Distance regularity of compositions of graphs

DRAGAN STEVANOVIĆ[1] stevanov@mathematik.uni-bielefeld.de
FSPM, University of Bielefeld, Germany and *SMG, University of Brussels, Belgium*

Abstract

We study preservation of distance regularity when taking strong sums and strong products of distance-regular graphs.

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1 Introduction

Let G denote a simple connected graph with vertex set $V = V(G)$ having at least two vertices, adjacency matrix $A = A(G)$ and the maximum vertex degree Δ_G. The *distance* between two vertices $u, v \in V$ is represented by $d_G(u,v)$. The *eccentricity* of a vertex u is $\mathrm{ecc}_G(u) := \max_{v\in V} d_G(u,v)$, and the *diameter* of G is $D(G) := \max_{u\in V} \mathrm{ecc}_G(u)$. Further, let $G^k(u)$, $0 \le k \le \mathrm{ecc}_G(u)$, denote the set of vertices at distance k from u, with $G^1(u)$ being simply written as $G(u)$. For the sake of completeness, let $G^k(u) := \emptyset$ if $k < 0$ or $\mathrm{ecc}_G(u) < k$.

A connected graph G is *distance-regular* if, for any two vertices u and $v \in G^k(u)$, $0 \le k \le D(G)$, the numbers $a_k^G(u,v) = |G^k(u)\cap G(v)|$, $b_k^G(u,v) = |G^{k+1}(u)\cap G(v)|$, and $c_k^G(u,v) = |G^{k-1}(u)\cap G(v)|$ do not depend on u and v, but only on k. In such case, we will simply write a_k^G, b_k^G and c_k^G instead of $a_k^G(u,v)$, $b_k^G(u,v)$ and $c_k^G(u,v)$, and call the set of these values the *parameters* of G. Note that $a_k^G + b_k^G + c_k^G = \Delta_G$ for all $0 \le k \le D(G)$. This definition also extends to disconnected graphs, by assuming that each of its components is distance-regular with the same parameters a_k^G, b_k^G and c_k^G for $0 \le k \le D(G)$. Some basic references dealing with this topic are [1, 2, 3]. For other undefined notions of graph theory, the reader is referred to [4].

Many compositions of graphs are defined on the Cartesian product of vertex sets of graphs using only equality and adjacency among corresponding vertices of these graphs. Most widely known are the sum and the product of graphs, which are special cases of the following very general graph composition. It is defined for the first time in [5], while the following definition is taken from [6, p. 66], with a minor modification.

Definition 1 *Let $\mathcal{B}$ be a set of binary n-tuples, i.e. $\mathcal{B} \subseteq \{0,1\}^n \setminus \{(0,\dots,0)\}$ such that for every $i = 1,\dots,n$ there exists $\beta \in \mathcal{B}$ with $\beta_i = 1$. The* non-complete extended p-sum (NEPS) *of graphs $G_1,\dots,G_n$ with basis $\mathcal{B}$, denoted by* $\mathrm{NEPS}\,(G_1,\dots,G_n;\mathcal{B})$, *is the graph with the vertex set $V(G_1)\times \dots \times V(G_n)$, in which two vertices $(u_1,\dots,u_n)$ and $(v_1,\dots,v_n)$ are adjacent if and only if there exists $(\beta_1,\dots,\beta_n) \in \mathcal{B}$ such that u_i is adjacent to v_i in G_i whenever $\beta_i = 1$, and $u_i = v_i$ whenever $\beta_i = 0$. In such case, we will say that the adjacency of $(u_1,\dots,u_n)$ and $(v_1,\dots,v_n)$ is determined by $(\beta_1,\dots,\beta_n)$.*

In particular, for $n = 2$ we have the following instances of NEPS: the *product* $G_1 \times G_2$, when $\mathcal{B} = \{(1,1)\}$; the *sum* $G_1 + G_2$, when $\mathcal{B} = \{(0,1),(1,0)\}$; the *strong sum* $G_1 \oplus G_2$, when $\mathcal{B} = \{(1,1),(1,0)\}$; and the *strong product* $G_1 \otimes G_2$, when $\mathcal{B} = \{(0,1),(1,0),(1,1)\}$. Despite the fact that the sum of graphs is also known as Cartesian product, while the product of graphs is also known as direct product, Kronecker product or tensor product of graphs, we adopted the terminology of [6, p. 66], because of the spectral properties of these operations. The eigenvalues of $G_1 + G_2$ are of the

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form $\lambda_1 + \lambda_2$, and the eigenvalues of $G_1 \times G_2$ are of the form $\lambda_1 \cdot \lambda_2$, where λ_1 is an eigenvalue of G_1, and λ_2 is an eigenvalue of G_2.

Preservation of distance-regularity under the sum and product of distance-regular graphs is studied in [7] and [8]. Here we consider preservation of distance-regularity under the remaining two cases of the strong sum and the strong product of graphs, and in the main theorem in Section 3 we summarize the results from [7], [8] and this paper.

Before passing to our results, we mention that the question of connectedness of NEPS is discussed in [9], where it is proved (see Corollary 2) that if $G_1, \ldots, G_k$ are connected bipartite graphs, and $G_{k+1}, \ldots, G_n$ are connected nonbipartite graphs, then the number of components of $NEPS(G_1, \ldots, G_n; \mathcal{B})$ is equal to $2^{k-\mathrm{rank}\,(\mathcal{B}')}$, where $\mathcal{B}'$ consists of the first k columns of $\mathcal{B}$, and $\mathrm{rank}\,(\mathcal{B}')$ denotes the rank of a 0-1 matrix taken over the binary field. Therefore, we see from definitions of the sum, strong sum and strong product that the resulting composition of connected graphs is always connected. Further, the product of connected graphs is connected if at least one of the graphs is not bipartite, while it has exactly two components if both graphs are bipartite.

2 Results

For a graph G and two vertices $u, v \in V(G)$ we define the *odd distance* $od_G(u, v)$ as the length of the shortest odd walk joining u and v in G, and the *even distance* $ed_G(u, v)$ as the length of the shortest even walk joining u and v in G. If no walk of odd (even) length exists between u and v, then we set $od_G(u, v) = \infty$ ($ed_G(u, v) = \infty$).

We begin with a lemma on distances in compositions of graphs.

Lemma 1 *Let G and H be two connected graphs, and let $u = (u_1, u_2)$, $v = (v_1, v_2) \in V(G) \times V(H)$. Then:*

$$\begin{aligned}
(1)\quad d_{G+H}(u,v) &= d_G(u_1,v_1) + d_H(u_2,v_2),\\
(2)\quad d_{G\otimes H}(u,v) &= \max\{d_G(u_1,v_1), d_H(u_2,v_2)\},\\
(3)\quad d_{G\oplus H}(u,v) &= \begin{cases}
d_G(u_1,v_1), & \textit{if } d_G(u_1,v_1) \geq d_H(u_2,v_2)\\
d_H(u_2,v_2), & \textit{if } d_G(u_1,v_1) < d_H(u_2,v_2)\\
& \quad \textit{and } d_G(u_1,v_1) \equiv d_H(u_2,v_2) \pmod 2\\
d_H(u_2,v_2), & \textit{if } d_G(u_1,v_1) < d_H(u_2,v_2),\\
& \quad d_G(u_1,v_1) \not\equiv d_H(u_2,v_2) \pmod 2\\
& \quad \textit{and } \max\{od_G(u_1,v_1), ed_G(u_1,v_1)\} \leq d_H(u_2,v_2)\\
d_H(u_2,v_2)+1, & \textit{if } d_G(u_1,v_1) < d_H(u_2,v_2),\\
& \quad d_G(u_1,v_1) \not\equiv d_H(u_2,v_2) \pmod 2\\
& \quad \textit{and } \max\{od_G(u_1,v_1), ed_G(u_1,v_1)\} > d_H(u_2,v_2),
\end{cases}
\end{aligned}$$

and if u and v are in the same component of $G \times H$, then

$$(4)\qquad d_{G\times H}(u,v) = \min\{\max\{od_G(u_1,v_1), od_H(u_2,v_2)\}, \max\{ed_G(u_2,v_2), ed_H(u_2,v_2)\}\}.$$

Proof We first show that in the corresponding compositions there exist walks between u and v of lengths given in (1)-(4), and then show that there are no shorter walks. Let $d_1 = d_G(u_1, v_1)$ and $d_2 = d_H(u_2, v_2)$, and let $u_1 = s_0, s_1, \ldots, s_{d_1-1}, s_{d_1} = v_1$ and $u_2 = t_0, t_1, \ldots, t_{d_2-1}, t_{d_2} = v_2$ be the shortest walks between u_1 and v_1 in G, and respectively, between u_2 and v_2 in H. The first coordinate needs at least d_1 steps, the second coordinate needs at least d_2 steps, and thus a walk between u and v may not be shorter than $\max\{d_1, d_2\}$ in any of these compositions.

The following walk between u and v in $G + H$ has length $d_1 + d_2$:

$$W_1:\ u = (s_0,t_0), (s_1,t_0), (s_2,t_0), \ldots, (s_{d_1},t_0), (s_{d_1},t_1), (s_{d_1},t_2), \ldots, (s_{d_1},t_{d_2}) = v.$$

Since any walk between u and v in $G + H$ changes exactly one coordinate at any step, we see that the number of steps needed to go from u to v is at least $d_1 + d_2$. Thus, $d_{G+H}(u, v) = d_1 + d_2$.

Suppose now that $d_1 \geq d_2$. The following walk between u and v has length d_1, and it belongs to both $G \otimes H$ and $G \oplus H$:

$$W_2:\ u = (s_0,t_0), (s_1,t_1), (s_2,t_2), \ldots, (s_{d_2},t_{d_2}), (s_{d_2+1},t_{d_2}), (s_{d_2+2},t_{d_2}), \ldots, (s_{d_1},t_{d_2}) = v.$$

Suppose now that $d_1 < d_2$. The following walk between u and v in $G \otimes H$ has length d_2:

$$W_3: \ u = (s_0, t_0), (s_1, t_1), (s_2, t_2), \ldots, (s_{d_1}, t_{d_1}), (s_{d_1}, t_{d_1+1}), (s_{d_1}, t_{d_1+2}), \ldots, (s_{d_1}, t_{d_2}) = v.$$

Since $\max\{d_1, d_2\}$ is the smallest possible length of any walk between u and v in $G \otimes H$, we conclude that $d_{G\otimes H}(u, v) = \max\{d_1, d_2\}$.

If $d_1 < d_2$ and $d_1 \equiv d_2 \pmod 2$, then the following walk between u and v in $G \oplus H$, which first reaches the vertex (s_{d_1}, t_{d_1}) and then the first coordinate oscillates until the second one reaches t_{d_2}, has length d_2:

$$W_4: \ u = (s_0, t_0), (s_1, t_1), \ldots, (s_{d_1}, t_{d_1}), (s_{d_1-1}, t_{d_1+1}), (s_{d_1}, t_{d_1+2}), (s_{d_1-1}, t_{d_1+3}), \ldots, (s_{d_1}, t_{d_2}) = v.$$

If $d_1 < d_2$ and $d_1 \not\equiv d_2 \pmod 2$, then the following walk between u and v in $G \oplus H$, in which, after reaching (s_{d_1}, t_{d_1}), the first coordinate oscillates until the second one reaches t_{d_2}, has length $d_2 + 1$:

$$W_5: \ u = (s_0, t_0), (s_1, t_1), \ldots, (s_{d_1}, t_{d_1}), (s_{d_1-1}, t_{d_1+1}), (s_{d_1}, t_{d_1+2}), \ldots, (s_{d_1-1}, t_{d_2}), (s_{d_1}, t_{d_2}) = v.$$

A shorter walk between u and v in $G \oplus H$ exists if and only if there exists a walk between u_1 and v_1 in G having a length at most d_2 and the same parity as d_2. Thus, we conclude that (3) also holds.

Finally, any walk between u and v in $G \times H$ must change both coordinates at the same time, therefore it must induce walks between u_1 and v_1 in G and between u_2 and v_2 in H having the same parity. This also shows that $G \times H$ has two components when both G and H are bipartite, because in that case exactly one of $od_G(u_1, v_1)$ and $ed_G(u_1, v_1)$ exists (the same holds for $od_H(u_2, v_2)$ and $ed_H(u_2, v_2)$). Whenever a graph is connected and not bipartite, then both odd and even distance exists for all pairs of its vertices. Thus, if u and v are in the same component, $d_{G\times H}(u, v) \leq \max\{od_G(u_1, v_1), od_H(u_2, v_2)\}$ and $d_{G\times H}(u, v) \leq \max\{ed_G(u_1, v_1), ed_H(u_2, v_2)\}$. Since by "oscillating" one of the coordinates we can construct the walks between u_1 and v_1 in $G \times H$ of lengths $\max\{od_G(u_1, v_1), od_H(u_2, v_2)\}$ and $\max\{ed_G(u_1, v_1), ed_H(u_2, v_2)\}$ (for those values not equal to ∞), we conclude that (4) also holds. ■

For the rest of this section, we suppose that G and H are connected, distance-regular graphs.

Lemma 2 *If either $G \otimes H$ or $G \oplus H$ is distance-regular, then either $D(G) = 1$ or $D(H) = 1$.*

Proof By contradiction. Suppose that $D(G) \geq 2$ and $D(H) \geq 2$. We will show that neither $G \otimes H$ nor $G \oplus H$ can be distance-regular by showing that the coefficients $c_2^{G\otimes H}(u, v)$ and $c_2^{G\oplus H}(u, v)$ vary with the choice of vertices $u, v \in V(G) \times V(H)$.

First, let $u_1, v_1 \in V(G)$, $u_2, v_2 \in V(H)$ be chosen such that $d_G(u_1, v_1) = 2$ and $d_H(u_2, v_2) = 2$. Then for $u = (u_1, u_2)$, $v = (v_1, v_2) \in V(G) \times V(H)$ we have from (2) and (3) that $d_{G\otimes H}(u, v) = 2$ and $d_{G\oplus H}(u, v) = 2$. A vertex $w = (w_1, w_2) \in V(G) \times V(H)$ is adjacent to both u and v if and only if the adjacency between u and w, and between w and v in $G \otimes H$, respectively $G \oplus H$, is determined by the vector $(1, 1)$ from bases of these compositions. Therefore, the vertex $w_1 \in V(G)$ must be adjacent to both u_1 and v_1 which gives c_2^G choices for w_1, while the vertex $w_2 \in V(H)$ must be adjacent to both u_2 and v_2 which gives c_2^H choices for w_2. Thus, we conclude that $c_2^{G\otimes H}(u, v) = c_2^{G\oplus H}(u, v) = c_2^G \cdot c_2^H$.

Next, let $v^* = (v_1, u_2) \in V(G) \times V(H)$. Then from (2) and (3) we have that $d_{G\otimes H}(u, v^*) = 2$ and $d_{G\oplus H}(u, v^*) = 2$. A vertex $w = (w_1, w_2) \in V(G) \times V(H)$ is adjacent to both u and v^* if and only if the adjacency between u and w, and between w and v^* in $G \otimes H$, respectively $G \oplus H$, is determined by one of the vectors $(1, 0)$ and $(1, 1)$ from bases of these compositions. Therefore, the vertex $w_1 \in V(G)$ must be adjacent to both u_1 and v_1 which gives c_2^G choices for w_1, while the vertex $w_2 \in V(H)$ is either equal to u_2 or adjacent to it, giving $\Delta_H + 1$ choices for w_2. Thus, we conclude that $c_2^{G\otimes H}(u, v^*) = c_2^{G\oplus H}(u, v^*) = c_2^G \cdot (\Delta_H + 1)$.

Since $D(G) \geq 2$ and $D(H) \geq 2$, we have that $c_2^G > 0$ and $c_2^H < \Delta_H + 1$, and we obtain that $c_2^{G\otimes H}(u, v) \neq c_2^{G\otimes H}(u, v^*)$ and $c_2^{G\oplus H}(u, v) \neq c_2^{G\oplus H}(u, v^*)$. Thus, neither $G \otimes H$ nor $G \oplus H$ is distance-regular. ■

Let K_m denote the complete graph on m vertices, and let $K_{m_1,m_2,\ldots,m_l}$ denote the complete multipartite graph with parts of sizes m_1, m_2, $\ldots$, m_l.

Theorem 1 *$G \otimes H$ is distance-regular if and only if $G \cong K_m$ and $H \cong K_n$ for some $m, n \in \mathbf{N}$.*

Proof On the one hand, if $G \cong K_m$ and $H \cong K_n$ for some $m, n \in \mathbf{N}$, then $G \otimes H \cong K_m \otimes K_n \cong K_{mn}$, which is distance-regular.

On the other hand, suppose that $G \otimes H$ is distance-regular. Based on Lemma 2 and the fact that the base of the strong product is symmetric with respect to columns, we may suppose, without loss of generality, that $H \cong K_n$ for some $n \in \mathbf{N}$. Choose $u_1, v_1 \in V(G)$ and $u_2, v_2 \in V(K_n)$ such that u_1 and v_1 are adjacent in G and $u_2 \neq v_2$.

First, consider vertices $u = (u_1, u_2)$ and $s = (u_1, v_2)$ of $G \otimes K_n$. They are adjacent in $G \otimes K_n$, and a vertex w adjacent to both u and s has either the form (u_1, w_2), with $w_2 \notin \{u_2, v_2\}$, giving $n-2$ choices for w_2, or the form (w_1, w_2), with w_1 adjacent to u_1 and w_2 being an arbitrary vertex of K_n, giving $\Delta_G \cdot n$ choices. We conclude that

$$a_1^{G\otimes K_n}(u,s) = (\Delta_G + 1)n - 2. \tag{5}$$

Next, consider vertices u and $t = (v_1, u_2)$, that are also adjacent in $G \otimes K_n$. A vertex w adjacent to both u and t has either the form (u_1, w_2), with $w_2 \neq u_2$, giving $n-1$ choices for w_2, or the form (v_1, w_2), with $w_2 \neq u_2$, giving again $n-1$ choices for w_2, or the form (w_1, w_2), with w_1 being adjacent to both u_1 and v_1, and w_2 being an arbitrary vertex of K_n, giving $a_1^G \cdot n$ choices. We conclude that

$$a_1^{G\otimes K_n}(u,t) = (a_1^G + 2)n - 2. \tag{6}$$

From (5) and (6), we have that $a_1^G = \Delta_G - 1$, which together with $c_1^G = 1$, shows that $b_1^G = 0$, i.e., that $D(G) = 1$ and $G \cong K_m$ for some $m \in \mathbf{N}$. ■

Theorem 2 *a) If G has at least three vertices, then $G \oplus H$ is distance-regular if and only if there exists $m, n, t \in N$ such that $G \cong K_{\underbrace{m,m,\ldots,m}_{t}}$ and $H \cong K_n$.*

b) The graph $K_2 \oplus H$ is distance-regular if and only if the parameters of H satisfy the following relations for all $j = 0, 1, \ldots, D(H) - 1$

$$b_j^H = 1 + a_{j+1}^H + b_{j+1}^H, \tag{7}$$
$$1 + a_j^H + c_j^H = c_{j+1}^H. \tag{8}$$

Proof of a) If $G \cong K_{\underbrace{m,m,\ldots,m}_{t}}$ and $H \cong K_n$ for some $m, n, t \in N$, then

$$G \oplus H \cong K_{\underbrace{m,m,\ldots,m}_{t}} \oplus K_n \cong K_{\underbrace{mn,mn,\ldots,mn}_{t}},$$

which is distance-regular.

On the other hand, if $G \oplus H$ is distance-regular, then it follows from Lemma 2 that either $D(G) = 1$ or $D(H) = 1$. If $D(G) = D(H) = 1$, then $G \cong K_m$ (with $t = 1$) and $H \cong K_n$ for some $m, n \in \mathbf{N}$. Thus, we can suppose that either $D(G) \geq 2$ or $D(H) \geq 2$.

First, suppose that $G \cong K_m$ (thus, $t = 1$) for some $m \in \mathbf{N}$, $m \geq 3$, and $D(H) \geq 2$. Let $u = (u_1, u_2)$, $s = (u_1, v_2)$ and $t = (v_1, v_2) \in V(K_m) \times V(H)$, such that $u_1 \neq v_1$ and $d_H(u_2, v_2) = 2$. Then $d_{K_m \oplus H}(u, s) = d_{K_m \oplus H}(u, t) = 2$. A vertex $w = (w_1, w_2)$ is adjacent to both u and s in $K_m \oplus H$ if and only if $w_1 \neq u_1$ and w_2 is adjacent to both u_2 and v_2 in H, showing that $c_2^{K_m \oplus H}(u, s) = (m-1) \cdot c_2^H$. Also, a vertex $w = (w_1, w_2)$ is adjacent to both u and t in $K_m \oplus H$ if and only if $w_1 \neq u_1$, $w_1 \neq v_1$ and w_2 is adjacent to both u_2 and v_2 in H, showing that $c_2^{K_m \oplus H}(u, t) = (m-2) \cdot c_2^H$. From $D(H) \geq 2$ it follows that $c_2^H > 0$, and we get that $c_2^{K_m \oplus H}(u, s) \neq c_2^{K_m \otimes H}(u, t)$, showing that $K_m \oplus H$ is not distance-regular, which is a contradiction.

We conclude that $D(H) = 1$, i.e., $H \cong K_n$ for some $n \in \mathbf{N}$ and that $D(G) \geq 2$. Again, let $u = (u_1, u_2)$, $s = (u_1, v_2)$ and $t = (v_1, v_2) \in V(G) \times V(K_n)$, such that $d_G(u_1, v_1) = 2$ and $u_2 \neq v_2$. Then $d_{G \oplus K_n}(u, s) = d_{G \oplus K_n}(u, t) = 2$. A vertex $w = (w_1, w_2)$ is adjacent to both u and s in $G \oplus K_n$ if and only if w_1 is adjacent to u_1 in G and w_2 is an arbitrary vertex of K_n, showing that $c_2^{G \oplus K_n}(u, s) = \Delta_G \cdot n$. On the other hand, a vertex $w = (w_1, w_2)$ is adjacent to both u and t in $G \oplus K_n$ if and only if w_1 is adjacent to both u_1 and v_1 in G and w_2 is an arbitrary vertex of K_n, showing that $c_2^{G \oplus K_n}(u, t) = c_2^G \cdot n$. It follows that $c_2^G = \Delta_G$, and that $D(G) = 2$ (since then $b_2^G = 0$).

Therefore, each vertex at distance two from u_1 in G must be adjacent to all neighbors of u_1. If we denote by S_{u_1} the set formed by u_1 and all vertices at distance two from u_1 in G, we conclude that each vertex from S_{u_1} is adjacent to all vertices from $V(G) \setminus S_{u_1}$, and the set S_{u_1} has the fixed size: $m = |S_{u_1}| = |V(G)| - \Delta_G$. Thus, each component of the complement $\overline{G}$ of G is a clique of size m, and we conclude that $G \cong K_{\underbrace{m, m, \ldots, m}_{t}}$. ∎

Proof of b) Let $V(K_2) = \{0, 1\}$. If $u_1, v_1 \in V(K_2)$, $u_2, v_2 \in V(H)$ then from (3) it follows that

$$d_{K_2 \oplus H}((u_1, u_2), (v_1, v_2)) = \begin{cases} d_H(u_2, v_2), & d_H(u_2, v_2) \text{ is even and } u_1 = v_1, \\ & d_H(u_2, v_2) \text{ is odd and } u_1 \neq v_1, \\ d_H(u_2, v_2) + 1, & d_H(u_2, v_2) \text{ is even and } u_1 \neq v_1, \\ & d_H(u_2, v_2) \text{ is odd and } u_1 = v_1. \end{cases}$$

Next, let $u = (0, u') \in V(K_2 \oplus H)$ and let $v' \in V(H)$. Then, $d_{K_2 \oplus H}(u, (0, v'))$ is always even and $d_{K_2 \oplus H}(u, (1, v'))$ is always odd (and vice versa if we would have $u = (1, u')$). Thus, $K_2 \oplus H$ is bipartite and we have that $a_l^{K_2 \oplus H}(u, v) = 0$ for every $l \in \mathbf{N}$ and and $v \in V(K_2) \times V(H)$.

Suppose that $d_H(u', v') = 2k$, $k \in \mathbf{N}$. Then $d_{K_2 \oplus H}(u, (0, v')) = 2k$ and $d_{K_2 \oplus H}(u, (1, v')) = 2k + 1$. All neighbors of $(0, v')$ have the form $(1, t)$, $t \in V(H)$, and $d_{K_2 \oplus H}(u, (1, t)) = 2k - 1$ holds if and only if $d_H(u', t) = 2k - 1$. Thus, we see that

(9) $$b_{2k}^{K_2 \oplus H}(u, (0, v')) = 1 + a_{2k}^H + b_{2k}^H \quad \text{and} \quad c_{2k}^{K_2 \oplus H}(u, (0, v')) = c_{2k}^H.$$

Further, all neighbors of $(1, v')$ have the form $(0, t)$, $t \in V(H)$, and $d_{K_2 \oplus H}(u, (0, t)) = 2k + 2$ holds if and only if $d_H(u', t) = 2k + 1$. Thus, we see that

(10) $$b_{2k+1}^{K_2 \oplus H}(u, (1, v')) = b_{2k}^H \quad \text{and} \quad c_{2k+1}^{K_2 \oplus H}(u, (1, v')) = 1 + a_{2k}^H + c_{2k}^H.$$

Similarly, if we suppose that $d_H(u', v') = 2k + 1$, $k \in \mathbf{N}$, we see that

(11) $$b_{2k+2}^{K_2 \oplus H}(u, (0, v')) = b_{2k+1}^H \quad \text{and} \quad c_{2k+2}^{K_2 \oplus H}(u, (0, v')) = 1 + a_{2k+1}^H + c_{2k+1}^H,$$

while

(12) $$b_{2k+1}^{K_2 \oplus H}(u, (1, v')) = 1 + a_{2k+1}^H + b_{2k+1}^H \quad \text{and} \quad c_{2k+1}^{K_2 \oplus H}(u, (1, v')) = c_{2k+1}^H.$$

If we assume $K_2 \oplus H$ is distance-regular, then equating (9) and (11) for corresponding values of k, as well as (10) and (12), we get (7) and (8). Conversely, if we assume relations (7) and (8) hold, we may use (10)-(8) to find that $K_2 \oplus H$ is distance-regular. ∎

Remark The hypercubes Q_n and complete graphs K_n, together with some other graphs (e.g. an octahedron), satisfy relations (7) and (8). However, at present we are not aware of a characterization of distance-regular graphs satisfying these relations. ∎

3 Conclusion

The Hamming graphs $Ham(d, n)$, $d \geq 2$, $n \geq 2$, of the diameter d have vertex set consisting of all d-tuples of elements taken from an n-element set, with two vertices adjacent if and only if they differ in exactly one coordinate. Notice that $Ham(d, n)$ is actually equal to $\underbrace{K_n + K_n + \ldots + K_n}_{d}$.

It was shown in [10, 11] that $Ham(d, n)$ is characterized by its parameters if and only if $n \neq 4$. Summarizing the results of [8, 7] and Theorems 1 and 2, we get the following theorem.

The Main Theorem *Let G and H be connected, distance-regular graphs. Then*

(i) $G + H$ is distance-regular if and only if G has the same parameters as $Ham(D(G), a_1^G + 2)$ and H has the same parameters as $Ham(D(H), a_1^G + 2)$ (see [7])*;*

(ii) $G \times H$ is distance-regular if and only if either $G \cong H \cong K_{n,n}$ or $G \cong H \cong K_n$ for some $n \in \mathbf{N}$ (see [8])*;*

(iii) $G \otimes H$ *is distance-regular if and only if* $G \cong K_m$ *and* $H \cong K_n$ *for some* $m, n \in \mathbf{N}$*;*

(iv) If G *has at least three vertices, then* $G \oplus H$ *is distance-regular if and only if there exists* $m, n, t \in N$ *such that* $G \cong K_{\underbrace{m, m, \ldots, m}_{t}}$ *and* $H \cong K_n$*;*

(v) $K_2 \oplus H$ *is distance-regular if and only if the parameters of* H *satisfy the following relations for all* $j = 0, 1, \ldots, D(H) - 1$

$$
\begin{aligned}
b_j^H &= 1 + a_{j+1}^H + b_{j+1}^H, \\
1 + a_j^H + c_j^H &= c_{j+1}^H.
\end{aligned}
$$

From this main theorem it can be seen that among these compositions, only the sum of graphs allows distance-regularity to be preserved by factors which both may have arbitrarily large diameters. That can be explained by the fact that it is the only one of these compositions for which the distance formula does not involve further relations among distances of coordinates (like max, min, *od*, *ed* etc.). We see that one could hope to achieve preservation of distance-regularity, only if the distance formula for the NEPS of graphs could be expressed as a symmetric function of the distances between coordinates and some further conditions on the factors of NEPS are met.

Finally, having such a variety of distance formulas just for the special cases when NEPS has only two factors, it is understandable that at the moment it is hardly possible to expect a nice distance formula for the general case of NEPS. Currently, it is only known that if all factors of the NEPS are connected and bipartite, and the NEPS itself is connected, then the diameter of NEPS does not exceed the sum of the diameters of the factors (see [12]).

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