A PAIR OF HIGHER ORDER SYMMETRIC NONDIFFERENTIABLE MULTIOBJECTIVE MINI-MAXMIXED PROGRAMMING PROBLEMS

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ABSTRACT
A pair of higher order symmetric non differentiable minimax mixed programming problem where each objective function contains support function of compact convex set in $R^n$, is formulated. Under higher order F-convexity assumption, weak, strong and converse symmetric duality theorems related to a properly efficient solution and self-duality are proved.

Keywords: efficient solution, self duality, F-convexity, support function, minimax-mixed, Higher-order symmetric

1. INTRODUCTION
Symmetric duality in nonlinear programming problem was first introduced by Dorn [1] who defined a mathematical programming problem and it’s dual to be symmetric if the dual is the primal problems. Later Dantzing, Eisinberg and Cottle [2] and Mond [3] formulated a pair of symmetric dual programs for scalar function $f(x, y)$ that is convex in the first variable and that is concave in the second variable respectively. Balas[4] generalized duality for linear and nonlinear mixed integer programming problems.


Higher order duality in nonlinear programs have been studied by some researchers. Mangasarian [13] formulated a class of higher order dual programs for nonlinear programming problem. Mond and Zhang obtained duality results for various higher order dual programming problems under higher order invexity assumption, such as higher order type-1, higher order pseudo type-1, and higher order quasi type -1 conditions. Mishra and Rueda [14] gave various duality results which included Mangasarian higher order duality and Mond-Weir higher order duality. Chen [15] also discussed the duality theorems under the higher order F-convexity for a pair of nondifferentiable programs. Chen first gave a pair of nondifferentiable multiobjective functions contains a support function of compact convex set in $R^n$ and discussed the symmetric duality for multiobjective minimax mixed integer programming problems.

2. PRESENT WORK
In this chapter, a pair of higher order symmetric nondifferentiable mini-max mixed programming problems by introducing a differentiable function is formulated, where each of objective functions contains a support function of compact convex set in $R^n$. For a differentiable function $h : R^n \times R^n \rightarrow R$ the definitions of the higher order F-convexity (F-pseudo convexity, F-pseudo convexity) with respect to $h$ are introduced. Then all known other generalized invexity, such that, type-1 invexity and higher order type-1 invexity can be put into the category of the higher order type-1 invexity can be put into the category of the higher order F-invex functions by taking certain appropriate transformations of $F$ and $h$. Under these the higher-order
F-convexity assumption, the higher order weak, higher order strong and higher converse symmetric duality theorems related to a properly efficient solution and self duality are proved.

3. NOTATION AND DEFINITION

Throughout this chapter $R^n$ and $R^m$ are $n$-dimensional and $m$-dimensional Euclidian spaces respectively. $R^n_+$ and $R^m_+$ are non negative orthants of $R^n$ and $R^m$ respectively. Let U and V be two arbitrary sets of integers in $R^n (0 \leq n \leq n)$ and $R^m (0 \leq m \leq m)$ respectively and $C_1$ and $C_2$ are closed convex cones in $R^n_+$ and $R^m_+$. Let $x \in R^n$ and $y \in R^m$. Without loss of generality, suppose the first $n_1$ components of $x$ and the first $m_1$ components of $y$ are constrained to be integers and write $(x, y) = (x_1, x_2, y^t, y^2)$ where $x_1 \in U$ and $y^t \in V$, $x_2 \in C_1$ and $y^2 \in C_2$, where $n = n_1 + n_2$ and $m = m_1 + m_2$. For a real-valued twice differentiable function $g(x, y)$ defined on an open set $R^n \times R^m$, denote by $\nabla_{x,y} g(\bar{x}, \bar{y})$ the gradient vector of $g$ with respect to $x^2$ at $(\bar{x}, \bar{y})$, $\nabla_{x,y}^t g(\bar{x}, \bar{y})$, the Hessian matrix with respect to $x^2$ at $(\bar{x}, \bar{y})$, similarly $\nabla_{x,y}^t g(\bar{x}, \bar{y})$ and $\nabla^2_{x,y} g(\bar{x}, \bar{y})$ are also defined.

Let $C$ be a compact convex set in $R^n$. The support function of $C$ is defined by

$$s(x \mid C) = \max \{ x^t \gamma \mid y \in C \}$$

A support function, being convex and everywhere finite, has a sub differential, that is there exists $z \in R^n$ such that $s(x \mid C) \geq s(x \mid C) + z^t (y - x) \mid C$ for all $y \in C$.

The sub differential of $s(x \mid C)$ is given by $\partial s(x \mid C) = \{ z \in C \mid z^t x = s(x \mid C) \}$.

For any set $D \subset R^n$, the normal cone to $D$ at a point $x \in D$ and is defined as

$$N_D(x) = \{ y \in R^n \mid y^t(z - x) \leq 0, \forall z \in D \}$$

It is obvious that for a compact convex set $C$, $y \in N_C(x)$ iff $s(x \mid C) = x^t y$, or equivalently, $x \in \partial s(y \mid C)$.

Consider the following multiobjective programming problem

(MOP) Minimize $f(x)$

subject to $g(x) \leq 0, x \in X$, where $f: R^n \rightarrow R^l$, $g: R^n \rightarrow R^m$ and $X \subset R^n$.

We denote the set of feasible solutions of (MOP) by $P = \{ x \in X \mid g(x) \leq 0 \}$.

**Definition 3.1:** A point $\bar{x} \in P$ is said to be an efficient solution of (MOP) if there exists no other $x \in P$ such that $f(\bar{x}) \leq f(x)$ in $R^l \setminus \{ 0 \}$, that is $f_i(x) \leq f_i(\bar{x})$ for all $i \in \{ 1, 2, 3, \ldots, k \}$, and at least one $j \in \{ 1, 2, 3, \ldots, k \}$, $f_j(x) < f_j(\bar{x})$, $\bar{x} \in P$ is said to be a weak efficient solution of (P) if there exists no other $x \in P$ such that for all $i \in \{ 1, 2, 3, \ldots, k \}$, $f_i(\bar{x}) > f_i(x)$.

**Definition 3.2:** $\bar{x} \in P$ is said to be a Geoffrion properly efficient solution of (P), if $\bar{x}$ is an efficient solution, and there exists a real number $M > 0$ such that for all $i \in \{ 1, 2, 3, \ldots, p \}$, $x \in P$ and $f_j(x) < f_j(\bar{x})$, then $f_j(x) - f_j(\bar{x}) \leq M[f_j(x) - f_j(x)]$ for some $j \in \{ 1, 2, 3, \ldots, k \}$ such that $f_j(\bar{x}) < f_j(x)$.

**Lemma 3.1:** If $x \in P$ is a properly efficient solution of (MOP), there exist $a = (a_1, a_2, \ldots, a_m) \in R^n$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_m)^T \in R^m$ such that

$$\sum_{i=1}^{n} a_i \nabla_{x,y} f_i(\bar{x}) + \sum_{j=1}^{m} \beta_j \nabla_{x,y} g_j(\bar{x}) = 0, a \geq 0, \beta \geq 0, (a^T, \beta^T) \neq 0.$$

**Definition 3.3:** A function $F: X \times X \times R^n \rightarrow R$ (where $X \subset R^n$) is sub linear with respect to the third variable if for all $(u, v) \in X \times X$

(i) $F(x, u; a_1 + a_2) \leq F(x, u; a_1) + F(x, u; a_2)$, for all $a_1, a_2 \in R^n$.

(ii) $F(x, u, \alpha a) = \alpha F(x, u; a)$, $a \geq 0, \alpha \geq 0$, for all $a \in R^n$.

**Definition 3.4:** Suppose that $h: X \times R^n \rightarrow R$ is a differentiable function, $F$ is sub linear with respect to the third argument. We say that

(i) $f$ is said to be higher order F-convexity in $u \in X$ with respect to $h$, if for all $(x, p) \in X \times R^n \Rightarrow f(x) - f(u) \geq F(x, u; \nabla x f(u) + \nabla y h(u, p)) - \gamma \{ \nabla y h(u, p) \}$

(ii) $f$ is said to be higher order F-pseudo-convexity in $u \in X$ with respect to $h$, if for all $(x, p) \in X \times R^n$, we have

$$F(x, u; \nabla x f(u) + \nabla y h(u, p)) \geq 0 \Rightarrow f(x) - f(u) \geq h(u, p) - \gamma \{ \nabla y h(u, p) \}$$

(iii) $f$ is said to be higher order F-quasi-convex in $u \in X$ with respect to $h$, if for all $(x, p) \in X \times R^n$, we have

$$f(x) \leq f(u) + \gamma h(u, p) - \gamma \{ \nabla y h(u, p) \} \Rightarrow F(x, u; \nabla x f(u) + \nabla y h(u, p)) - \gamma \{ \nabla y h(u, p) \} \leq 0$$

**Definition 3.5:** Let $f: R^n \times R^n \rightarrow R$ and $h: X \times R^n \rightarrow R$ be differentiable function where $X \subset R, F: X \times R^n \rightarrow R$ be sub linear with respect to its third argument. We say that

(i) $f_{\gamma}(x, y)$ is higher-F-convex at $u \in X$, with respect to some function $h$, if for all $(x, p) \in X \times R^n$ and for fixed $y \in Y \subset R^m$ we have

$$f(x, y) - f(u, y) \geq F(x, u; \nabla x f(u, y) + \nabla y h(u, p))$$

(ii) $f_{\gamma}(x, y)$ is said to be higher-order F-pseudo-convex at $u \in X$ with respect to $h$, if for fixed $y \in Y \subset R^m$ and for all $(x, p) \in X \times R^n$ we have

$$f(x, y) \leq f(u, y) + \gamma h(u, p) - \gamma \{ \nabla y h(u, p) \} \Rightarrow F(x, u; \nabla x f(u, y) + \nabla y h(u, p)) - \gamma \{ \nabla y h(u, p) \} \leq 0$$
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(iii) \( f(., y) \) is said to be higher-order F-quasi-convex at \( u \in X \) with respect to \( h \) if for \( (x, p) \in X \times R^m \) and for fixed \( y \in Y \subset R^n \) we have

\[
f(x, y) - f(u, y) \leq h(u, p) - p^T \nabla h(u, p)
\]

\[
F(x, u : \nabla f(x, y) + \nabla h(u, p)) \leq 0
\]

If \(-f(., y)\) is higher order function \( F \) (pseudo-convex or F quasi-convex) at \( u \) with respect to \( h \) for all \( (x, p) \in X \times R^m \) and for fixed \( y \in Y \subset R^n \) then \( f(y, .) \) is higher order \( F \) concave (F pseudo-concave or F quasi-concave) at \( u \) with respect to \( h \) for all \( (x, p) \in X \times R^m \) and for fixed \( y \in Y \subset R^n \).

Remark 1: (i) When \( h(u, p) = \frac{1}{2} p^T (\nabla_w w(f(u))) p \) and \( F(x; u; a) = \eta(x, u)^T a, \) where \( h : X \times X \rightarrow R^n \), the higher-order convexity (higher-order F pseudo-convexity, higher order F quasi-convexity) reduces to \( \eta \)-convexity (\( \eta \)-pseudo-convexity, \( \eta \)-quasi-convexity) in [9].

(ii) When \( h(u, p) = \frac{1}{2} p^T (\nabla_w w(f(x))) p \) and \( F(x; u; a) = \eta(x, u)^T a, \) where \( a : X \times X \rightarrow R^n \) \{0\}, \( \eta : X \times X \rightarrow R^m \) are positive functions and \( F(x; u; a) = \eta(x, u)^T a, \) where \( a : X \times X \rightarrow R^n \) \{0\}, \( \eta : X \times X \rightarrow R^m \) is differentiable function, then the higher order \( F \) convexity (higher order F pseudo-convexity, higher order F quasi-convexity) function becomes the higher-order type 1 (higher order pseudo-type 1, higher order quasi-type 1) function.

From now on, suppose that the sub linear function \( F \) satisfies the following condition \( F(x; y; a + a^\prime y \geq 0, \) for all \( a \in R^m. \)

Definition 3.6: A real valued function \( \phi(x^1, x^2, ... x^l) \) will be called additionally separable with respect to \( x \) 1 if there exist real valued functions \( \xi(x^l) \) independent of \( x^1, x^2, ... x^l \) such that \( \phi(x^1, x^2, ... x^l) = \xi(x^l) + \xi(x^2, x^3, ... x^l) \)

4. HIGHER-ORDER SYMMETRIC DUALITY

In this section, we consider twice differentiable functions \( f_i : R^n \times R^m \rightarrow R, g_i : R^n \times R^m \rightarrow R, h_i : R^n \times R^m \times \cdots \rightarrow R \), and compact convex sets \( C_i \subset R^{n_i} \) and \( D_i \subset R^{n_i} \) for \( i = 1, 2, ... k \).

We formulate the following higher order symmetric nondifferentiable multiobjective Minimax mixed integer symmetric primal and dual problems.

Primal problem (MOP).

\[
\max_{x, y, \lambda} \min_{x, y} \left[ (f_i(x, y) + s(x^i | C_i) - (y^j)^T z + h_i(x, y, p^i) - (p^j)^T \nabla_p h_i(x, y, p^i)) \right] \\
\min_{x, y} \left[ (f_j(x, y) + s(x^j | C_j) - (y^j)^T z + h_j(x, y, p^j) - (p^j)^T \nabla_p h_j(x, y, p^j)) \right]
\]

subject to

\[
\sum_{i=1}^{k} \lambda_i \left[ \nabla_{y^i} f_i(x, y) - z^i + \nabla_{y^i} h_i(x, y, p^i) \right] \leq 0
\]

\[
(y^j)^T \sum_{i=1}^{k} \lambda_i \left[ \nabla_{y^i} f_i(x, y) - z^i + \nabla_{y^i} h_i(x, y, p^i) \right] \geq 0
\]

\[
x^i \in U_i, y^j \in V_j, x^j \in R^{n_j}, z \in D_i, p^i \in R^{n_i}, \quad i = 1, 2, ... k, \lambda > 0, \lambda^T e = 1,
\]

Dual Problem (MOD)

\[
\min_{x, y, \lambda} \max_{x, y} \left[ (f_i(u, v) - s(x^i | D_i) - (y^j)^T w^i + g_i(u, v, r^i) - (r^j)^T \nabla g_i(u, v, r^i)) \right] \\
\min_{x, y} \left[ (f_j(u, v) - s(x^j | D_j) + (y^j)^T w^i + g_i(u, v, r^i) - (r^j)^T \nabla g_i(u, v, r^i)) \right]
\]

subject to

\[
\sum_{i=1}^{k} \lambda_i \left[ \nabla_{y^i} f_i(u, v) - w^i + \nabla_{y^i} g_i(u, v, r^i) \right] \geq 0,
\]

\[
(y^j)^T \sum_{i=1}^{k} \lambda_i \left[ \nabla_{y^i} f_i(u, v) - w^i + \nabla_{y^i} g_i(u, v, r^i) \right] \leq 0,
\]

\[
u^i \in U_i, v^j \in V_j, u^j \in R^{n_j}, w^i \in C_i, \quad r^i \in R^{m_j}, i = 1, 2, ... k, \lambda > 0, \sum_{i=1}^{k} \lambda_i = 1
\]

Since the objective functions of (MOP) and (MOD) contain the support function \( s(x^2 | C) \) and \( s(x^2 | D) \), \( i = 1, 2, 3, ... k \), they are non-differentiable multiobjective programming problems.

Remark 2: (i) If \( U = \phi, V = \phi \) then (MOP) and (MOD) become the problems considered by X. Chen [15].

(ii) If \( h_i(x, y, p^i) = \frac{1}{2} (p^j)^T \nabla_{y^i} f_i(x, y)p^j + p_i - s(x^i | D_i), \) \( g_i(u, v, r^i) = \frac{1}{2} (r^j)^T \nabla_{y^i} f_i(u, v)r^i, \) \( r_i = r, \) and \( k = 1, \) then (MOP) and (MOD) can be changed into the following problems.

Primal:

\[
\max_{x, y, \lambda} \min_{x, y} \left[ (f(x, y) + s(x^i | C_i) - (y^j)^T z + \frac{1}{2} (p^j)^T \nabla_{y^i} f(x, y)p^j) \right]
\]

Subject to

\[
\nabla_{y^i} f(x, y) - z + \frac{1}{2} (p^j)^T \nabla_{y^i} f(x, y)p^j \leq 0,
\]

\[
(y^j)^T \nabla_{y^i} f(x, y) - z + \frac{1}{2} (r^j)^T \nabla_{y^i} f(x, y)p^j \geq 0,
\]

\[
x^i \in U_i, y^j \in V_j, x^j \in R^{n_j}, z \in D, p \in R^{m_j}.
\]

Dual:

\[
\min_{x, y, \lambda} \max_{x, y} \left[ (f_i(u, v) - s(x^i | D_i) + (y^j)^T w + \frac{1}{2} (r^j)^T \nabla_{y^i} f(u, v)r^j) \right]
\]
Subject to \[ \nabla_{\lambda} f(u, v) + w + \nabla_{\lambda} f(u, v) r \geq 0, \]
\[ (u^T)^T [\nabla_{\lambda} f(u, v) + w + \nabla_{\lambda} f(u, v) r] \leq 0, \]
\[ u^T \in U, v^T \in V, v^2 \geq 0, w \in D, r \in R_{-n}, \]
which are the generalized forms in Hou and Yang [12].

In the sequel we shall establish the weak strong and converse duality theorem under the higher order \( F \)-convexity assumptions.

For this we suppose that the function \( F : R^n \times R^n \rightarrow R \)
and \( G : R^m \times R^m \rightarrow R \) are sub linear and satisfy the condition
\[ \begin{align*}
(1) & \quad F(x, u; a) + a^T y \geq 0, \forall a \in R^n, \\
(2) & \quad G(v, y; a) + a^T y \geq 0, \forall a \in R^n.
\end{align*} \]

Also suppose that the following condition are satisfied:
\[ \begin{align*}
(i) & \quad \text{The function } f_1(x, y) + f_2(x, y) \text{ are higher order F-convex at } x \text{ with respect to } (u^T, v^T) \text{ and } \ \&(u^T, v^T) \text{ are higher order G-convex at } y \text{ with respect to } h_i(x, y, p) \text{ for } i = 1, 2, ..., k. \\
(ii) & \quad \text{For all } i \in \{1, 2, 3, ..., k\}
\end{align*} \]
\[ f(x, y) + s(x^2 | C) - (y^T)^T Z + h(x, y, p) - (p^T)^T [\nabla_{\lambda} h(x, y, p)] \leq 0, \]
\[ f(u, v) - s(\phi^2 | D) + (u^T)^T w + g(u, v, r) - r^T [\nabla_{\lambda} g(u, v, r)] \leq 0, \]
\[ f(x, y) + s(\phi^2 | C) - (y^T)^T Z + h(x, y, p) - (p^T)^T [\nabla_{\lambda} h(x, y, p)] \leq 0, \]
\[ f(u, v) - s(\phi^2 | D) + (u^T)^T w + g(u, v, r) - r^T [\nabla_{\lambda} g(u, v, r)] \leq 0. \]

Proof: Since \( f(x, y) \) and \( h(x, y, p) \) are additively separable with respect to \( x_i \) or \( y_i \) (say with respect to \( x^2 \)), it holds
\[ f_i(x, y) = f_{i1}(x^2) + f_{i2}(x, y), \]
\[ h_i(x, y, p) = h_{i1}(x^2) + h_{i2}(x, y, p), \]
\[ \nabla_{\lambda} f_i(x, y) = \nabla_{\lambda} h_{i1}(x^2, y) \]
\[ \text{and } \nabla_{\lambda} h_i(x, y, p) = \nabla_{\lambda} h_{i2}(x^2, y, p), i = 1, 2, 3, ..., k. \]

Thus (MMP) can be rewritten as
\[ Z = \max \min_{\lambda, \phi} \sum_{i=1}^k \lambda_i \left[ \nabla_{\lambda} f_{i1}(x^2, y) + h_{i1}(x^2, y) + s(x^2 | C) - (y^T)^T Z + h_{i2}(x^2, y, p) - (p^T)^T [\nabla_{\lambda} h_{i2}(x^2, y, p)] \right]. \]
Remark 3:

(i) From now on, without loss of generality we can assume that $f_i(x, y), h_i(x, y, p_i)$, and $g_i(x, y, p_i)$ are additively separable with respect to $x_i, i = 1, 2, k$.

(ii) From the process of the proof in theorem-1, we can also obtain that (A) and (B) cannot hold simultaneously if sub linear functions $F$ and $G$ satisfy the condition (a) and (b) and for each feasible solution $(x, y, \lambda, z^1, z^2, ..., z^p, p^1, p^2, ..., p^m)$ of (MOP) and each feasible solution $(u, v, \lambda, w^1, w^2, ..., w^p, r^1, r^2, ..., r^m)$ of (MOD), one of the following conditions holds.

(i) $f_i(u^1, v) + (\gamma)^T w^i$ is higher order $F$-pseudo convex at $u^2$ with respect to $g_2(u^2, v, r)$ and $f_i(x, y^1, . - (\gamma)^T z^i$ is higher order $G$-pseudo-concave at $y^2$ with respect to $h_2(x^2, y, p^i)$.

(ii) $f_i(u^1, . , v) + (\gamma)^T w^i$ is higher order $F$-quasi-convex at $u^2$ with respect to $g_2(u^2, v, r)$ and $f_i(x, y^1, . - (\gamma)^T z^i$ is higher order $G$-quasi-concave at $y^2$ with respect to $h_2(x^2, y, p^i)$.

The following result indicates that under some conditions, a properly efficient solution of (MOP) is also the ones of (MOD) and the two objective values are correspondingly equal.

**Theorem 4.2 (Strong Duality):**

Let $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}^1, \overline{z}^2, \overline{z}^3, ... \overline{z}^k, \overline{p}^1, \overline{p}^2, ..., \overline{p}^m)$ be an efficient solution of (MMP), $f_i: R^n \times R^m \rightarrow R$ is twice differentiable at $(\overline{x}, \overline{y})$, $h_i: R^n \times R^m \rightarrow R$, $g_i: R^n \times R^m \rightarrow R$, is twice differentiable at $(\overline{x}, \overline{y}, \overline{p}^i)$, is twice differentiable at $(\overline{x}, \overline{y}, \overline{p}^i)$, for $i = 1, 2, 3, ... k$. Assume that the following conditions hold;

$h_i(\overline{x}^2, \overline{y}, 0) = 0, g_i(\overline{x}^2, \overline{y}, 0) = 0, \nabla_{p^i} h_i(\overline{x}^2, \overline{y}, 0) = 0$

(i) $h_i(\overline{x}^2, \overline{y}, 0) = 0, \nabla_{p^i} h_i(\overline{x}^2, \overline{y}, 0) = 0$

$\nabla_{p^i} h_i(\overline{x}^2, \overline{y}, 0) = \nabla_{p^i} h_i(\overline{x}^2, \overline{y}, 0), i = 1, 2, 3, ..., k$

(ii) for all $i \in \{1, 2, 3, ..., k\}$, the linear independent vectors

$\begin{align*}
\{ \nabla_{p^i} f_i(\overline{x}^2, \overline{y}, \overline{p}^i) - \overline{z}^i + \nabla_{p^i} h_i(\overline{x}^2, \overline{y}, \overline{p}^i) \}^T
\end{align*}$

is linearly independent,

(iii) The set of vectors

$\{ \nabla_{p^i} f_i(\overline{x}^2, \overline{y}, \overline{p}^i) - \overline{z}^i + \nabla_{p^i} h_i(\overline{x}^2, \overline{y}, \overline{p}^i) \}^T$

Then $p^i = 0, i = 1, 2, 3, ..., k$; And there exists $x^{i'} \in C_i$ such that $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}^1, \overline{w}^2, ..., \overline{w}^k, \overline{r}^1 = 0, \overline{r}^2 = 0, ..., \overline{r}^k = 0)$ is a feasible solution of (MOD). Furthermore, if the hypotheses in Theorem 3.1 are satisfied and $h_i(\overline{x}^i) = g_i(\overline{x}^i), i = 1, 2, 3, ..., k$, then $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}^1, \overline{w}^2, ..., \overline{w}^k, \overline{r}^1 = 0, \overline{r}^2 = 0, ..., \overline{r}^k = 0)$ is an efficient solution of (MOD), and the two objective values are equal.

**Proof:** If $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}^1, \overline{z}^2, ..., \overline{z}^k, \overline{p}^1, \overline{p}^2, ..., \overline{p}^m)$ is a properly efficient solution for (MMP) then $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}^1, \overline{z}^2, ..., \overline{z}^k, \overline{p}^1, \overline{p}^2, ..., \overline{p}^m)$ is also efficient for (MP). Thus, under the condition in this theorem we obtain from theorem-2 in X. Chen[15] that there exist $\overline{w}^i \in C_i, i = 1, 2, 3, ..., k$, such that $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}^1, \overline{w}^2, ..., \overline{w}^k, \overline{r}^1 = \overline{r}^2 = ... = \overline{r}^k = 0$ is a feasible solution of (MD). It is obvious that it is also feasible for (MMD).

Furthermore, if the hypotheses of higher order $F$-convexity in theorem-1 are satisfied, then the objective values of (MP) and (MD) are equal by [15], that is

$f_i(\overline{x}^i, \overline{y}) = s(\overline{y}^i, \overline{y}) - h_i(\overline{y}^i, \overline{y}, \overline{p}^i) - (\gamma)^T \overline{z}^i$ and $f_i(\overline{x}^i, \overline{y}) = s(\overline{y}^i, \overline{y}) + g_i(\overline{y}^i, \overline{y}, \overline{p}^i) - (\gamma)^T \overline{z}^i$

(i) $i = 1, 2, 3, 4, ..., k$.

Note that $h_i(\overline{x}^i) = g_i(\overline{x}^i), \nabla_{p^i} h_i(\overline{x}^i) = 0, \nabla_{p^i} g_i(\overline{x}^i) = 0$.

From theorem-1, $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}^1, \overline{w}^2, ..., \overline{w}^k, \overline{r}^1 = \overline{r}^2 = ... = \overline{r}^k = 0$ is an efficient solution of (MMD).

It is similar to the method of the proof of theorem-2 in X. Chen-2004[15] that it is also a properly efficient solution of (MMD). Similarly, we have the following converse Duality.

**Theorem 4.3: (Converse Duality):** Let $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{w}^1, \overline{w}^2, ..., \overline{w}^k, \overline{r}^1 = \overline{r}^2 = ... = \overline{r}^k = 0$ be a properly efficient solution of (MMD), $f_i: R^n \times R^m \rightarrow R$ is twice differentiable at $(\overline{u}, \overline{v}), g_i: R^n \times R^m \rightarrow R$ is twice differentiable at $(\overline{u}, \overline{v}, \overline{r}^i)$, $h_i: R^n \times R^m \rightarrow R$ is differentiable at $(\overline{u}, \overline{v}, \overline{r}^i)$ if the following conditions hold

(i) $h_i(\overline{u}^i, \overline{v}, 0) = 0, g_i(\overline{u}^i, \overline{v}, 0) = 0, \nabla_{p^i} g_i(\overline{u}^i, \overline{v}, 0) = 0, \nabla_{p^i} h_i(\overline{u}^i, \overline{v}, 0) = 0, i = 1, 2, 3, ..., k$;

(ii) For some $a \in R^i (a > 0)$ and $p \in R^{m}, p^i \neq 0, i = 1, 2, 3, ... k$ we have

$\sum_{i=1}^{k} a_i (p^i)^T \left[ \nabla_{p^i} f_i(\overline{u}^i, \overline{v}) - \overline{z}^i + \nabla_{p^i} h_i(\overline{u}^i, \overline{v}, \overline{p}^i) \right] = 0$

Then $p^i = 0, i = 1, 2, 3, ..., k$; And there exists $\overline{w}^i \in C_i$ such that $\{ \nabla_{p^i} f_i(\overline{u}^i, \overline{v}) - \overline{z}^i + \nabla_{p^i} h_i(\overline{u}^i, \overline{v}, \overline{p}^i) \}^T$ is linearly independent.
(iv) For some \( \alpha \in R (\alpha > 0) \) and \( r^i \in R^n, r^i \neq 0 \) \( (i = 1, 2, 3, ..., k) \) implies that
\[
\sum_{i=1}^k \alpha_i r^i \epsilon \{ \mathbf{v}_x^i f_{\alpha,i} (\mathbf{v}^2_iz, \mathbf{v}) + \mathbf{w}^i + \mathbf{v}_x^i \mathbf{g}_{\alpha,i} (\mathbf{v}^2_iz, \mathbf{v}, \mathbf{F}) \} \neq 0
\]
Then (i) \( \mathbf{r}^i = 0, i = 1, 2, 3, ..., k \).

(ii) There exists \( \mathbf{z}^i \in C \), such that \( \{ \mathbf{v}, \mathbf{v}^2_iz, \mathbf{z}^i, ..., \mathbf{z}^i, \mathbf{r}^i, ..., \mathbf{r}^i \} = 0 \) is feasible solution of (MMP).

Furthermore, if the hypotheses in Theorem 3.1 are satisfied and \( \mathbf{g}_{\alpha,i} (\mathbf{v}^2_i z) = h_{\alpha,i} (\mathbf{v}^2_i z), i = 1, 2, ... k \) then \( \{ \mathbf{v}, \mathbf{v}^2_iz, \mathbf{z}^i, ..., \mathbf{z}^i, \mathbf{r}^i, ..., \mathbf{r}^i \} = 0 \) is properly efficient solution of (MMP), and the two objective values are correspondingly equal.

5. HIGHER-ORDER SELF DUALITY

A mathematical programming problem is said to be self-dual, if when the primal is recast in the form of the dual, the new problem obtained is the same as the dual problem.

First, we give the following definition.

**Definition 5.1:** The function \( h : I \times R^n \times R^n \times R^n \rightarrow R \) is said to be skew-symmetric with respect to \( x \) and \( y \) if for all \( x \) and \( y \) in the domain of \( k \) such that \( h(x^i_1, x^i_2, y^i_1, y^i_2, y^i_3) = -h(y^i_1, y^i_2, y^i_3, x^i_1, x^i_2) \) where \( x^i_1 \in U, x^i_2 \in R^n \) and \( y^i_2 \in R^n \) and \( U \) is an arbitary set of integers in \( R^n \).

**Theorem 5.1 (Self-duality):** If \( f_i \) and \( h_i \) in (MMP) are skew symmetric function with respect to \( x \) and \( y \) and \( n = n_1 + n_2 = n \), \( U = V, C = D, z^i = w^i, p^i = r^i \) and \( h(x, y, p) = g(x, y, r), i = 1, 2, 3, ..., k \). Then (MMP) is self-dual, that is, the dual problem of (MMP) is itself, and the proper efficiency of \((\overline{x}, \overline{y}, \overline{z}, ... \overline{z}^i, ... \overline{p}^i, ... \overline{p}^i)\) for (MMP) and the converse. Furthermore, under the conditions of theorem (4.2) and (4.3), if \((\overline{x}, \overline{y}, \overline{z}, ... \overline{z}^i, ... \overline{p}^i, ... \overline{p}^i)\) is a properly efficient solution of (MMP), then \((\overline{y}, \overline{x}, \overline{z}, ... \overline{z}^i, ... \overline{p}^i, ... \overline{p}^i)\) is a proper efficient solution of (MMD), the common optimal values is zero and the converse.

**Proof:** The problem (MMP) may be represented as a max-min problem
\[
\max \min_{x \epsilon x, y \epsilon y} \{ \begin{array}{c}
-\{f_i(x, y) + s(\alpha_i \mathbf{C}_i - (\mathbf{C}_i)^{-1} \mathbf{z}^i - h_i(x, y, p^i)) + (p^i)^i [\mathbf{v}_x^i h_i(x, y, p^i)]
+ (p^i)^i [\mathbf{v}_y^i h_i(x, y, p^i)]
\end{array} \}
\]
subject to \( \sum_{i=1}^k \lambda_i \{ \mathbf{v}_x^i f_i(x, y) - \mathbf{z}^i + \mathbf{v}_y^i h_i(x, y, p^i) \} \leq 0 \)
\( (\mathbf{y}^i)^i \sum_{i=1}^k \lambda_i \{ \mathbf{v}_x^i f_i(x, y) - \mathbf{z}^i + \mathbf{v}_y^i h_i(x, y, p^i) \} \geq 0 \)
\( x^i \epsilon U, y^i \epsilon V, x^i \epsilon R^n, y^i \epsilon R^n, z^i \epsilon D, \)
\( p^i \epsilon R^n, i = 1, 2, ..., k, \lambda > 0, \lambda_i^2 e = 1, \)
\( s \epsilon S \).

Since \( f_i \) and \( h_i \) is skew-symmetric function with respect to \( x \) and \( y \) and \( C = D \), \( z^i = w^i, p^i = r^i \) and \( h(x, y, p^i) = g(x, y, r^i) \), \( i = 1, 2, 3, ..., k \) it holds
\[
\min \max_{x \epsilon x, y \epsilon y} \{ \begin{array}{c}
-\{f_i(x, y) + s(\alpha_i \mathbf{C}_i - (\mathbf{C}_i)^{-1} \mathbf{z}^i - h_i(x, y, p^i)) + (p^i)^i [\mathbf{v}_x^i h_i(x, y, p^i)]
+ (p^i)^i [\mathbf{v}_y^i h_i(x, y, p^i)]
\end{array} \}
\]
\( (\mathbf{y}^i)^i \sum_{i=1}^k \lambda_i \{ \mathbf{v}_x^i f_i(x, y) + \mathbf{w}^i + \mathbf{v}_y^i g_i(x, y, r^i) \} \geq 0 \)
\( y^i \epsilon U, y^i \epsilon V, x^i \epsilon R^n, y^i \epsilon R^n, w^i \epsilon D, \)
\( i = 1, 2, ..., k, \lambda > 0, \lambda_i^2 e = 1, \)
\( s \epsilon S \).

By theorem (4.2), (4.3) and (5.1) we have
\[
\min \max_{x \epsilon x, y \epsilon y} \{ \begin{array}{c}
-\{f_i(x, y) + s(\alpha_i \mathbf{C}_i - (\mathbf{C}_i)^{-1} \mathbf{z}^i - h_i(x, y, p^i)) + (p^i)^i [\mathbf{v}_x^i h_i(x, y, p^i)]
+ (p^i)^i [\mathbf{v}_y^i h_i(x, y, p^i)]
\end{array} \}
\]

(18)

6. CONCLUSION

In the above, we formulate a pair of the higher-order symmetric non-differentiable multiobjective min-max mixed programming problem in which the objective functions contain a support function of a compact convex set in \( R^n \) or \( R^m \).

Under the higher-order \( F \)-convexity (higher-order \( F \)-pseudo-convexity, higher-order \( F \)-quasi-convexity) assumption, we give the higher-order weak., higher-order strong, higher-order converse duality, and self duality. In our models, \( U = \phi, V = \phi \), then (MMP) and (MMD) become the problems considered by X. Chen[15].

If \( h(x, y, p) = 1/2^p^i \mathbf{v}^i g_i(x, y)p \), \( g_i(u, v, r) = 1/2 r^i \mathbf{v}^i f_i(u, v) \) and \( k = 1, U = \phi, V = \phi \) then (MMP) and (MMD) reduce to the second-order symmetric models of Hou and Yang [12].
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REFERENCE