Completeness of Temporal Logics over Infinite Intervals

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Abstract

Interval Temporal Logics over infinite intervals are studied. First, the ordinary possible worlds models are extended to infinite possible world models. Accordingly, an axiomatic system is proposed and it has been proved complete. Secondly, infinite intervals are included in a logic over abstract intervals. A corresponding axiomatic system is given and proven to be complete also.

Keywords: Interval Temporal Logic, Completeness, Henkin’s construction

1 Introduction

Temporal logics have been widely studied and used in specification and verification of computer systems since it was first proposed by Pnueli [10]. Usually, models of temporal logics are point-based (e.g., [6]), but there is also a class of temporal logics that are interval-based [7]. The latter, known as Interval Temporal Logic (ITL), is suitable for expressing temporal properties that are associated with a period of time [11, 13]. In particular, ITL has formed the basis of Duration Calculus [15, 5], a logic for real-time systems.

Generally, above ITLs are logics about finite intervals. This limits the logics in expressing properties that are related to infinite behaviours. For example, traditional ITLs are not able to express liveness properties. But some behaviours of real-time systems may take infinite time. As an example, consider the following toy program, which is the sequential composition of a loop command followed by an assignment statement

\[
\text{while true do delay 1; } x := -x \text{ od ; } y := 2
\]

During the execution of this command, the loop statement will repeat forever, and the assignment to \(y\) is subsequently never executed. It is difficult to describe the behaviours of such programs in a logic over finite intervals.

Several ITLs over infinite intervals have been proposed and applied to reasoning about non-terminating systems ([14, 8, 9]). The foundation of such logics has become an interesting issue. In this paper, we study the axiomatization of ITL over infinite intervals. This follows the work of Dutertre Dut95:tr on traditional ITL, and the method is the classical Henkin’s construction. Dutertre gave two complete proof systems for ITL on possible worlds and abstract intervals respectively. We extended both systems to infinite cases and prove that the resulting systems are complete.

*Hanpin Wang, the corresponding author with phone: +86 10 62765818, is supported in part by the National Science Foundation of China under grant No. 69973003 and grant No. 60173002, and the National Grand Fundamental Research of China (973 project) under grant 1999032726.
2 Interval Temporal Logic over finite intervals

In this section we briefly review ITL over finite intervals, and the axiomatisation by Dutertre [2].

2.1 Syntax

As usual, a first-order ITL language \( \mathcal{L} \) (with equality) contains two kinds of symbols: a set of logical symbols and a set of non-logical symbols. The set of non-logical symbols consists of function symbols and predicate symbols. Each non-logical symbol is associated with an non-negative integer as its arity. Function and predicate symbols of arity 0 are respectively individual constants and propositions. The set of logical symbols contains: an infinite, denumerable set of variables, the existential quantifier \( \exists \), the connectives \( \neg \) and \( \land \), the symbol \( = \), and a binary modal connective \( ; \), which is used to divide an interval into two sub-intervals. Each symbol is either flexible or rigid. Rigid symbols are intended to represent fixed, global entities. Their interpretation will be the same in all worlds or intervals of a model. Conversely entities which may vary in different worlds or intervals are represented by flexible symbols. Every variable is rigid, and so is \( = \).

Terms of \( \mathcal{L} \) are defined by the following BNF
\[
t ::= x_i | c | f(t_1, \ldots, t_n)
\]
where \( x_i \) is a variable, \( c \) a constant, \( f \) an \( n \)-ary function, \( t_1, \ldots, t_n \) terms.

Formulas are defined by
\[
\alpha ::= P | F(t_1, \ldots, t_n) | t_1 = t_2 | \neg \alpha | \alpha_1 \land \alpha_2 | \alpha_1 ; \alpha_2 | (\exists x_i) \alpha
\]
where \( P \) is a proposition, \( F \) an \( n \)-ary predicate, \( t_1, \ldots, t_n \) terms, \( \alpha_1 \) and \( \alpha_2 \) formulas. The other standard logic connectives \( \lor \), \( \Rightarrow \), \( \Leftrightarrow \) and the universal quantifier can be derived as usual.

2.2 Semantics

The semantics of ITL is given firstly in a general model of possible worlds, and afterwards in an interval model which is a special case of the former.

A possible worlds model \( \mathcal{M} \) for an ITL-language \( \mathcal{L} \) is a quadruple \((W, R, D, I)\), where

- \( W \) is a non-empty set of possible worlds and \( R \) is a ternary accessibility relation on \( W \). The pair \((W, R)\) is called the frame of \( \mathcal{M} \);
- \( D \) is a non-empty set, called the domain of \( \mathcal{M} \);
- \( I \) is a function which assigns to each symbol \( s \) of \( \mathcal{L} \) and each world \( w \in W \) an interpretation \( I(s,w) \) such that the values of rigid symbols are the same in all worlds, and

  - if \( s \) is an \( n \)-ary function symbol, then \( I(s,w) \) is a function from \( D^n \) to \( D \);
  - if \( s \) is an \( n \)-ary predicate symbol, then \( I(s,w) \) is an \( n \)-ary relation on \( D \), that is, \( I(s,w) \subseteq D^n \) or, equivalently, \( I(s,w) \) is function from \( D^n \) to \( \{0,1\} \).

where \( D^n \) is the \( n \)-ary Cartesian product of \( D \).

An \( \mathcal{M} \)-valuation \( v \) is a mapping from variables to \( D \). Given a model \( \mathcal{M} \) as above, an \( \mathcal{M} \)-valuation \( v \) and a possible world, the interpretation \( I'_w \) of terms in \( w \) under \( v \) is defined as usual. So is the satisfaction of formulas in \( w \) under \( v \), denoted by \( \mathcal{M}, w, v \models \alpha \). We mention the satisfaction only for two kinds of formulas as below.

- \( \mathcal{M}, w, v \models P \) iff \( I(P,w) = 1 \),
- \( \mathcal{M}, w, v \models \alpha_1 ; \alpha_2 \) iff there are two worlds \( w_1 \) and \( w_2 \) such that \( R(w_1,w_2,w) \), \( \mathcal{M}, w_1, v \models \alpha_1 \) and \( \mathcal{M}, w_2, v \models \alpha_2 \).
2.3 System $S$

S-models
Of particular interest is a class of ITL languages which contain a flexible constant $\ell$, denoting the length of the interval. We shall from now on only consider such languages. A possible worlds model $M = (W; R, D, I)$ for an ITL-language $L$ is an $S$-model if for any worlds $w, w_1, w_2, w'_1$ and $w'_2$ of $W$ such that $R(w_1, w_2, w)$ and $R(w'_1, w'_2, w)$,

- if $I(\ell, w_1) = I(\ell, w'_1)$ then $w_2 = w'_2$, and
- if $I(\ell, w_2) = I(\ell, w'_2)$ then $w_1 = w'_1$.

Axioms of $S$

$S$ contains the following modal axioms:

A1 \[(\alpha; \beta) \wedge \neg(\alpha; \gamma)) \Rightarrow (\alpha; (\beta \wedge \neg\gamma)), \]
\[(\alpha; \beta) \wedge \neg(\gamma; \beta)) \Rightarrow (\alpha \wedge \neg\gamma); \beta), \]

R \[(\alpha; \beta) \Rightarrow \alpha \quad \text{if}\ \alpha \text{ is a rigid formula}, \]
\[(\alpha; \beta) \Rightarrow \beta \quad \text{if}\ \beta \text{ is a rigid formula}, \]

B \[ ((\exists x)\alpha; \beta) \Rightarrow (\exists x)(\alpha; \beta) \quad \text{if}\ \alpha \text{ is not free in} \ \beta, \]
\[ (\alpha; (\exists x)\beta) \Rightarrow (\exists x)(\alpha; \beta) \quad \text{if}\ \beta \text{ is not free in} \ \alpha, \]

L1 \[ (\ell = x; \alpha) \Rightarrow \neg(\ell = x; \neg\alpha), \]
\[ (\alpha; \ell = x) \Rightarrow \neg(\neg\alpha; \ell = x). \]

In addition, $S$ contains the first order axioms. They can be chosen from any system for first order logic, although some care must be taken in the instantiation of universally quantified formulas. For example, we can choose the following quantification axioms:

Q1 \[ (\forall x)\alpha(x) \Rightarrow \alpha(t) \quad \text{if}\ \alpha \text{ is not free in} \ \alpha \text{ and,} \ t \text{ is rigid or} \ \alpha \text{ is chop-free}, \]

Q2 \[ (\forall x)(\alpha \vee \beta) \Rightarrow (\forall x)\alpha \vee \beta \quad \text{if}\ \beta \text{ is not free in} \ \beta. \]

Rules of $S$

MP: \[ \frac{\alpha}{\alpha \Rightarrow \beta} \]
N: \[ \frac{\alpha}{\neg(\alpha; \beta)} \quad \text{and} \quad \frac{\alpha}{\neg(\beta; \neg\alpha)} \]

G: \[ \frac{\alpha}{(\forall x)\alpha} \quad \text{Mono:} \quad \frac{\alpha \Rightarrow \beta}{(\alpha; \gamma) \Rightarrow (\beta; \gamma)} \quad \text{and} \quad \frac{\alpha \Rightarrow \beta}{(\gamma; \alpha) \Rightarrow (\gamma; \beta)}. \]

Soundness and completeness of $S$

Theorem 1 (Dutertre [2]) For any formulas $\alpha$ of an ITL-language $L$, $\alpha$ is a theorem of $S$ if and only if $\alpha$ is valid in all $S$-models of $L$. \[ \blacksquare \]

2.4 System $S'$

Axiomatic system $S'$ concentrates on reasoning about intervals rather than just possible worlds. Since interval model is a special case of possible worlds model, $S'$ should be an extension of $S$. Indeed, $S'$ is obtained from $S$ by adding some new axioms.

An interval language is an ITL-language containing, in addition to the flexible constant $\ell$, two rigid symbols $+$ and $0$.

Temporal and duration domain

A temporal domain is a pair $(T, \leq)$ where $T$ is a non-empty set and $\leq$ is a total order relation on $T$. An interval on $T$ is a pair of elements $[t_1, t_2]$ of $T$ such that $t_1 \leq t_2$. To define a measure on intervals (intuitively, the length) we need a duration domain. A duration domain $D$ is a non-empty set equipped with a binary operation $+$ and at least one element $0$ which satisfy the conditions D1–D5 below.
D1 \((x + y) + z = x + (y + z)\),  
D2 \(0 + x = x + 0 = x\),  
D3 \((x + y = x + z) \Rightarrow y = z, , \(y + x = z + x) \Rightarrow y = z\),  
D4 \(x + y = 0 \Rightarrow ((x = 0) \land (y = 0))\),  
D5 \((\exists z)(x + z = y \lor y + z = x), \(\exists z)(z + x = y \lor z + y = x)\).

\[S'\]-models

From a temporal domain \(T\), we can derive an interval frame \((W, R)\), where

- \(W\) is the set of intervals on \(T\),
- \(R\) is the following ternary relation on \(W\)
  \[R([t_1, t'_1], [t_2, t'_2], [t, t']) \text{ if } t = t_1, t'_1 = t_2, t'_2 = t'.\]

Then the measure \(m\) mentioned above can be defined as a function from the intervals to the duration domain with the following properties.

\(M1\) If \(m[t, u] = m[t, u']\) then \(u = u'\), and if \(m[u, t] = m[u', t]\) then \(u = u'\);

\(M2\) \(m[t, t] = 0\);

\(M3\) \(m[t, u] + m[u, t'] = m[t, t']\) for \(t \leq u \leq t'\);

\(M4\) If \(m[t, t'] = x + y\) then there is \(u \in T\): \(t \leq u \leq t'\), such that \(m[t, u] = x\) and \(m[u, t'] = y\).

For an interval language \(\mathcal{L}\), an interval model or \(S'\)-model is a possible worlds model \(\mathcal{M} = (W, R, D, I)\) of \(\mathcal{L}\) if

- the frame of \(\mathcal{M}\) is the interval frame derived from a temporal domain \(T\),
- the domain of \(\mathcal{M}\) is just the duration domain \(D\),
- the interpretation of the symbols \(\ell, +\) and \(0\) is, for any interval \([t, t']\),
  \[I(\ell, [t, t']) = m[t, t'] \]
  \[I(+, [t, t']) = + \]
  \[I(0, [t, t']) = 0 \]

where \(m\) is a measure function from intervals on \(T\) to duration domain \(D\).

**Axiomatic system of \(S'\)**

In addition to axioms and rules of \(S\), the axiomatic system of \(S'\) contains the following axioms

- about chop "\(;\)"
  \[A2: ((\alpha; \beta); \gamma) \Leftrightarrow (\alpha; (\beta; \gamma)), \]
  \[L2: \ell = x + y \Leftrightarrow (\ell = x; \ell = y), \]
  \[L3: \alpha \Rightarrow (\alpha; \ell = 0), \]
  \[\alpha \Rightarrow (\ell = 0; \alpha); \]

- about domain: \(D1-D5\) above.

**Soundness and completeness of \(S'\)**

**Theorem 2 (Dutertre [2])** For any formula \(\alpha\) of an interval language \(\mathcal{L}\), \(\alpha\) is a theorem of \(S'\) if and only if \(\alpha\) is valid in all interval models for \(\mathcal{L}\).
3 ITL on infinite possible worlds models

3.1 Syntax

To specify infinite possible worlds, we need a special symbol to distinguish infinite possible worlds from all possible worlds. So an infinite ITL-language contains, in addition to the symbols of a finite ITL-language, a propositional symbol $I$. The construction rules of terms and formulas are left unchanged except the following addition: $I$ is a (atomic) formula.

3.2 Semantics

An infinite possible worlds model for an infinite ITL-language $L$, is obtained by adding to a finite possible worlds model a component which is used to assign the meaning of $I$. It is a quintuple $M = (W, \tilde{W}, R, D, I)$, where $W, R, D$ and $I$ are the same as in an $S$-model, and $\tilde{W} \subseteq W$. Intuitively, $\tilde{W}$ denotes the set of all infinite possible worlds.

The interpretation of the symbols of an infinite ITL-language $L$ is the same as for a finite ITL-language except the following addition

- $I(w, I) = 1$ iff $w \in \tilde{W}$.

The satisfaction relation of a formula in an infinite possible worlds model is defined in the same way as in a finite ITL model.

An infinite possible worlds model $M = (W, \tilde{W}, R, D, I)$ is called an $S_\infty$-model if the following conditions hold:

- For any worlds $w, w_1$ and $w_2$ such that $R(w_1, w_2, w)$,
  - $w_1 \not\in \tilde{W}$,
  - $w_2 \in \tilde{W}$ iff $w \in \tilde{W}$;
- For any worlds $w, w_1, w_2, w_1'$ and $w_2'$ such that $R(w_1, w_2, w)$ and $R(w_1', w_2', w)$,
  - if $I(\ell, w_1) = I(\ell, w_1')$ then $w_1 = w_1'$ and $w_2 = w_2'$, and
  - if $I(\ell, w_2) = I(\ell, w_2')$ and $w_2, w_2' \not\in \tilde{W}$, then $w_1 = w_1'$.

It is easy to see that any $S$-model $M = (W, R, D, I)$ corresponds to an $S_\infty$-model $M = (W, \tilde{W}, R, D, I)$ where $\tilde{W} = \emptyset$.

3.3 System $S_\infty$

$S_\infty$ contains all axioms A1,R, B, L1, Q1 and Q2, and inference rules MP, N, G and Mono of $S$ except L1 should be changed. Additionally, $S_\infty$ still has axiom S1 and other three axioms about infinity, which are listed below.

- $L1 \quad (\ell = x; \alpha) \Rightarrow \neg(\ell = x; \neg\alpha)$,
  $\quad (\alpha; (\ell = x \land \neg\mathcal{I})) \Rightarrow \neg(\neg\alpha; (\ell = x \land \neg\mathcal{I}))$,
- $S1 \quad ((\ell = x \land \alpha); \beta) \Rightarrow \neg((\ell = x \land \neg\alpha); \gamma)$,
- $P1 \quad \neg(I; \alpha)$,
- $P2 \quad (\alpha; \mathcal{I}) \Rightarrow \mathcal{I}$,
- $P3 \quad (\alpha; \neg\mathcal{I}) \Rightarrow \neg\mathcal{I}$,

The first part of axiom L1 is the same as in $S$ but the second part is different. This is because the semantics of the chop operator is defined in such a way that if an interval is chopped into two sub-intervals, then the first sub-interval must be finite while the second sub-interval may or may not be finite. Therefore, when an interval is chopped into two sub-intervals in two ways, if the
length of the two first parts are the same, then these two parts as well as the two second parts are the same, but if the length of the two second parts are the same, we can only reach the same conclusion if the second interval is infinite. Axiom S1 follows the idea of L1, that is, chopping of an interval is uniquely decided by the length of the first part. While L1 specifies the result about the part other than the one controlled by the length, S1 is about the part controlled by it. Axioms P1—P3 are new, which we think reflecting some of the properties of infinite possible worlds. Axiom P1 says that any interval cannot be chopped in a way that the first sub-interval is infinite. And P2 (P3) says that if an interval is chopped into two parts and the second part is finite (infinite), then the whole interval is also finite (infinite).

It is easy to prove that \( S_\infty \) is sound.

### 3.4 Completeness

**Theorem 3 (Completeness of \( S_\infty \))** For any formula \( \alpha \) in an infinite ITL-language \( \mathcal{L} \), if \( \alpha \) is valid in all infinite possible worlds models of \( \mathcal{L} \), then \( \alpha \) is a theorem of \( S_\infty \).

The theorem is proved by classical “Henkin’s construction” as the proof for \( S \) by Dutertre [2]. Some of the technical mechanisms developed by Dutertre can still be used. We will concentrate on aspects which are different from the proof for \( S \) in [2]. More details can be found in [12].

**Step 1:** As for classic logics (see for example, [4]), Theorem 3 is equivalent to

For any sentence \( \alpha \) in an infinite ITL-language \( \mathcal{L} \), if \( \neg \alpha \) is not a theorem of \( S_\infty \), then \( \alpha \) can be satisfied in an infinite possible worlds model of \( \mathcal{L} \).

The notions of consistent set of sentences and maximal consistent set of sentences are defined as usual. To prove Theorem 3, it is enough to show that

*Any consistent set of sentences of \( \mathcal{L} \) has an infinite possible worlds model of \( \mathcal{L} \).*

**Step 2:** Let \( \mathcal{L} \) be an infinite ITL-language, and \( B = \{b_0, b_1, b_2, \ldots \} \) be an infinite countable set of constant symbols not occurring in \( \mathcal{L} \). Denote by \( \mathcal{L}^+ \) the infinite ITL-language obtained by adding to \( \mathcal{L} \) all the symbols of \( B \) as rigid constants. A set \( \Gamma \) of sentences of \( \mathcal{L}^+ \) is said to have witnesses in \( B \), if for every sentences of \( \Gamma \) of the form \((\exists x)\alpha(x)\), where \( x \) is the only free variable of \( \alpha(x) \), there exists a constant \( b_i \) of \( B \) such that \( \alpha(b_i) \) is also in \( \Gamma \).

*Any consistent set \( \Gamma \) of \( \mathcal{L} \) can be extended to a maximal consistent set \( \Gamma^+ \) of \( \mathcal{L}^+ \) which has witness in \( B \).*

Theorem 3 can be reduced to the following:

*Any maximal consistent set \( \Gamma^+_0 \) of \( \mathcal{L}^+ \) which has witnesses in \( B \) has an infinite possible worlds models of \( \mathcal{L}^+ \).*

**Step 3:** Let \( \Gamma^+_0 \) be a maximal consistent set \( \Gamma^+_0 \) of \( \mathcal{L}^+ \) which has witnesses in \( B \). Denote by \( \Sigma \) the set of rigid sentences of \( \Gamma^+_0 \). We construct an infinite possible worlds model \( \mathcal{M} = (W, \bar{W}, R, D, I) \) as follows:

- The set of worlds \( W \) is the set of all maximal consistent sets \( \Delta \) of \( \mathcal{L}^+ \) which have witnesses in \( B \) and \( \Sigma \subseteq \Delta \).
- \( \bar{W} = \{ \Delta \in W \mid I \in \Delta \} \).
- The relation \( R \) is defined by, for all \( \Delta_1, \Delta_2 \) and \( \Delta \) in \( W \),

\[
R(\Delta_1, \Delta_2, \Delta) \iff \Delta_1 \ast \Delta_2 \subseteq \Delta
\]

where \( \Delta_1 \ast \Delta_2 = \{ \alpha_1; \alpha_2 \mid \alpha_1 \in \Delta_1, \alpha_2 \in \Delta_2 \} \).
- The domain \( D \) is defined by

\[
D = \{ [b_i] \mid b_i \in B \}
\]

where \([b_i]\) is the equivalence class of \( b_i \) under the following equivalence relation “\( \equiv \)” on \( B \)

\[
b_i \equiv b_j \iff b_i = b_j \in \Sigma.
\]
• The interpretation $I$ is defined as:
  - For an $n$-ary function symbol $f$ and $n$ elements $[b_{i_1}], \ldots, [b_{i_n}]$ in $B$,
    \[
    I(f, \Delta)([b_{i_1}], \ldots, [b_{i_n}]) = [b_j]
    \iff
    f(b_{i_1}, \ldots, b_{i_n}) = b_j \in \Delta
    \]
    where $b_j$ is a witness of formula $(\exists x)(f(b_{i_1}, \ldots, b_{i_n}) = x)$ in $B$, and it is easy to prove that such $[b_j]$ is unique. Particularly, if $c$ is a constant symbol, then
    \[
    I(c, \Delta) = [b] \iff c = b \in \Delta.
    \]
  - For an $n$-ary predicate symbol $F$ and $n$ elements $[b_{i_1}], \ldots, [b_{i_n}]$ in $B$,
    \[
    I(F, \Delta)([b_{i_1}], \ldots, [b_{i_n}]) = 1 \iff F(b_{i_1}, \ldots, b_{i_n}) \in \Delta.
    \]
    Particularly, if $P$ is a propositional symbol, then
    \[
    I(P, \Delta) = 1 \iff P \in \Delta.
    \]

From the definition of $I$, we can easily see that
\[
I(\mathcal{I}, \Delta) = 1 \iff \Delta \in \overline{W}
\]

**Step 4:** To finish the proof of the completeness theorem it is only necessary to show that
\[
\Gamma_0^* \text{ can be satisfied by the world } \Gamma_0^* \text{ in the model } \mathcal{M} \text{ constructed as above.}
\]

This can be proved by the following a series of lemmas. Given a non-empty set of sentences $\Gamma$, let
\[
\hat{\Gamma} = \{\gamma_1 \land \ldots \land \gamma_n \mid n \geq 1, \gamma_1, \ldots, \gamma_n \in \Gamma\},
\]
\[
\Gamma = \{\gamma \mid \vdash (\alpha \Rightarrow \gamma), \alpha \in \hat{\Gamma}\}.
\]

**Lemma 4** Let $\Gamma$ be a maximal consistent set of sentences. If $\Gamma_1$ and $\Gamma_2$ are non-empty sets of sentences and $\hat{\Gamma}_1 \ast \hat{\Gamma}_2 \subseteq \hat{\Gamma}$, then $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ are consistent and $\Gamma_1 \ast \Gamma_2 \subseteq \Gamma$.

An important mechanism in the proof is the following two functions invented by Dutertre [2]:
\[
\delta_1(\Gamma, \Gamma_1) = \{\neg \beta \mid \neg (\alpha; \beta) \in \Gamma, \alpha \in \Gamma_1\},
\]
\[
\delta_2(\Gamma, \Gamma_2) = \{\neg \alpha \mid \neg (\alpha; \beta) \in \Gamma, \beta \in \Gamma_2\}
\]
where $\Gamma$, $\Gamma_1$ and $\Gamma_2$ are sets of sentences.

**Lemma 5** Given a maximal consistent set $\Gamma$ and two non-empty sets $\Gamma_1$ and $\Gamma_2$ such that $\hat{\Gamma}_1 \ast \hat{\Gamma}_2 \subseteq \hat{\Gamma}$, let $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ be $\Gamma_1 = \Gamma \cup \delta_2(\Gamma, \Gamma_2)$ and $\Gamma_2 = \Gamma \cup \delta_1(\Gamma, \Gamma_1)$. Then $\hat{\Gamma}_1 \ast \hat{\Gamma}_2 \subseteq \hat{\Gamma}$ and $\Gamma_1 \ast \Gamma_2 \subseteq \Gamma$.

**Lemma 6** If $\Gamma$ is maximal consistent and $\Gamma_1$ and $\Gamma_2$ are two non-empty sets of sentences such that $\Gamma_1 \ast \Gamma_2 \subseteq \Gamma$ then there are two maximal consistent sets $\Gamma_1^*$ and $\Gamma_2^*$ such that $\Gamma_1 \subseteq \Gamma_1^*$, $\Gamma_2 \subseteq \Gamma_2^*$, and $\Gamma_1^* \ast \Gamma_2^* \subseteq \Gamma$.

**Lemma 7** Let $\Delta$ be a world of $\mathcal{M}$ and $\Gamma_1^*$ and $\Gamma_2^*$ be two maximal consistent sets of sentences of $\mathcal{L}^+$. If the following conditions are satisfied,
  - $\Gamma_1^* \ast \Gamma_2^* \subseteq \Delta$,
  - there is an element $b_i$ of $B$ such that $\ell = b_i \in \Gamma_1^*$,
  - there is an element $b_j$ of $B$ such that $\ell = b_j \in \Gamma_2^*$,
then $\Gamma_1^*$ and $\Gamma_2^*$ are two worlds of $\mathcal{M}$.

**Proof:** The fact that $\Sigma$ is included in both $\Gamma_1^*$ and $\Gamma_2^*$ can be verified easily.

To show $\Gamma_1^*$ has witnesses in $B$, let $(\exists x)\alpha(x)$ be a sentence of $\Gamma_1^*$. Then $((\exists x)\alpha(x) \land \ell = b_i); \ell = b_j \in \Delta$. Since
\[
\vdash ((\exists x)\alpha(x) \land \ell = b_i) \Rightarrow (\exists x)(\alpha(x) \land \ell = b_i)
\]
we have
\[ \vdash \left( (\exists x)\alpha(x) \land \ell = b_i; \ell = b_j \right) \Rightarrow \left( (\exists x)(\alpha(x) \land \ell = b_i); \ell = b_j \right) \]
and subsequently
\[ \vdash \left( (\exists x)(\alpha(x) \land \ell = b_i); \ell = b_j \right) \Rightarrow (\exists x)\left( (\alpha(x) \land \ell = b_i); \ell = b_j \right). \]
Therefore \((\exists x)(\alpha(x) \land \ell = b_i); \ell = b_j \) must be in \(\Delta\). Since \(\Delta\) has witnesses in \(B\) there exists a constant \(b_k\) of \(B\) such that
\[ (\alpha(b_k) \land \ell = b_i); \ell = b_j \in \Delta. \]
By axiom S1, we have
\[ \vdash \left( (\alpha(b_k) \land \ell = b_i); \ell = b_j \right) \Rightarrow \neg\left( (\neg\alpha(b_k) \land \ell = b_i); \ell = b_j \right). \]
Therefore \(\neg\left( (\neg\alpha(b_k) \land \ell = b_i); \ell = b_j \right)\) is a sentence of \(\Delta\). Since \(\Gamma^*_1 \ast \Gamma^*_2 \subseteq \Delta\), \(\neg\alpha(b_k)\) cannot belong to \(\Gamma^*_1\). Hence \(\alpha(b_k)\) must be in \(\Gamma^*_1\). So \(\Gamma^*_1\) has witnesses in \(B\). A easier discussion shows that \(\Gamma^*_2\) has witnesses in \(B\) also.

**Theorem 8** Let \(\alpha(x_1, \ldots, x_n)\) be a formula of \(L^+\) with free variables among \(x_1, \ldots, x_n\). For any world \(\Delta\) of \(W\), any valuation \(v\) and any constants \(b_1, \ldots, b_n\) in \(B\) such that \(v(x_k) = [b_k]\) for \(1 \leq k \leq n\), we have
\[ \mathcal{M}, \Delta, v \models \alpha(x_1, \ldots, x_n) \text{ iff } \alpha(b_1, \ldots, b_n) \in \Delta. \]

**Proof:** The proof is by induction on the construction of \(\alpha(x_1, \ldots, x_n)\). We only show it for the case that \(\alpha\) is \(\beta; \gamma\).

If \(\mathcal{M}, \Delta, v \models \beta(x_1, \ldots, x_n); \gamma(x_1, \ldots, x_n)\), there are two worlds \(\Delta_1\) and \(\Delta_2\) such that
\[ \mathcal{M}, \Delta_1, v \models \beta(x_1, \ldots, x_n), \]
\[ \mathcal{M}, \Delta_2, v \models \gamma(x_1, \ldots, x_n), \]
\[ \Delta_1 \ast \Delta_2 \subseteq \Delta. \]
By the induction hypothesis, this implies that
\[ \beta(b_1, \ldots, b_n) \in \Delta_1 \text{ and } \gamma(b_1, \ldots, b_n) \in \Delta_2. \]
Since \(\Delta_1 \ast \Delta_2 \subseteq \Delta\), we have \(\beta(b_1, \ldots, b_n); \gamma(b_1, \ldots, b_n) \in \Delta\), that is \(\alpha(b_1, \ldots, b_n) \in \Delta\).

Conversely, assume
\[ \beta(b_1, \ldots, b_n); \gamma(b_1, \ldots, b_n) \in \Delta. \]
Let \(\beta'\) and \(\gamma'\) denote the sentences \(\beta(b_1, \ldots, b_n)\) and \(\gamma(b_1, \ldots, b_n)\) respectively. Let \(x\) and \(y\) be two variables. We can derive that
\[ \vdash (\beta'; \gamma') \Rightarrow (\exists x)(\exists y)((\beta' \land \ell = x); (\gamma' \land \ell = y)). \]
The sentence \((\exists x)(\exists y)((\beta' \land \ell = x); (\gamma' \land \ell = y))\) belongs to \(\Delta\). Since \(\Delta\) has witnesses in \(B\), there are two elements \(b_i\) and \(b_j\) such that
\[ (\beta' \land \ell = b_i); (\gamma' \land \ell = b_j) \in \Delta. \]
Let
\[ \Gamma_1 = \{\beta'; \ell = b_i\} \text{ and } \Gamma_2 = \{\gamma'; \ell = b_j\}. \]
Then \(\Gamma_1 \ast \Gamma_2 \subseteq \Delta\). By Lemma 7, there are two maximal consistent sets \(\Gamma_1^*\) and \(\Gamma_2^*\) such that
\[ \Gamma_1 \subseteq \Gamma_1^*, \text{ } \Gamma_2 \subseteq \Gamma_2^*, \text{ } \Gamma_1^* \ast \Gamma_2^* \subseteq \Delta. \]
By Lemma 4, the two sets $\Gamma_1^*$ and $\Gamma_2^*$ are worlds of $\mathcal{M}$. By induction hypothesis, we have

$$\mathcal{M}, \Gamma_1^*, v \models \beta(x_1, \ldots, x_n) \quad \text{and} \quad \mathcal{M}, \Gamma_2^*, v \models \gamma(x_1, \ldots, x_n).$$

Therefore

$$\mathcal{M}, \Delta, v \models \beta(x_1, \ldots, x_n); \gamma(x_1, \ldots, x_n).$$

To conclude the proof of the completeness theorem, we only need to show that the above $\mathcal{M}$ is a $S_\infty$-model, and this can be done easily.

4 ITL on infinite interval models

In addition to the symbols in an ITL-language, an ITL-language over infinite intervals includes two rigid constants symbols “0” and “$\infty$”, and a rigid binary function symbol “+”.

4.1 Infinite interval models

Interval frames

Definition 9 A temporal domain with infinite time is a total order set $(T, \leq, \infty)$ with a unique maximal element $\infty$.

The unique maximal element is called the infinite time point.

Definition 10 For a given temporal domain with infinite time $(T, \leq, \infty)$, an interval on $T$ is a pair of time points $[t_1, t_2]$ such that $t_1 \leq t_2$ and $t_1 \neq \infty$.

If the second point of an interval is $\infty$ then the interval is called an infinite interval. The set of intervals on $T$ is denoted by $I(T)$.

Definition 11 Given a temporal domain with infinite time $(T, \leq, \infty)$, the interval frame derived from $T$ is a pair $(W, R)$, where

- $W$ is a non-empty set of intervals on $T$, and satisfies,

  - $[t, t] \in W$ for every $t \neq \infty$ in $T$, and
  - if $[t, u]$ and $[u, t']$ are in $W$, then $[t, t'] \in W$ for all $t, u, t' \in T$.

- $R$ is a ternary relation on $W$ such that, for any intervals $[t_1, t_1'], [t_2, t_2']$ and $[t, t']$ of $W$,

  $$R([t_1, t_1'], [t_2, t_2'], [t, t']) \iff t = t_1, \ t_1' = t_2 \text{ and } t_2' = t'.$$

Note that the first time point of an interval cannot be infinity, that is, $(\infty, \infty)$ is not an element of $I(T)$ for any temporal domain $T$ with infinite time. Therefore, for any intervals $[t_1, t_1']$, $[t_2, t_2']$ and $[t, t']$ of $I(T)$, if $R([t_1, t_1'], [t_2, t_2'], [t, t'])$, then $t_1' = t_2 \neq \infty$. In other words, any interval cannot be chopped into two intervals with the first one being infinite.

In the definition of interval frame over $T$ above, $W$ is not limited to be the set of all intervals on $T$. This will allow us to deal with both finite and infinite intervals.

Duration domain

Let $D$ be an algebra with a binary operation $+$ and two distinct constants 0 and $\infty$. $D$ is called a duration domain if the algebra satisfies the following conditions

1. $(x + y) + z = x + (y + z)$;
2. $0 + x = x + 0 = x;$
(3) \( x + y = \infty \iff x = \infty \) or \( y = \infty \);

(4) If \( (x \neq \infty) \) and \( (x + y = x + z) \) then \( y = z \), and
    if \( (x \neq \infty) \) and \( (y + x = z + x) \) then \( y = z \);

(5) If \( x + y = 0 \) then \( x = 0 \) and \( y = 0 \);

(6) There exists \( z \) such that \( x + z = y \) or \( y + z = x \), and
    there exists \( z \) such that \( z + x = y \) or \( z + y = x \).

Measure of intervals

Given a temporal domain with infinite time \( T \) and a duration domain \( D \), a measure is a function \( m \) from \( W \) over \( T \) to \( D \) that has the following properties:

(M1) If \( m[t, u] = m[t, u'] \) then \( u = u' \), and
    if \( m[u, t] = m[u', t] \) and \( t \neq \infty \) then \( u = u' \);

(M2) \( m[t, t] = 0 \);

(M3) \( m[t, u] = \infty \) iff \( u = \infty \);

(M4) \( m[t, u] + m[u, t'] = m[t, t'] \) for \( t \leq u \leq t' \);

(M5) If \( m[t, t'] = x + y \) and \( x \neq \infty \) then there is \( u \in T: t \leq u \leq t' \), such that \( [t, u] \) and \( [u, t'] \)
    are in \( W \), \( m[t, u] = x \) and \( m[u, t'] = y \).

Models over infinite intervals

Let \( T \) be a temporal domain with infinity time, \((D, +, 0, \infty)\) a duration domain and \( m \) a measure from \( W \) to \( D \). We say a possible worlds model \( \mathcal{M} = (W, \tilde{W}, R, D, I) \) is an infinite interval model, or \( S'_{\infty} \) for short, if

- the frame of \( \mathcal{M} \) is the frame defined by \( T \);
- \( \tilde{W} \) is the set of infinite intervals in \( W \);
- the domain of \( \mathcal{M} \) is the duration domain;
- the interpretation in \( \mathcal{M} \) of the symbols \( \ell \), +, 0 and \( \infty \) is, for any interval \([t, t']\),

\[
\begin{align*}
I(\ell, [t, t']) &= m[t, t'], \\
I(+, [t, t']) &= +, \\
I(0, [t, t']) &= 0, \\
I(\infty, [t, t']) &= \infty.
\end{align*}
\]

Note \( W \) is only required to be a non-empty set of intervals on \( T \) (subject to conditions associated with \( M2, M4 \) and \( M5 \) of measure functions), it may or may not contain any infinite intervals, and subsequently \( \tilde{W} \) may or may not be empty. Clearly, if \( \tilde{W} \) is empty, the infinite interval model corresponds to a finite interval model.

Comparing with \( S' \), \( S'_{\infty} \)-dodel contains infinite ingredients both in the time domain and the frame derived.
4.2 System $S'_\infty$

System $S'_\infty$ for infinite interval models is obtained by adding the following new axioms to $S_\infty$:

- about chop “;”
  
  \begin{align*}
  A2: & \quad ((\alpha; \beta); \gamma) \Leftrightarrow (\alpha; (\beta; \gamma)), \\
  L2: & \quad (\ell = x; \ell = y) \Rightarrow \ell = x + y, \\
  & \quad (\ell = x + y \land \neg(x = \infty)) \Rightarrow (\ell = x; \ell = y), \\
  L3: & \quad (\alpha \land \neg(\ell = \infty)) \Rightarrow (\alpha; \ell = 0), \\
  & \quad \alpha \Rightarrow (\ell = 0; \alpha);
  \end{align*}

- about Duration domain
  
  \begin{align*}
  D1 & \quad (x + y) + z = x + (y + z), \\
  D2 & \quad 0 + x = x + 0 = x, \\
  D3 & \quad x + y = \infty \Leftrightarrow ((x = \infty) \lor (y = \infty)), \\
  D4 & \quad ((x + y = x + z) \land \neg(x = \infty)) \Rightarrow y = z, \\
  & \quad ((y + x = z + x) \land \neg(x = \infty)) \Rightarrow y = z, \\
  D5 & \quad x + y = 0 \Rightarrow ((x = 0) \land (y = 0)), \\
  D6 & \quad (\exists x)(x + z = y \lor y + z = x), \\
  & \quad (\exists x)(z + x = y \lor z + y = x);
  \end{align*}

- about infinity
  
  \begin{align*}
  I1 & \quad I \Leftrightarrow \ell = \infty, \\
  I2 & \quad \ell = 0 \Rightarrow \neg(\ell = \infty).
  \end{align*}

It is easy to prove that $S'_\infty$ is sound.

4.3 Completeness

**Theorem 12** For any formula $\alpha$ of an infinite interval language $L$, $\alpha$ is a theorem of $S'_\infty$ if $\alpha$ is valid in all infinite interval models for $L$.

For a consistent set $\Gamma_0$ of sentences in an infinite interval language $L$ with respect to $S'_\infty$, below we construct an $S'_\infty$-model for it.

First, as in the previous section, we extend $L^+$ by a new set of rigid constants $B$, and let $\Gamma_0^*$ be a maximal consistent extension of $\Gamma_0$ with witnesses in $B$. Let $\Sigma_0$ be the set of rigid sentences in $\Gamma_0^*$. Second, following the steps 1-4 in the last section, we construct an $S_\infty$-model $M_0 = (W_0, \tilde{W}_0, R_0, D_0, I_0)$. By the completeness proof for $S_\infty$, we have $\Gamma_0^* \subseteq W_0$ and $M, \Gamma_0^*, v \models \alpha$ if $\alpha \in \Gamma_0^*$, where $v$ is an (arbitrary) assignment of the variables of $L^+$ ($\mathcal{L}$).

Finally, we construct the desired $S'_\infty$-model. It is done for two cases.

**Case 1:** $I \in \Gamma_0^*$

When $I \in \Gamma_0^*$ holds, that is, $\ell = \infty \in \Gamma_0^*$, we can imagine that $\Gamma_0^*$ can only be satisfied by an infinite interval (possible word). So $\Gamma_0^*$ will serve as the infinite time point.

The time domain is defined as

$$T = \{ \Delta \in W_0 \mid \text{there exists } \Delta' \in W_0 \text{ such that } \Delta \ast \Delta' \subseteq \Gamma_0^* \} \cup \{ \Gamma_0^* \}.$$ 

Now we define an order “$\leq$” on $T$: for any two time points $\Delta_1$ and $\Delta_2$ in $T$, $\Delta_1 \leq \Delta_2$ if and only if there are $b_1$ and $b_2$ in $B$ such that $\ell = b_1 \in \Delta_1$ and $\ell = b_1 + b_2 \in \Delta_2$. It is trivial to prove
that \(\leq\) is a total order on \(T\). Noting that \(\ell = \infty \in \Gamma^*_0\), we can easily prove that \(\Gamma^*_0\) is the only maximal element, that is, the infinite time point.

Let \(W\) be the set of all intervals on \(T\).

To establish a relation between \(W\) and \(W_0\), we define a map \(\mu : W \rightarrow W_0\) as follows:

\[
\mu([\Delta_1, \Delta'_1]) = \Delta, \quad \text{where} \; \Delta \; \text{is the only world in} \; W_0 \; \text{such that} \; R_0(\Delta_1, \Delta, \Delta'_1)
\]

The existence and the uniqueness of \(\Delta\) can be ensured by the following lemma.

Lemma 13 Let \(\Delta_1\) and \(\Delta'_1\) be two elements in \(T\), Assume \(\Delta_1 \leq \Delta'_1\) and there is a world \(\Delta_2\) of \(W_0\) such that \(\Delta_1 \ast \Delta_2 \subseteq \Gamma^*_0\).

1. If there is also a world \(\Delta'_2\) of \(W_0\) such that \(\Delta'_1 \ast \Delta'_2 \subseteq \Gamma^*_0\), then there is a unique world \(\Delta\) such that \(R_0(\Delta_1, \Delta, \Delta'_1)\) (and \(R_0(\Delta, \Delta_2, \Delta_2)\)).

2. If \(\Delta'_1 = \Gamma^*_0\) then here is a unique world \(\Delta\) of \(W_0\) such that \(R_0(\Delta_1, \Delta, \Delta'_1)\).

Proof: (1) Suppose there are two worlds \(\Delta_2\) and \(\Delta'_2\) in \(W_0\) such that

\[
\Delta_1 \ast \Delta_2 \subseteq \Gamma^*_0 \; \text{and} \; \Delta'_1 \ast \Delta'_2 \subseteq \Gamma^*_0
\]

Choose four constants \(b_1, b_2, b'_1\) and \(b'_2\) from \(B\) such that

\[
\ell = b_1 \in \Delta_1, \quad \ell = b_2 \in \Delta_2, \quad \ell = b'_1 \in \Delta'_1, \quad \ell = b'_2 \in \Delta_2
\]

Since \(\Delta_1 \leq \Delta'_1\), there is also an element \(b\) of \(B\) such that \(b_1 + b = b'_1 \in \Sigma_0\). Particularly, \(b_1 + b = b'_1 \in \Delta'_1\).

Define

\[
A = \{\ell = 0\} \cup \delta_1(\Delta'_1, \Delta_1) \cup \delta_2(\Delta_2, \Delta'_2)
\]

Then we can show that \(A\) is consistent. Let \(\Delta\) be a maximal consistent set of sentences in \(\mathcal{L}^+\) which includes \(A\). By lemma 4.8, since \(\delta_1(\Delta'_1, \Delta_1) \subseteq \Delta\) and \(\delta_2(\Delta_2, \Delta'_2) \subseteq \Delta\), the set \(\Delta\) is a world of \(W_0\) and \(R_0(\Delta_1, \Delta, \Delta'_1)\) and \(R_0(\Delta, \Delta'_2, \Delta_2)\).

Uniqueness of \(\Delta\) is due to the fact that \(\mathcal{M}_0\) is an \(S_{\infty}\)-model.

(2) If \(\Delta'_1 = \Gamma^*_0\) but \(\Delta_1 \neq \Gamma^*_0\), by definition of \(\Delta_1\), there is a world \(\Delta\) of \(W_0\) such that \(\Delta_1 \ast \Delta = \Gamma^*_0\), that is, \(\Delta_1 \ast \Delta = \Delta'_1\). So there is a world \(\Delta\) of \(W_0\) such that \(R_0(\Delta_1, \Delta, \Delta'_1)\). The uniqueness of such \(\Delta\) is also can be derived from the property that \(\mathcal{M}_0\) is an \(S_{\infty}\)-model.

Now we define a new infinite possible world model \(\mathcal{M}\) be \((W, W, R, D, I)\) which will be used as the infinite interval model modal satisfying \(\Gamma^*_0\).

- \((W, R)\) is the interval frame on \(T\) defined as above;
- \(\tilde{W}\) is a subset of \(W\) such that an interval \([u, u']\) of \(W\) is in \(\tilde{W}\) if and only if \(\ell = \infty \in u'\);
- Duration domain \(D\) is the same as \(D_0\);
- The interpretation function \(I\) is defined by

\[
I(s, [u, u']) = I_0(s, \mu([u, u']))
\]

for any symbol of \(\mathcal{L}\) and any interval \([u, u']\) of \(W\).

The completeness proof for case 1 is concluded with the following two lemmas and a proof of the fact that \(\mathcal{M}\) is indeed an \(S_{\infty}\)-model.

Lemma 14 Let \([u, u']\) be an interval of \(\mathcal{M}\) and \(\alpha\) a formula of \(\mathcal{L}\), then

\[
\mathcal{M}, [u, u'], v \models \alpha \quad \text{iff} \quad \mathcal{M}_0, \mu([u, u']), v \models \alpha.
\]
Lemma 15 There is an interval $[u, u']$ of $W$ such that $\mu([u, u']) = \Gamma_0^*$.  

Case 2: $\mathcal{I} \not\in \Gamma_0^*$  
In this case, roughly speaking, $\Gamma_0^*$ would be finite time point. So we need put an infinite time point into the time domain to fit the definition of infinite time domain. But this infinite time point is “virtual”, because $\Gamma_0^*$ can be satisfied by a finite interval.

Now the time domain $T$, having an infinite time point, is defined as

$$T = \{ \Delta \in W_0 \mid \text{there exists } \Delta' \in W_0 \text{ such that } \Delta * \Delta' \subseteq \Gamma_0^* \} \cup \{ \infty \}$$

where $\infty$ is an arbitrary element not occurring in $T_1$;

$$T_1 = \{ \Delta \in W_0 \mid \text{there exists } \Delta' \in W_0 \text{ such that } \Delta * \Delta' \subseteq \Gamma_0^* \}.$$  

We still need define an order “$\leq$” on $T$: for any two time points $\Delta_1$ and $\Delta_2$ in $T$, $\Delta_1 \leq \Delta_2$ if and only if $\Delta_1 \in T_1$ and, either $\Delta_2 = \infty$ or $\Delta_2 \in T_1$ and there are $b_1$ and $b_2$ in $B$ such that $\ell = b_1 + b_2 \in \Delta_2$.

The set $W$ is chosen to be all the finite intervals on $T$, that is $W = \{ [\Delta_1, \Delta_2] \mid \Delta_1 \in T_1, \Delta_2 \in T_1, \Delta_1 \leq \Delta_2 \}$. The map from $W$ to $W_0$ is defined as in Case 1.

The new infinite possible world model $M$ be $(W, \mathcal{W}, R, D, I)$ that be used as the infinite interval model modal satisfying $\Gamma_0^*$ now is as follows.

- $(W, R)$ is the interval frame on $T$ defined as above;
- $\mathcal{W}$ is a subset of $W$ such that an interval $[u, u']$ of $W$ is in $\mathcal{W}$ if and only if $\ell = \infty \in u'$, that is $\mathcal{W} = \emptyset$;
- Duration domain $D$ is the same as $D_0$;
- The interpretation function $I$ is defined by
  $$I(s, [u, u']) = I_0(s, \mu([u, u']))$$
  for any symbol of $L$ and any interval $[u, u']$ of $W$.

As proved in Case 1, we can find a finite interval $[u, u']$ of $W$ such that $\mu([u, u']) = \Gamma_0^*$ and, $u' \neq \infty$ if $\ell = \infty \not\in \Gamma_0^*$. The proof is finished by a easy showing og $[u, u']$ satisfies $\Gamma_0^*$.

5 Conclusion

In this paper, we have extended Interval Temporal Logics to infinite models. The ITL proof systems for finite possible worlds and abstract intervals models are extended and shown to be complete by the classical “Henkin’s construction” method.

Our work provides a foundation to some logics on infinite intervals [8, 9]. ITL on infinite intervals is related to Interval Calculus when the latter is also extended to infinite intervals [3, 1]. Interval Calculus is formed in set theory and corresponds to the semantics of an ITL. A number of theorems are proven in Interval Calculus and can be used to derive other theorems. Therefore, they serve as axioms and rules of a proof systems. However, we are not aware of any work on the completeness of the rules. Therefore, our completeness results may also be of interest to Interval Calculus.

Acknowledgements

A large part of the research was done at International Institute for Software Institute, the United Nations University. The authors wish to thank Professor Zhou Chaochen, Michael R. Hansen for their helpful suggestions, and the anonymous referees for suggesting improvements.
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