a presentation for its fundamental group in terms of a geometric basis in a local reference fibre subjected to a single relation.

Next we address the hard problem – which lies at the heart of our argument – to bring those local relations together globally: Basically we can start with a path connecting a global reference fibre to a local disc transversal to $\mathcal{A}$. The trivialisation along the path induces an isomorphism between the fundamental groups of the local and global reference fibres at the endpoints.

In practise we are just able to determine unambiguously the images of local relations along a very restricted set of paths, which we have to construct with great care.

Finally we notice that a geometric element associated to $\mathcal{A}$ comes naturally with each path and the boundary of the corresponding local disc. In fact it is our final task for this section to show that the geometric elements thus obtained suffice to apply the results of section 3.

5.1 affine and local models

To describe the geometry of the projected discriminant locally at a point of $\mathcal{A}$ we consider first the plane affine curves C_n

$$C_n \subset \mathbb{C}^2 \quad : \quad y(yx^n - 1) \quad = \quad 0.$$

We call C_n an *asymptotic curve* of order n, since its equation can be rewritten as $y(y - \frac{1}{x^n})$ to show that the y-axis is a vertical asymptote and n is its pole order.

Lemma 5.1 *Suppose b_0, b form a geometric basis in a vertical fibre associated to $y = 0$ and $yx^n = 1$ respectively. Then the complement of the curve C_n in $\mathbb{C}^2$ has fundamental group presentable as*

$$\pi_1(\mathbb{C}^2 - C_n) \quad \cong \quad \langle b_0, b \,|\, b^n b_0 = b_0 b^n \rangle$$

Proof: The complement of C_n is isomorphic to the complement of C'_n, a plane affine curve given by

$$z(x^n - z),$$

since both are complements of a quasi-projective curve in $\mathbb{C} \times \mathbf{P}^1$ given by $yz(x^n y - z)$.

The projection along z exhibits the complement of C'_n to be a fibre bundle over the punctured complex line $\mathbb{C}^*$. The fibre is diffeomorphic to a 2-punctured complex line, which we denote by $\mathbb{C} - [2]$. Since the projection map has a section, we get a short exact sequence with split surjection

$$1 \to \pi_1(\mathbb{C} - [2]) \to \pi_1(\mathbb{C}^* \times \mathbb{C} - C'_n) \to \pi_1(\mathbb{C}^*) \to 1.$$

With base points $z_0 > 1, x_0 = 1$ we may choose elements

– a_0, b_0 in $\pi_1(\mathbb{C} - [2])$ represented by a geometric elements associated to $z = 0$ and $z = x^n$ respectively,

– c in $\pi_1(\mathbb{C}^* \times \mathbb{C} - C'_n)$ represented by a geometric element in the line $z = z_0$ associated to $x = 0$.

All relations in $\pi_1(\mathbb{C}^2 - C'_n)$ follow from the conjugacy action of c on a_0, b_0 in $\pi_1(\mathbb{C} \times \mathbb{C}^* - C'_n)$, which can be determined as a braid monodromy, and the triviality of c in $\pi_1(\mathbb{C}^2 - C'_n)$.

Indeed our claim is derived from the conjugacy action given explicitly as

$$ca_0c^{^{-1}} = (a_0b_0)^n a_0 (a_0b_0)^{-n}, \quad cb_0c^{^{-1}} = (a_0b_0)^n b_0 (a_0b_0)^{-n},$$

which is an immediate consequence of the braid monodromy given by n full twists σ_1^{2n}. Since either $a_0b_0b = 1$ or $b_0a_0b = 1$, we should replace a_0b_0 by $b^{^{-1}}$ or $b_0bb_0^{^{-1}}$ and we get the claim in either case. □

Remark: Note that the complement of C_n can be strongly retracted to any fibred neighbourhood of the y-axis, since C_n is invariant under $(x, y) \mapsto (\lambda^n y, \lambda^{^{-1}} x)$.

Definition A curve $C \subset D \times \mathbb{C}$ is called an *asymptotic germ* of fibre degree m and pole order n if

i) the fibration to D is locally trivial onto the punctured disc D^* with fibre a m-punctured complex line, which we denote by $\mathbb{C} - [m]$,

ii) the projective closure of C in $D \times \mathbf{P}^1$ is a topological cover of D of degree m and intersects the line at infinity with multiplicity n.

An example with fibre degree $m+1$ and pole order n is provided by the plane curve

$$(y^m - 1)(x^n y - 1).$$

A neighbourhood of the y-axis in $\mathbb{C} \times \mathbf{P}^1$ is covered by two bi-disc $D \times D^0$ and $D \times D^\infty$ centered at 0 and ∞, which intersect in the product of D with an annulus A. Given any asymptotic germ $C_{n,m}$ of fibre degree $m + 1$ and pole order n, that kind of open cover can be chosen to cover a sufficiently small neighbourhood $D \times \mathbf{P}^1$ such that

– $D \times A$ and $C_{n,m}$ are disjoint,

– $D \times D^\infty$ contains only the branch of $C_{n,m}$ through ∞,

– $D \times D^0$ contains all other branches of $C_{n,m}$.

For D small enough $D \times D^0$ meets $C_{n,m}$ in m disjoint horizontal copies of D, thus we can identify the complement with the product of D with the m-punctured disc D^0, which we denote by $D^0 - [m]$.

On the other hand $C_{n,m}$ meets $D \times D^\infty$ in a single smooth branch, which intersects the line at infinity with multiplicity n. And so does the branch of our model curve C_n in the complement of the x-axis $y = 0$. Therefore we can discard the branch of $C_{n,m}$ and the line at infinity from $D \times D^\infty$ to get a complement, which can be identified with the complement of C'_n in $D \times \mathbb{C}$. Hence we get a decomposition

$$(D \times \mathbb{C}) - C_{n,m} = (D \times \mathbb{C} - C_n) \cup_{D \times A} (D \times (D^0 - [m])).$$

Proposition 5.2 *If C is an asymptotic germ of fibre degree $m+1$ and pole order n then there is a geometric basis $b_0, b_1, ..., b_m$ in a vertical fibre of the complement such that*

$$\pi_1(D \times \mathbb{C} - C) \quad \cong \quad \langle b_0, b_1, ..., b_m \,|\, (b_1 \cdots b_m)^n b_0 = b_0 (b_1 \cdots b_m)^n \rangle$$

with b_0 associated to the branch going to infinity and D sufficiently small.

Proof: By the hypothesis on D we can employ the decomposition discussed above. We can choose a geometric basis such that the b_i belong to $D \times (D^0 - [m])$ except for b_0, which belongs to the part around infinity. Then $\pi_1(D \times (D^0 - [m]))$ is the free group generated freely by $b_1, ..., b_m$ and $\pi_1(D \times \mathbb{C} - C_n)$ is isomorphic to $\pi_1(\mathbb{C}^2 - C_n)$ and can by lemma 5.1 be presented as

$$\langle b_0, b \,|\, b^n b_0 = b_0 b^n \rangle.$$

Here b is the geometric generator associated to $y = 0$, which is to be identified with the core of the annulus A.

We can then apply the van Kampen theorem to our decomposition with the observation that $b_1 \cdots b_m$ and b represent the same generator of the fundamental group of the intersection, and we get

$$\pi_1 \quad \cong \quad \langle b_0, b_1, ..., b_m, b \,|\, b^n b_0 = b_0 b^n,\ b = b_1 \cdots b_m \rangle$$

from which the claim is immediate by removing b. □

A geometric basis as in the proposition is called *adapted* to the asymptote and provides a tool to make the second set of relations in lemma 3.6 more explicit using the element δ_0 introduced in lemma 4.5.

5.2 a sufficient subfamily

In this subsection we consider a suitable family $\mathcal{G}$ of polynomials in which our parallel transport will take place.

$$\begin{aligned} f \quad = \quad & y^3 - 3\lambda_0 y + \textstyle\sum_i \left(x_i^{3d} - 3d\lambda_i x_i \right) \\ + \quad & x_n^{3d} - 3d\lambda_n x_n - \tfrac{3d}{3d-1} \lambda'_n x_n^{3d-1} - 3\lambda \left(\lambda_0 y x_n^{2d} + d \textstyle\sum_i \lambda_i x_i x_n^{3d-1} \right) \end{aligned}$$

We use the following claims to trace a distinguished critical value along suitable paths.

Lemma 5.3 *The value of f at critical points is given by*

$$f|_{\nabla f = 0} \quad = \quad -2\lambda_0 y - (3d-1)\lambda_n x_n - \tfrac{1}{3d-1} \lambda'_n x_n^{3d-1} - (3d-1) \sum_i \lambda_i x_i.$$

Proof: First the vanishing gradient condition may be expressed by the following system of equations:

$$\begin{aligned} y^2 &= \lambda_0(1+\lambda x_n^{2d}) \\ x_i^{3d-1} &= \lambda_i(1+\lambda x_n^{3d-1}) \\ x_n^{3d-1} &= \lambda_n + \lambda_n' x_n^{3d-2} + \lambda(2\lambda_0 y x_n^{2d-1} + (3d-1)\sum_i \lambda_i x_i x_n^{3d-2}) \end{aligned} \tag{3}$$

Next we use them to replace the pure monomials in f:

$$\begin{aligned} f &= yv(1+\lambda x_n^{2d}) - 3vy + \sum_i \big(x_i(\lambda_i - \lambda d\lambda_i x_n^{3d-1}) - 3d\lambda_i x_i\big) \\ &\quad + x_n(\lambda_n + \lambda_n' x_n^{3d-2} + 2\lambda v y x_n^{2d-1} + (3d-1)\lambda\lambda_i x_i x_n^{3d-2}) \\ &\quad - 3d\lambda_n x_n - \tfrac{3d}{3d-1}\lambda_n' x_n^{3d-1} - 3\lambda\big(v y x_n^{2d} + d\sum_i \lambda_i x_i x_n^{3d-1}\big) \\ &= -2\lambda_0 y - (3d-1)\lambda_n x_n - \tfrac{1}{3d-1}\lambda_n' x_n^{3d-1} - (3d-1)\sum_i \lambda_i x_i. \end{aligned}$$

□

Lemma 5.4 *Suppose $\lambda_0^3 \in \mathbb{R}^+$, $\lambda_i^{3d} \in \mathbb{R}^+$ and $\lambda_n' \in \mathbb{R}^+$. Then there is a positive real λ_{crit} such that the number of critical points (counted with multiplicity) is maximal for $\lambda \in [0, \lambda_{crit}[$ and drops by one at $\lambda = \lambda_{crit}$.*

Proof: From the equations (3) we deduce a polynomial equation for the last coordinate x_n of all critical points. To eliminate the other coordinates we note, that the following expression with ξ_i, ξ_0 primitiv roots of unity of order $3d-1$ resp. 2,

$$\prod_{\rho=1}^{2}\prod_{\rho_i=1}^{3d-1}\big(x_n^{3d-1} - \lambda_n - \lambda_n' x_n^{3d-2} - 2\lambda\xi_0^{\rho}\lambda_0 y x_n^{2d-1} - (3d-1)\lambda\sum_i \xi_i^{\rho_i}\lambda_i x_i x_n^{3d-2}\big)$$

is zero on critical points by the last equation of (3). Due to the invariance under $x_i \mapsto \xi_i x_i$, $y \mapsto \xi_0 y$, our expression is a polynomial y^2 and x_i^{3d-1}. We may thus insert the expressions given by the right hand sides of (3) to get a polynomial in x_n only.

To understand the leading and subleading coefficient, we may neglect lower order terms in the elimination process. Then we get the formal expression

$$\prod_{\rho=1}^{2}\prod_{\rho_i=1}^{3d-1}\big(x_n^{3d-1} - \lambda_n' x_n^{3d-2} - 2\lambda^{\frac{3}{2}}\xi_0^{\rho}\lambda_0^{\frac{3}{2}} x_n^{3d-1} - (3d-1)\lambda^{\frac{3d}{3d-1}}\sum_i \xi_i^{\rho_i}\lambda_i^{\frac{3d}{3d-1}} x_n^{3d-1}\big),$$

where the λ_i are uniquely determined as positive real numbers thanks to the hypotheses λ_0^3, $\lambda_i^{3d} \in \mathbb{R}^+$. So the leading coefficient is given by

$$\prod_{\rho=1}^{2}\prod_{\rho_i=1}^{3d-1}\big(1 - 2\lambda^{\frac{3}{2}}\xi_0^{\rho}\lambda_0^{\frac{3}{2}} - (3d-1)\lambda^{\frac{3d}{3d-1}}\xi_i^{\rho_i}\lambda_i^{\frac{3d}{3d-1}}\big)$$

Let λ_{crit} be its smallest positive real root in λ, which is in fact that of the factor with $\xi_0, \xi_i = 1$. Then the algebraic number of critical points drops at λ_{crit} for the first time. Moreover the next coefficient is given by the sum of all the different possibilities to neglect one factor of the product up to the constant $\lambda'_n > 0$, hence it is non-zero at $\lambda = \lambda_{\text{crit}}$ if and only if no other factor except the obvious one vanishes, which is obviously the case. □

Lemma 5.5 *Suppose $\lambda'_n, \lambda_n \in \mathbb{R}^{\geq 0}$, $\lambda_0, \lambda_i \neq 0$. Then there is no degenerate critical point with $\lambda_0 y$, $\lambda_i x_i$, $x_n \in \mathbb{R}^+$, $\lambda \in [0, \lambda_{crit}[$, except if all λ, λ_n and λ'_n vanish.*

Proof: It suffices to show that the gradient vectors to the equations (3) are linearly independent at solutions with $\lambda_0 y$, $\lambda_i x_i \in \mathbb{R}^+$.

$$\begin{vmatrix} 2y & 0 & \cdots & -2d\lambda\lambda_0 x^{2d-1} \\ 0 & (3d-1)x_i^{3d-2} & \cdots & -\lambda(3d-1)\lambda_i x_n^{3d-2} \\ \vdots & \vdots & \ddots & \vdots \\ -2\lambda\lambda_0 x_n^{2d-1} & -(3d-1)\lambda\lambda_i x_n^{3d-2} & \cdots & \begin{matrix}(3d-1)x_n^{3d-2} - (3d-2)\lambda'_n x_n^{3d-3} \\ -2(2d-1)\lambda\lambda_0 y x_n^{2d-2} \\ -(3d-1)(3d-2)\lambda\sum_i \lambda_i x_i x_n^{3d-3}\end{matrix} \end{vmatrix} \neq 0$$

We multiply each column by the corresponding variable, which is non-vanishing by assumption to get an equivalent claim

$$\begin{vmatrix} 2y^2 & 0 & \cdots & -2d\lambda\lambda_0 x^{2d} \\ 0 & (3d-1)x_i^{3d-1} & \cdots & -\lambda(3d-1)\lambda_i x_n^{3d-1} \\ \vdots & \vdots & \ddots & \vdots \\ -2\lambda\lambda_0 y x_n^{2d-1} & -(3d-1)\lambda\lambda_i x_i x_n^{3d-2} & \cdots & \begin{matrix}(3d-1)x_n^{3d-1} - (3d-2)\lambda'_n x_n^{3d-2} \\ -2(2d-1)\lambda\lambda_0 y x_n^{2d-1} \\ -(3d-1)(3d-2)\lambda\sum_i \lambda_i x_i x_n^{3d-2}\end{matrix} \end{vmatrix} \neq 0$$

Next we apply the equations (3) and simplify the entry in the right bottom corner

$$\begin{vmatrix} 2\lambda_0(1+\lambda x_n^{2d}) & 0 & \cdots & -2d\lambda\lambda_0 x^{2d} \\ 0 & (3d-1)\lambda_i(1+\lambda x_n^{3d-1}) & \cdots & -\lambda(3d-1)\lambda_i x_n^{3d-1} \\ \vdots & \vdots & \ddots & \vdots \\ -2\lambda\lambda_0 y x_n^{2d-1} & -(3d-1)\lambda\lambda_i x_i x_n^{3d-2} & \cdots & \begin{matrix}(3d-1)\lambda_n + \lambda'_n x_n^{3d-2} \\ +2d\lambda\lambda_0 y x_n^{2d-1} \\ +(3d-1)\lambda\lambda_i x_i x_n^{3d-2}\end{matrix} \end{vmatrix} \neq 0$$

We factor in each row but the last the corresponding λ_0, resp. λ_i and add d times the first column, and all following ones to the last:

$$\begin{vmatrix} 2(1+\lambda x_n^{2d}) & 0 & \cdots & 2d \\ 0 & (3d-1)(1+\lambda x_n^{3d-1}) & \cdots & (3d-1) \\ \vdots & \vdots & \ddots & \vdots \\ -2\lambda\lambda_0 y x_n^{2d-1} & -(3d-1)\lambda\lambda_i x_i x_n^{3d-2} & \cdots & (3d-1)\lambda_n + \lambda'_n x_n^{3d-2} \end{vmatrix} \neq 0$$

By hypothesis each summand of the determinant is real and non-negative real, and at least one summand is positive. □

Proposition 5.6 *Suppose $\lambda_n' \in \mathbb{R}^{\geq 0}, \lambda_0^3, \lambda_i^{3d}, \lambda_n \in \mathbb{R}^+$. Then for $\lambda \in [0, \lambda_{crit}[$ there is a unique critical point with $\lambda_0 y, \lambda_i x_i \in \mathbb{R}^+, x_n \in \mathbb{R}^+$, and it has maximal critical value.*

Proof: By assumption the coordinate change $\check{y} = \lambda_0 y$, $\check{x}_i = \lambda_i x_i$ transforms equations (3) into an equivalent system of equations:

$$\begin{aligned} \check{y}^2 &= \lambda_0^3(1 + \lambda x_n^{2d}) \\ \check{x}_i^{3d-1} &= \lambda_i^{3d}(1 + \lambda x_n^{3d-1}) \\ x_n^{3d-1} &= \lambda_n + \lambda_n' x_n^{3d-2} + \lambda(2\check{y}x_n^{2d-1} + (3d-1)\sum_i \check{x}_i x_n^{3d-2}) \end{aligned} \tag{4}$$

For $\lambda = 0$ there is a unique solution with $\check{y}, \check{x}_i, x_n \in \mathbb{R}^+$, because in that case the last equation in (4) has a unique solution in $\mathbb{R}^+$ by the sign rule.

Since solutions depend continuously on the parameter λ, there are only the following transitions, which lead to a change of the number of positive real solutions:

i) a solution tends to infinity, which actually happens for the critical parameter, but not before, cf. lemma 5.4

ii) positive real solutions become complex and vice versa,

iii) positive real solutions become semi-positive or vice versa, but with the give hypotheses there is never a semipositive real solution, since $x_n = 0$ implies $\lambda_n = 0$ and $\check{y} = 0$ or $\check{x}_i = 0$ implies $x_n = 0$ by (4).

The uniqueness claim follows since also the second case can be excluded:

Suppose a complex solution tends to a positive real solution, then there is another complex solution tending to the same real solution, since equations (4) are real, thus complex conjugation acts on solutions. Therefore the limit solution is positive real and degenerate. But with the given hypotheses such degenerate solutions do not exit due to lemma 5.5.

To prove the maximality claim for the value of the distinguished solution we notice:

i) a critical point with last coordinate x_n of smaller modulus than the distinguished critical point has by (4) all coordinates smaller in modulus and hence smaller critical value by lemma 5.3.

ii) for $\lambda = 0$ the distinguished critical point has maximal last coordinate and uniquely so if $\lambda_n' \neq 0$.

iii) For $\lambda_n' \in \mathbb{R}^+$ there is no second critical point with last coordinate equal in modulus to that of the distinguished critical point, since we may argue as follows: We get the last equation of (4) for the last coordinate and for its absolute value,

$$\begin{aligned} x_n^{3d-1} &= \lambda_n + \lambda_n' x_n^{3d-2} + \lambda(2\check{y}x_n^{2d-1} + (3d-1)\sum_i \check{x}_i x_n^{3d-2}) \\ |x_n^{3d-1}| &= \lambda_n + \lambda_n' |x_n^{3d-2}| + \lambda(2\check{y}'|x_n^{2d-1}| + (3d-1)\sum_i \check{x}_i' |x_n^{3d-2}|) \end{aligned}$$

where $\check{y}', \check{x}_i'$ denote the coordinates of the distinguished critical point.

By the other equations in (4) the absolute values of $\check{y}$ and $\check{x}_i$ are bounded by $\check{y}', \check{x}_i'$, hence we may deduce that all summand in the first equation are positive real. In particular $x_n^{3d-1}, x_n^{3d-2} \in \mathbb{R}^+$ and we conclude $x_n \in \mathbb{R}^+$ and consequently $\check{y}, \check{x}_i \in \mathbb{R}^+$.

By the continuity of critical points we may conclude, that for all admissible λ and λ_n' the last coordinate of a critical point is bounded by that of the distinguished critical point. Therefore the distinguished critical points has maximal value according to the first observation. □

5.3 asymptotic arcs, paths, and induced paths

We recall that the parameter space $\mathbb{C}^{N-1}$ of pencils of polynomials is naturally identified – setting $x_0 = 0$ – with a parameter space of polynomials contained in $\mathbb{C}[y, x_1, ..., x_n]$ with vanishing constant coefficient. Hence arcs and paths in $\mathbb{C}^{N-1}$ are given by families of polynomials parameterised by a real interval.

We consider arcs $\alpha_{\boldsymbol{j}}$ in the family $\mathcal{G}$ in bijection to tuples $j_1, ..., j_{n-1}$ starting at the Hefez-Lazzeri base point $\lambda_\kappa = v_\kappa$, $\lambda_n' = 0$, $\lambda = 0$, which corresponds to a polynomial $f = y^3 - 3v_0 y + \sum_\kappa^n (x_\kappa^{3d} - 3dv_\kappa x_\kappa)$ admitting a Hefez-Lazzeri geometric basis. They are composed of two parts each

i) $\lambda, \lambda_n' = 0$, λ_κ moves from v_κ to $v_\kappa \xi^{j_\kappa - 1}$ and λ_0 from v_0 to $v_0 \xi^{d(j_0-1)}$, cf. the construction of the paths ω_j in 4.1.

ii) $\lambda = 0, \lambda_\kappa$ stay fixed, λ_n' increases to some small finite value.

iii) $\lambda_\kappa, \lambda_n'$ stay fixed, λ increases from 0 to λ_{crit}.

By lemma 5.4 each $\alpha_{\boldsymbol{j}}$ leads to a point of $\mathcal{A}$ without intersecting $\mathcal{A}$ elsewhere, so we get well defined geometric elements associated to $\mathcal{A}$. An arbitrarily small isotopy yields the same geometric element represented by a path $\gamma_{\boldsymbol{j}}$ in the complement of $\mathcal{A} \cup \mathcal{B}$.

Proposition 5.7 *The relation imposed on the generators along the path $\gamma_{\boldsymbol{j}}$ is*

$$(t_{j1}^{-1}\delta_0)^{3d-1} = (\delta_0 t_{j1}^{-1})^{3d-1}.$$

Proof: From lemma 2.4 we deduce that over a transversal disc to $\mathcal{A}$ the discriminant is an asymptotic germ of pole order $3d-1$.

So from prop. 5.2 we get a relation between elements of a geometric basis in a local reference fibre adapted to the asymptote. The point is, that we have to identify the significant elements b_0, b with elements in terms of the Hefez-Lazzeri geometric basis along the arc $\alpha_{\boldsymbol{j}}$.

Now the difficult construction of this section shows, that a representative of t_j is first transported to a representative of $t_{1..1}(\boldsymbol{j})$ according to lemma 4.6. That representative is further transported to b_0, since its enclosed puncture remains the puncture of largest modulus according to proposition 5.6 over the second and third part of $\alpha_{\boldsymbol{j}}$.

Of course a loop around all punctures is transported to a loop around all punctures, hence parallel transport identifies δ_0 with $b_0 b$. We conclude $b = t_j^{-1}\delta_0$ and get

$$\begin{aligned} (t_j^{-1}\delta_0)^{3d-1} t_j &= t_j (t_j^{-1}\delta_0)^{3d-1} \\ \iff (t_j^{-1}\delta_0)^{3d-1} &= \delta_0 (t_j^{-1}\delta_0)^{3d-2} t_j^{-1}, \end{aligned}$$

which yields the claim. □

Proposition 5.8 *The set of paths $\gamma_{\boldsymbol{j}}$ generates $\pi_1(\mathbb{C}^{N-1} - \mathcal{A})$.*

Proof: Since $\mathcal{A}$ is defined by $\ell_{n,d} = p_{n-1,d}^{3d-1} \in \mathbb{C}[u'_\nu] = \mathbb{C}[u_\nu | \nu_0 = 0]$, the complement of $\mathcal{A}$ projects to the complement of $\mathcal{D}_{n-1,d}$ via linear projection along all $u_\nu, \nu_0 > 0$. It thus suffices to prove that the projected paths $\gamma'_{\boldsymbol{j}}$ generate $\pi_1(\mathbb{C}^{N'} - \mathcal{D}_{n-1,d})$.

We observe that under projection the first two pieces of each path contract to a point, but the crucial observation is, that the tail of each $\gamma'(\boldsymbol{j})$ is homotopic to a path $t_{1\cdots 1}(\boldsymbol{j})$ conjugated by a path from the base point of $\gamma'(\boldsymbol{j})$ to that of $t_{1\cdots 1}(\boldsymbol{j})$ given by

$$\varpi(\boldsymbol{j}): \quad \lambda \;\mapsto\; \left(\lambda v_0 \xi^{d(j_0-1)}, \lambda v_1 \xi^{j_1-1}, \cdots, \lambda v_{n-1}\xi^{j_{n-1}-1}, 1\right),$$

where the entries correspond to the coefficients of the monomials x_κ and the constant term in the perturbation of the Brieskorn Pham polynomial $f' = y^3 + \sum_{\kappa=1}^{n-1} x_\kappa^{3d}$.

The claim then follows from two additional observations: First – as is seen by methods similar to those in 4.1 – the paths $\omega^*_{\boldsymbol{j}}(t_{1\cdots 1}(\boldsymbol{j}))$ generate $\pi_1(L_{v'} - \mathcal{D}'_L)$, where $v' = (v_0, v_1, ..., v_{n-1})$ and $\mathcal{D}' = \mathcal{D}_{n-1}$.

Second, each $t_{1\cdots 1}(\boldsymbol{j})$ conjugated by $\varpi(\boldsymbol{j})$ is homotopic to $\omega^*_{\boldsymbol{j}}(t_{1\cdots 1}(\boldsymbol{j}))$ conjugated by $\varpi(1\cdots 1)$, since each $\varpi(\boldsymbol{j})$ is homotopic to the concatenation of $\omega_{\boldsymbol{j}}$ – cf. 4.1 – and $\varpi(1\cdots 1)$. □

6 moduli quotient

It is time now to recall that our interest is in the fundamental group of the moduli stack $\mathcal{M}_{n,d}$ of Weierstrass fibrations. Of course we have first to define the appropriate moduli problem and then to show that we can construct $\mathcal{M}_{n,d}$ as a quotient of $\mathcal{U}'_{n,d}$ by a suitable group action.

Definition Suppose W_1, W_2 are Weierstrass fibrations. An isomorphism $\phi : W_1 \to W_2$ is called an *isomorphism of Weierstrass fibrations*, if ϕ preserves the section at infinity and fits into a commutative diagram

$$\begin{array}{ccc} W_1 & \xrightarrow{\phi} & W_2 \\ \downarrow & & \downarrow \\ \mathbf{P}^n & \longrightarrow & \mathbf{P}^n \end{array}$$

Over the parameter space $\mathcal{U}'_{n,d}$ there is the tautological Weierstrass fibration $\mathcal{W}_{n,d}$. The factor group $G = \mathbb{C}^* \times \mathrm{GL}_{n+1} / \mathbb{C}^*$ by the central subgroup given by elements $\left(\lambda^{-d/2}, \lambda \mathbb{1}\right)$ acts faithfully on $\mathcal{W}_{n,d}$ by the action induced from

$$(\lambda, A) \cdot (\mathbf{x}, y_0, y, y_2) \quad = \quad (A \cdot \mathbf{x}, \lambda^2 y_0, y, \lambda^{-1} y_2).$$

Proposition 6.1 *Every isomorphism of Weierstrass fibrations is induced from an automorphism in G of the tautological Weierstrass fibration $\mathcal{W}_{n,d}$.*

Proof: Given an isomorphism ϕ of smooth Weierstrass fibrations we get immediately an induced automorphism of $\mathbf{P}^n$ and we may pick some $A \in \mathrm{GL}_{n+1}$ inducing it.

Therefore we may assume without loss of generality, that ϕ induces the identiy on the base. Hence ϕ induces abstract isomorphisms of plane cubic curves in Weierstrass normal form, mapping points at infinity to points at infinity.

Now it is well known, that each such isomorphism is induced by an automorphism of $\mathbf{P}^2$ of the form determined by a non-vanishing complex number λ acting on the coordinates by $\lambda \cdot (y_0, y, y_2) = (\lambda^2 y_0, y, \lambda^{-1} y_2)$.

Hence we get a map from $\mathbf{P}^n$ to a subgroup of PGL_2 isomorphic to $\mathbb{C}^*$. Thus it must be constant and our claim is proved. □

According to this result it is natural to conceive the following definition:

Definition The moduli stack $\mathcal{M}_{n,d}$ of smooth Weierstrass fibrations is the quotient of the base space $\mathcal{U}'_{n,d}$ by the induced action of G.

Proposition 6.2 *There is an exact sequence*

$$\pi_1(\mathbb{C}^* \times \mathrm{GL}_{n+1}, 1) \longrightarrow \pi_1(\mathcal{U}_{n,d}, u), \longrightarrow \pi_1(\mathcal{M}_{n,d}, [u]) \longrightarrow 1.$$

where the action of $\mathbb{C}^*$ is defined by $\lambda \cdot (p, q) = (\lambda^4 p, \lambda^6 q)$ and GL_{n+1} acts by linear coordinate change $A \cdot (p, q) = (p \circ A, q \circ A)$.

Proof: First we note that the affine group $\mathbb{C}[x_0, ..., x_n]_d$ acts on $\mathcal{W}_{n,d}$ by

$$s(x) \cdot (\mathbf{x}, y_0, y, y_2) \quad = \quad (\mathbf{x}, y_0, y - s(x) y_0, y_2)$$

and the induced action on $\mathcal{U}_{n,d}$ is free and faithful and has $\mathcal{U}'_{n,d}$ as a transversal section. Hence we may replace $\pi_1(\mathcal{U}_{n,d}, u)$ by $\pi_1(\mathcal{U}'_{n,d}, u)$. But then the claim is obviously just the exact sequence for orbifold fundamental groups and the action of G on $\mathcal{U}_{n,d}$ inducing the action of G on $\mathcal{U}'_{n,d}$ made explicit. □

In the following discussion we are going to combine results on fundamental groups of $\mathcal{U}_{n,d}$ and various of its subspaces which do *not* have the same base point.

Still all these base points are contained in a ball in $\mathcal{U}_{n,d}$, so our convention is that all occurring fundamental groups are identified using a connecting path for their base points inside this ball, which makes the identification unambiguous.

Let u now be the parameter point corresponding to the Brieskorn-Pham hypersurface

$$u: \quad y^3 + x_0^{3d} + x_1^{3d} + \cdots + x_n^{3d}.$$

Its $\mathbb{C}^*$ orbit belong entirely to the affine Brieskorn-Pham family $\mathcal{F}$ to which we shift our attention for the moment

$$y^3 + a_0 x_0^{3d} + a_1 x_1^{3d} + \cdots + a_n x_n^{3d}.$$

Definition The element δ_k is defined as the element represented by the path in the Brieskorn-Pham family $y^3 + \sum a_i x_i^{3d}$ given by $a_i = 1$ for $i \neq k$ and $a_k = e^{it}$.

According to our remark above δ_0 may be identified with elements represented by a loop in the coefficient $a_0 = z$ of x_0^{3d} for any sufficiently small perturbation of the Brieskorn-Pham polynomial.

In particular δ_0 identifies with the element of the same name from lemma 4.5, and therefore can be expressed in the geometric basis t_i of $\pi_1(L_0 - \mathcal{D}_L)$ using the enumeration function $\boldsymbol{i}_0 : \{1, ..., 2(3d-1)^n\} \to I_{n,d}$:

$$\delta_0 \quad = \quad \prod_{k=1}^{2(3d-1)^n} t_{\boldsymbol{i}_0(k)}.$$

Remember that $\boldsymbol{i}_0$ has been defined above as the lexicographical order $\prec_0$ of $I_{n,d}$ derived from the *reverse* order of the factors $\{1,2\}$, $\{1, ..., 3d-1\}$ of $I_{n,d} = \{1,2\} \times \{1, ..., 3d-1\}^n$.

$$i_0 i_1 \cdots i_n \prec_0 j_0 j_1 \cdots j_n \quad \Longleftrightarrow \quad \exists k : \, i_\nu = j_\nu \forall \nu < k, i_k > j_k.$$

To give analogous expressions for δ_k we employ similar enumeration functions

$$\boldsymbol{i}_\kappa \quad : \quad \{1, ..., 2(3d-1)^n\} \to I_{n,d}$$

which again are most conveniently described by the linear order $\prec_\kappa$ they induce on $I_{n,d}$:

$$i_0 i_1 \cdots i_n \prec_\kappa j_0 j_1 \cdots j_n \quad \Longleftrightarrow \quad i_\kappa < j_\kappa \quad \vee \quad i_\kappa = j_\kappa, \;\; i_0 i_1 \cdots i_n \prec_0 j_0 j_1 \cdots j_n$$

For $n = 1$ in particular $\prec_1$ is the order on $\{1,2\} \times \{1, ..., d-1\}$ given by

$$(2,1) \prec_1 (1,1) \prec_1 (2,2) \prec_1 (2,1) \prec_1 \cdots \prec_1 (2, 3d-1) \prec_1 (1, 3d-1).$$

Let us resume now our argument to get expressions for the δ_k:

Lemma 6.3 *The δ_k all commute with each other and*

$$\prod \delta_k^6 \quad = \quad 1 \in \pi_1(\mathcal{U}_{n,d}/\mathbb{C}^*).$$

Proof: The family $y^3 + \sum a_i x_i^{3d}$ has discriminant given by the normal crossing divisor $\prod a_i$ and hence the fundamental group of the complement is abelian.

Since the δ_k are geometric generators associated to the $n+1$ hyperplanes the fundamental group of the quotient of the complement $\mathcal{U}_{n,d}$ by $\mathbb{C}^*$ acting with multiplicity 6 is the free abelian group generated by the δ_k modulo the subgroup generated by the sixth power of their product.

Of course this relation maps homomorphically to $\pi_1(\mathcal{U}_{n,d}/\mathbb{C}^*)$. □

Lemma 6.4 *Consider the Hefez Lazzeri family*

$$y^3 - v_0 y x_0^{2d} + x_1^{3d} - v_1 x_1 x_0^{3d-1} + z x_0^{3d}.$$

Suppose t_i and t'_i form geometric Hefez-Lazzeri bases for positive real $v_0 > v_1$ respectively $v'_0 = v_1, v'_1 = v_0$ of sufficiently distinct magnitude, then there is a path connecting the base points such that the associated isomorphism on fundamental groups is given by

$$t_{i_0 i_1} \quad \mapsto \quad t'_{i_1 i_0}$$

Proof: We first convince ourselves that $t_{11} = t'_{11}$ which follows immediately if we change v_0, v_1 continuously in the real line swapping places since the extremal real puncture will keep that property and hence the corresponding geometric element will not be changed.

To move $t_{i_0 i_1}$ we first proceed along a path $\omega_{i_0 i_1}$ as in lemma 4.6 so that it becomes the t_{11} in the new system, then do the same as above and finally employ the same path $\omega_{i_0 i_1}$ back again to come to the final position.

The paths thus needed all connect the two base points with v_0, v_1 interchanged albeit in different ways. But all possible concatenated paths are trivial in the complement of the cuspidal bifurcation component. They may still be non-trivial in the complement of the Maxwell bifurcation component, but the induced isomorphism on the fundamental group is trivial, since the corresponding braid transformation just imposes the commutation relation, which is needed to establish the isomorphism. □

Lemma 6.5 *Consider the Hefez Lazzeri family*

$$y^3 - v_0 y x_0^{2d} + \sum (x_\kappa^{3d} - v_\kappa x_\kappa x_0^{3d-1}) + z x_0^{3d}.$$

Suppose π is a permutation such that t_i and t'_i are geometric generators for positive real $v_0 \gg v_1 \gg \cdots \gg v_n$ respectively $v'_{\pi(0)} = v_0, v'_{\pi(\kappa)} = v_\kappa$ of sufficiently distinct magnitude then there is a path connecting the respective base points such that the associated isomorphism on fundamental groups is given by

$$t_{i_0 i_1 i_2 \cdots i_n} \quad \mapsto \quad t'_{i_{\pi(0)} i_{\pi(1)} i_{\pi(2)} \cdots i_{\pi(n)}}$$

Proof: The same idea of proof as above applies here. □

Lemma 6.6 *The element $\delta_1^{\mathcal{F}}$ in the Fermat family $\mathcal{F}: a_0 x_0^{3d} + a_1 x_1^{3d}$ can be expressed in the geometric basis t_i as*

$$\delta_1 \quad = \quad t_1 t_2 \cdots t_{3d-1}.$$

Proof: Let $\mathcal{U}_{\mathbf{P}^1,l}$ as in theorem 1 denote the discriminant complement associated to the complete linear system of degree l on $\mathbf{P}^1$. It is the quotient of the subset $\tilde{\mathcal{U}}_l \in \mathbb{C}[x_0, x_1]_l$ of homogeneous polynomials of degree l defining l distinct points in $\mathbf{P}^1$ modulo the diagonal $\mathbb{C}^*$ action. Hence there is an exact sequence

$$1 \longrightarrow \pi_1(\mathbb{C}^*) \longrightarrow \pi_1(\tilde{\mathcal{U}}_l) \longrightarrow \pi_1(\mathcal{U}_{\mathbf{P}^1,l}) \longrightarrow 1$$

Now $\delta_1 \delta_0 \in \pi_1(\tilde{\mathcal{U}})$ is the image of a generator of $\pi_1(\mathbb{C}^*)$ and with $\delta_0 = t_{l-1} \cdots t_1$ and Zariski's result (cf. theorem 1)

$$t_1 \cdots t_{l-1} t_{l-1} \cdots t_1 \quad = \quad 1$$

we conclude $\delta_1 = t_1 \cdots t_{3d-1}$ for $l = 3d$ as claimed. □

Lemma 6.7 *In case $n = 1$ the element δ_1 can be expressed in the geometric basis t_i:*

$$\delta_1 \quad = \quad t_{2,1} t_{1,1} t_{2,2} t_{1,2} \cdots t_{2,3d-1} t_{1,3d-1}.$$

We postpone the proof to be able to use some ideas of the proof of the following corollary:

Lemma 6.8 *In case of general n the element δ_1 can be expressed in the geometric basis t_i using the enumeration $\boldsymbol{i}_1 : \{1, ..., 2(3d-1)^n\} \to I_{n,d}$ as*

$$\delta_1 \quad = \quad \prod_{k=1}^{2(3d-1)^n} t_{\boldsymbol{i}_1(k)}.$$

Proof: Since the case $n = 1$ is given in lemma 6.7 we may assume inductively that the claim is proved for $n-1$:

$$\delta_1^{(n-1)} \quad = \quad \prod_{k=1}^{2(3d-1)^{n-1}} t_{i'_1(k)}.$$

Hence there is a homotopy H in the space $\mathcal{U}_{n-1,d}$ between the representing path of $\delta_1^{(n-1)}$ and the concatenation of the representing paths of the $t_{i'_1(k)}$.

We consider the map defined on $\mathcal{U}_{n-1,d}$ by

$$f' \quad \mapsto \quad f = f' + x_n^{3d}$$

which maps H to $\mathcal{U}_{n,d}$. Its image provides a homotopy between the representing path of $\delta_1^{(n)}$ and the concatenation of the images of the paths representing the $t_{i'_1(k)}$.

These images represent elements $t^+_{i'_1(k)}$ which by lemma 4.4 can be expressed as $t^+_{i'_1(k)} = t_{i'_1(k)\,(d-1)} \cdots t_{i'_1(k)\,2} t_{i'_1(k)\,1}$. Therefore the image of H establishes the relation

$$\delta_1^{(n)} \quad = \prod_{k=1}^{2(3d-1)^{n-1}} t^+_{i'_1(k)} \quad = \prod_{k=1}^{2(3d-1)^{n-1}} t_{i'_1(k)\,(d-1)} \cdots t_{i'_1(k)\,2} t_{i'_1(k)\,1} \quad = \prod_{k=1}^{2(3d-1)^n} t_{\boldsymbol{i}_1(k)}. \qquad \square$$

Proof of lemma 6.7: Here we make use of a homotopy H in $\mathcal{U}_{\mathbf{P}^1,3d}$ between the representing paths of $\delta_1^{\mathcal{F}}$ and the concatenated path $t_1 \cdots t_{3d-1}$. We map the homotopy to $\mathcal{U}_{2,d}$ using the map $f' \mapsto f = y^3 + f'$. By an argument as above we thus get

$$\delta_1 \quad = \quad t_1^+ \cdots t_{3d-1}^+ \quad = \quad t'_{1,2} t'_{1,1} \cdots t'_{3d-1,2} t'_{3d-1,1},$$

where the t' form a Hefez-Lazzeri base for a perturbation with $v_0 \ll v_1$. We therefore apply the lemma 6.4 to get

$$\delta_1 \quad = \quad t_{2,1} t_{1,1} \cdots t_{2,3d-1} t_{1,3d-1},$$

$\square$

Lemma 6.9 *In case of general n an expression for δ_κ is given by*

$$\delta_\kappa \quad = \quad \prod_{m=1}^{2(3d-1)^n} t_{\boldsymbol{i}_\kappa(m)}.$$

Proof: We get the expression for general δ_κ using a transposition $\pi = (1\kappa)$ in lemma 6.5 on the expression for δ_1. Then it is obvious that the non-reversed order is transferred from the first entry into the κ-th entry. □

Proposition 6.10 *The image of a generator of $\pi_1(\mathbb{C}^*)$ in the fundamental group $\pi_1(\mathcal{U}_{n,d})$ for the natural map to the orbit of the base point is given in the Hefez-Lazzeri geometric generators t_i as*

$$\prod_{\kappa=0}^{n}\left(\prod_{m=1}^{2(3d-1)^n} t_{\boldsymbol{i}_\kappa(m)}\right)^6$$

Proof: Immediate from the previous. □

Lemma 6.11 *In the fundamental group $\pi_1(\mathcal{U}_{n,d}/\,\mathrm{GL}_n)$ the following relation holds:*

$$\delta_\kappa^{3d} \quad = \quad 1.$$

Proof: Each element δ_κ is the trace of the Brieskorn-Pham point transported by the $\mathbb{C}^*$ action on the coefficient a_κ. The $\mathbb{C}^*$ action on the variable x_κ has the same effect as the $\mathbb{C}^*$ action with multiplicity $3d$ on the coefficient of x_κ^{3d}, hence the claim. □

7 Conclusion

Finally we are in the position to prove the new theorems of the introduction with an actually more explicit claim:

Recall the definition of the index set $I_{n,d} = \{\,(i_0,...,i_n)\,|\,1 \le i_\nu \le 3d-1, i_0 \le 2\,\}$, and its reverse lexicographical order $\prec_0$. We define a graph $\Gamma_{n,d}$ with vertex set $I_{n,d}$ and edge set

$$E(\Gamma_{n,d}) \quad = \quad \big\{(\boldsymbol{i},\boldsymbol{j})\,\big|\,\boldsymbol{i}\neq\boldsymbol{j},\, i_\nu - j_\nu \in \{0,1\}\,\forall\,\nu\,\big\}.$$

Example of such graphs we have given in the introduction. The Main Theorems are thus made precise with the definition of $\Gamma_{n,d}$ just given and the enumeration functions $\boldsymbol{i}_\kappa$ of the previous section defining the distinguished elements δ_κ

Main Theorem for discriminant complement The complement $\mathcal{U}_{n,d}$ of the discriminant $\mathcal{D}_{n,d}$ has fundamental group π_1 finitely presented by generators $t_{\boldsymbol{i}}$, $\boldsymbol{i} \in I_{n,d}$ and relations

i) $t_{\boldsymbol{i}}t_{\boldsymbol{j}} = t_{\boldsymbol{j}}t_{\boldsymbol{i}}$, for all $(\boldsymbol{i},\boldsymbol{j}) \notin E_{n,d}$,

ii) $t_{\boldsymbol{i}}t_{\boldsymbol{j}}t_{\boldsymbol{i}} = t_{\boldsymbol{j}}t_{\boldsymbol{i}}t_{\boldsymbol{j}}$, for all $(\boldsymbol{i},\boldsymbol{j}) \in E_{n,d}$,

iii) $t_{\boldsymbol{i}}t_{\boldsymbol{j}}t_{\boldsymbol{k}}t_{\boldsymbol{i}} = t_{\boldsymbol{j}}t_{\boldsymbol{k}}t_{\boldsymbol{i}}t_{\boldsymbol{j}}$, for all $\boldsymbol{i} \prec \boldsymbol{j} \prec \boldsymbol{k}$ such that $(\boldsymbol{i},\boldsymbol{j}),(\boldsymbol{i},\boldsymbol{k}),(\boldsymbol{j},\boldsymbol{k}) \in E_{n,d}$,

iv) for all $\boldsymbol{i} \in I_{n,d}$

$$t_{\boldsymbol{i}}\left(t_{\boldsymbol{i}}^{-1}\prod_{\boldsymbol{i}\in\mathcal{V}_{n,d}^{\prec}} t_{\boldsymbol{i}}\right)^{3d-1} \quad = \quad \left(t_{\boldsymbol{i}}^{-1}\prod_{\boldsymbol{i}\in\mathcal{V}_{n,d}^{\prec}} t_{\boldsymbol{i}}\right)^{3d-1} t_{\boldsymbol{i}} \qquad (5)$$

Main Theorem for moduli stacks The moduli stack $\mathcal{M}_{n,d}$ for Weierstrass hypersurfaces of type (n,d) has fundamental group π_1 finitely presented by generators $t_{\boldsymbol{i}}$, $\boldsymbol{i} \in I_{n,d}$ and relations i), ii), iii) and iv) as above, and additionally v) and vi):

v)
$$\delta_0^6 \delta_1^6 \cdots \delta_n^6 \quad = \quad 1 \tag{6}$$

vi)
$$\delta_0^{3d} \quad = \quad 1 \tag{7}$$

Proof of both theorems: By lemma 4.9 there is a fibre L' for some projection $p_v : \mathbb{C}^{N-1} \to \mathbb{C}^{N-2}$ such that lemma 3.4 applies; elements r_b of a geometric basis for L' and elements r_a generating $\pi_1(\mathbb{C}^{N-1} - \mathcal{A})$ form a generating set for $\pi_1(\mathbb{C}^{N-1} - \mathcal{A} - \mathcal{B})$.

So in an abstract sense a presentation is obtained with the help of lemma 3.6, but we still have to make explicit the unspecified relations.

In fact we choose generators $t_{\boldsymbol{i}}, \boldsymbol{i} \in I_{n,d}$ represented by a Hefez-Lazzeri geometric basis. By prop. 5.8 we may choose to take the generators r_a to be represented by the paths $\gamma_{\boldsymbol{j}}$. Then the relations of second type in 3.6 can be replaced using those of prop. 5.7, so we get the relations in (5).

From prop. 4.10 we infer that we may replace the first set of relations by any other set normally generating the kernel of the map from the free group to the fundamental group of the discriminant complement in the singularity unfolding, in particular by the set given in below in the result cited from [Lö3].

Proceeding now to the quotient of $\mathcal{U}'_{n,d}$ it suffices to refer to proposition 6.2 to add just two relations, relation (6), as we have shown in prop. 6.10 and relation (7) due to lemma 6.11. □

Theorem 5 *The fundamental group of the discriminant complement for any versal unfolding of a Brieskorn Pham polynomial $y^3 + x_1^{3d} + \cdots x_n^{3d}$ is finitely presented by generators $t_{\boldsymbol{i}}$, $\boldsymbol{i} \in I_{n,d}$ and relations*

i) $t_{\boldsymbol{i}} t_{\boldsymbol{j}} = t_{\boldsymbol{j}} t_{\boldsymbol{i}}$, *for all $\boldsymbol{i}, \boldsymbol{j}$ such that $(\boldsymbol{i}, \boldsymbol{j}) \notin E_{n,d}$,*

ii) $t_{\boldsymbol{i}} t_{\boldsymbol{j}} t_{\boldsymbol{i}} = t_{\boldsymbol{j}} t_{\boldsymbol{i}} t_{\boldsymbol{j}}$, *for all $\boldsymbol{i}, \boldsymbol{j}$ such that $(\boldsymbol{i}, \boldsymbol{j}) \in E_{n,d}$,*

iii) $t_{\boldsymbol{i}} t_{\boldsymbol{j}} t_{\boldsymbol{k}} t_{\boldsymbol{i}} = t_{\boldsymbol{j}} t_{\boldsymbol{k}} t_{\boldsymbol{i}} t_{\boldsymbol{j}}$, *for all $\boldsymbol{i} \prec \boldsymbol{j} \prec \boldsymbol{k}$ such that $(\boldsymbol{i}, \boldsymbol{j}), (\boldsymbol{i}, \boldsymbol{k}), (\boldsymbol{j}, \boldsymbol{k}) \in E_{n,d}$,*

7.1 special cases for small invariants

The case of moduli stack of elliptic curves In case $n = 0$ we consider the set $\mathcal{U}'$ of three distinct points in $\mathbb{C}$ with center of mass in 0.

Accordingly the moduli stack is that of elliptic curves and our theorem just reproduces the well-know result, that

$$\begin{aligned} \pi_1(\mathcal{M}_0) &\cong \langle\, t_1, t_2 \,|\, t_1 t_2 t_1 \;=\; t_2 t_1 t_2, \quad \delta_0^6 = (t_1 t_2)^6 = 1 \,\rangle \\ &\cong \mathrm{SL}_2\mathbb{Z}\,. \end{aligned}$$

The case of elliptic curves in the Hirzebruch surface $\mathbf{F}_1$ In case $n = 1, d = 1$ we consider the set $\mathcal{U}_{1,1}$ of curves in $\mathbf{F}_1$, which is the subset of all smooth elliptic curves in the class of a triple positive section.

That set can be understood as the set of elliptic curves in $\mathbf{P}^2$ not passing through a distinguished point (the blow down of the exceptional line in $\mathbf{F}_1$). Therefore its π_1 should be a central extension of π_1 of the set of all smooth elliptic curves in $\mathbf{P}_2$.

Our results yield a finite presentation of $\pi_1(\mathcal{U}_{1,1})$ by generators $t_{11}, t_{12}, t_{21}, t_{22}$ and relations

- $t_2 t_3 = t_3 t_2$,
- $t_i t_j t_i = t_j t_i t_j$ if $(ij) \in \{(12), (13), (24), (34)\}$,
- $t_i t_j t_k t_i = t_j t_k t_i t_j$ if $(ijk) \in \{(124), (134)\}$,
- $t_4 t_3 t_2 t_4 t_3 t_2 t_1 = t_1 t_4 t_3 t_2 t_4 t_3 t_2$,
- $t_3 t_2 t_1 t_3 t_2 t_1 t_4 = t_4 t_3 t_2 t_1 t_3 t_2 t_1$,

In fact $\pi_1(\mathcal{U}_{1,1})$ is a central extension of $\pi_1(\mathcal{U}_{\mathbf{P}^2,3})$ which is finitely presented with the same generators and relations except for one additional relation

- $t_4 t_3 t_2 t_1 t_2 t_1 t_4 t_3 t_3 t_1 t_4 t_2 = 1$

Moreover $\pi_1(\mathcal{U}_{1,1})$ is isomorphic to $\pi_1(\tilde{\mathcal{U}}_{\mathbf{P}^2,3})$, for the affine cone $\tilde{\mathcal{U}}_{\mathbf{P}^2,3}$ of $\mathcal{U}_{\mathbf{P}^2,3}$ and the central extension is analogous to that of the proof of lemma 6.6.

The case of smooth rational Weierstrass fibrations In case $n = 1, d = 2$ we consider the moduli stack $\mathcal{M}_{1,2}$ of smooth Weierstrass fibrations which are rational. In fact every rational elliptic fibration with a section has a unique Weierstrass model, which is smooth if and only if all fibres are irreducible. Hence our fundamental group is also that of the moduli stack of rational elliptic fibrations with a section and irreducible fibres only. $\pi_1(\mathcal{M}_{1,2})$ is finitely presented by generators

$$t_{1,1}, t_{1,2}, t_{1,3}, t_{1,4}, t_{1,5}, t_{2,1}, t_{2,2}, t_{2,3}, t_{2,4}, t_{2,5},$$

subjected to relations

$$\begin{aligned}
t_{i_0,i_1} t_{j_0,j_1} t_{i_0,i_1} &= t_{j_0,j_1} t_{i_0,i_1} t_{j_0,j_1} && \text{if } |i_1 - j_1| \leq 1, (i_0 - j_0)(i_1 - j_1) \geq 0 \\
t_{i_0,i_1} t_{j_0,j_1} &= t_{j_0,j_1} t_{i_0,i_1} && \text{if } |i_1 - j_1| \geq 2 \text{ or } (i_0 - j_0)(i_1 - j_1) < 0 \\
t_{1,i_1} t_{2,i_1+1} t_{1,i_1+1} t_{1,i_1} &= t_{1,i_1+1} t_{1,i_1} t_{2,i_1+1} t_{1,i_1+1} \\
t_{1,i_1} t_{2,i_1+1} t_{2,i_1} t_{1,i_1} &= t_{2,i_1} t_{1,i_1} t_{2,i_1+1} t_{2,i_1} \\
(t_i^{-1} \delta_0)^5 &= (\delta_0 t_i^{-1})^5 \\
\delta_0^6 = 1 &= \delta_1^6
\end{aligned}$$

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