Parallel Possibility Results of Preference Aggregation and Strategy-proofness by using Prolog

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Abstract: Classical social choice theory provides axiomatic modeling for collective decision making in multi-agent situations as functions of a set of profiles (i.e., tuples of transitive orderings). The celebrated Arrow's impossibility theorem (for unanimity-and-independence-obeying preference aggregation) and the Gibbard–Satterthwaite theorem (for strategy-proof voting procedures) assume the unrestricted domain as well as the transitivity of orderings. This paper presents a distribution map of all Arrow-type aggregation rules without the unrestricted domain axiom for the two-individual three-alternative case in parallel with non-imposed strategy-proof voting procedures by using a Prolog program that systematically removes profiles in the super-Arrovian domains.

1 INTRODUCTION

Social choice theory studies axiomatic collective decision making in multi-agent situations by assuming the transitivity of individual preference orderings (i.e., rankings), which is mapped into certain collective decision outcomes.

Kenneth J. Arrow's general impossibility theorem is a classical result regarding the social aggregation of a tuple of individual orderings (i.e., a profile) into the ordering of society as a whole (Arrow, 1951/63). A social welfare function (SWF) is required to satisfy the following five axioms: unrestricted individual orderings (**U**), the transitivity of the ordering of society as a whole (**T**), unanimity, namely the weak Pareto principle (**P**), independence of irrelevant alternatives (**I**), and non-dictatorship (**D**). Arrow proved that any aggregation rule that satisfies the first four axioms should be dictatorial, and therefore it is impossible to satisfy all five axioms. Allan Gibbard and Mark Satterthwaite independently proved that if there are three or more candidates any voting procedure is non-imposed and strategy-proof, namely every candidate has a possibility to win and no individual can manipulate the outcome of a vote by falsely reporting his or her own preference, the procedure should be dictatorial (Gibbard, 1973; Satterthwaite, 1975). These classical results of social choice theory assume an unrestricted domain.

In order to prove new possibility results as well as classical impossibility theorems, this paper adopts a computational step to axiomatically model social choice under restricted domains instead of using pure mathematics. Prolog language is useful to program intelligent processing systems in the AI research and industry. In addition, Prolog uses a basic technology that stems from automated theorem proving (Robinson, 1965) based on predicate logic.

Social choice theory has recently drawn the attention of computer scientists, as it is the foundation of mechanism design for multi-agent systems as well as game theory. Tang and Lin (2008) and Lin and Tang (2009) provided computer-aided proofs of Arrow's impossibility theorem and the Gibbard–Satterthwaite theorem for two individuals and three alternatives (which Lin and Tang called the "base case") by using a SAT solver, and they proved the general case through mathematical induction. Without exposure to the source code, Tang and Lin also insisted that the base case could be proven by using Prolog. Independent of their work, Indo (2007) introduced a complete Prolog program that proves Arrow's theorem and Wilson's theorem for the "base case" in linear ordering. Indo (2009) also extended this approach to

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several classical results of social choice theory including the Gibbard–Satterthwaite theorem and domain restrictions. In order to generate a possible domain and a social choice rule, Indo (2010) developed a Prolog program that implemented the systematic removal of a pair of profiles in the super-Arrovian domains.

In this paper, a generalized version of the profile elimination that removes arbitrary subsets in the union of two super-Arrovian domains is applied to prove the comprehensive distribution of Arrow-type preference aggregation rules without the unrestricted domain axiom for the base case, in parallel to non-dictatorial strategy-proof social choice functions (SCFs).

The remainder of this paper is organized as follows. Section 2 describes the two classical impossibility results and the domain conditions in order to escape from impossibility. Section 3 introduces the alternative domain condition by eliminating profiles in the paired super-Arrovian domains. Section 4 explains a computational version of social choice theory and the profile elimination implemented in Prolog. Section 5 summarizes the experimental results (i.e., automated proofs) of the possibility of Arrow-type aggregation parallel to non-imposed strategy-proof voting. Section 6 concludes.

2 CLASSICAL RESULTS

2.1 Preference Model

Given a set of individuals, $N = \{1, 2, \ldots, n\}$, and a set of a finite number of alternatives, $A = \{x, y, z, \ldots\}$. A is called *agenda*. A (weak) preference ordering or *ranking* is defined as a complete and transitive binary relation on A. R is complete if for all x and y, either xRy, yRx, or both. R is transitive if for all x, y, and z, if $x\ R\ y$ and yRz, then xRz. A binary relation $\{xRy, yRz, zRx\}$ is intransitive.

The indifference relation w.r.t. a relation R, which means xRy and yRx, is denoted by x*I*y. R is anti-symmetric if for all x and y, if xIy, then $x = y$. P stands for the strict part of R, namely $x\ P\ y$ if xRy and *not* yRx. A preference ordering R is called a linear ordering if $R = P(R)$, i.e., R is strict.

Throughout this paper, we assume linear ordering. In this case, we can consider any ordering as a permutation of agenda A. Let *profile* $R_N = (R_1, R_2, \ldots, R_n)$ be a combination of all individuals' orderings. The set of all possible profiles U is called the *unrestricted domain* (or universal domain). Moreover, we assume that there exists a ranking of society as a whole, R_S, as well as individual rankings.

2.2 Preference Aggregation

Definition. An SWF is a function defined on a subset of profiles $D \subseteq U$ to the set of social rankings. For any profile $R_N \in D$, we say R_N is *permissible*. The ranking of society, $R_S = f(R_N)$, aggregated by an SWF f should satisfy the following five conditions:

(**U**) The SWF is defined for every possible profile (unrestricted domain).

(**T**) The social ranking R_S should be transitive.

(**P**) For any pair of alternatives, x and y, if xR_iy for every individual i, then xR_Sy (unanimity).

(**I**) For any pair of alternatives, x and y, if every individual has the same ranking regarding this pair in two profiles R_N and R_N', then xR_Sy if $xR_S'y$ (the so-called independence of irrelevant alternatives).

An individual i is called a dictator if for any pair of alternatives, x and y, if xR_iy, then xR_Sy.

(**D**) There is no dictator (i.e., non-dictatorship).

An SWF is termed resolute if the social ordering is linear for every profile. We abuse the notion of SWF when its domain is restricted to a subset of profiles, thereby dropping the conditions **U** and **D**.

Theorem (Arrow's Impossibility Theorem). If there are one or more individuals and more than two alternatives, then any SWF that satisfies **U**, **T**, **P**, and **I** is dictatorial.

2.3 Voting Procedure

Definition. A (resolute) SCF is a function that selects a single alternative from each non-empty subset of alternatives (i.e., the agenda) for every permissible profile.

An SCF is manipulable if an individual can report a false ordering to establish a more preferable outcome for herself/himself:

(**A**) An SCF is defined for every profile and the agenda is restricted to A.

(**S**) An SCF is not manipulable (strategy-proofness or non-manipulability).

(**C**) There is no alternative x as x is never selected (as a single winner) for any profile or agenda (non-imposition or a citizen's sovereignty).

A single individual whose top-ranked alternative is always selected as a winner is called a dictator. An SCF is dictatorial if there is a dictator.

Theorem (The Gibbard–Satterthwaite Theorem). If there are one or more individuals and more than two alternatives, then any (resolute) SCF that satisfies **A**,

S, and **C** is dictatorial.

2.4 Restricted Domain

Restricting permissible orderings for each individual may help society escape from impossibilities. Many classical domain conditions are known for a pairwise-majority vote. Kalai and Muller (1977) proposed decomposability into a relation that is closed under a decisiveness implication as a necessary and sufficient condition for the existence of non-dictatorial SWFs, paralleling strategy-proof voting procedures, assuming that every individual has the same possible orderings (i.e., a common admissible domain). Blair and Muller (1983) modified this for an individual's possible set of orderings. Kalai, Muller, and Satterthwaite (1979) extended decomposability to economic environments (i.e., saturated domains). Arrow proposed the "free triple" condition, while Kelly (1994) and Ozdemir and Sanver (2007) elaborated on this condition.

All the above literature argues that mathematical conditions that restrict the domain of social choice rule at most the possible set of orderings for each individual. We depart from classical mathematical approaches to social choice in two aspects. First, this paper proposes finer-grained conditions that restrict *profiles* as the permissible inputs of collective decision making (see Section 5) and, second, to do so we adopt computational proofs instead of standard mathematical proofs, as we demonstrate in the following sections.

3 ELIMINATION OF PROFILES FROM THE SUPER-ARROVIAN DOMAINS

This section introduces an alternative way in which to find domains to avoid impossibility. By using the backtrack mechanism, we can find all the possible aggregation rules, at least in principle. Note that even for two-individual three-alternative cases, a naive backtrack is not computationally efficient for generating all those functions. The cumulative constraint in the recursive program plays a crucial role in the negative proof for unrestricted domains. However, this is not enough for analyzing restricted domains. This section therefore introduces a profile elimination procedure for removing a set of profiles in order to generate versions of the SWF in restricted domains more efficiently.

3.1 Super-Arrovian Domain

Let agenda A = {a, b, c}. If we take a subset of profiles instead of the unrestricted domain, an aggregation rule may satisfy all the axioms except for **U**. Indeed, we can deliberately select a set of profiles to be eliminated in order to escape from the Arrow-type impossibility. These profiles have been termed the super-Arrovian domain in the literature (Fishburn and Kelly, 1997):

P_1. (a P_4 c P_4 b, c P_4 b P_4 a) = ((a, c, b), (c, b, a)),
P_2. (a P_5 b P_5 c, c P_5 a P_5 b) = ((a, b, c), (c, a, b)),
P_3. (b P_6 a P_6 c, a P_6 c P_6 b) = ((b, a, c), (a, c, b)),
P_4. (b P_1 c P_1 a, a P_1 b P_1 c) = ((b, c, a), (a, b, c)),
P_5. (c P_2 b P_2 a, b P_2 a P_2 c) = ((c, b, a), (b, a, c)),
P_6. (c P_3 a P_3 b, b P_3 c P_3 a) = ((c, a, b), (b, c, a)).

These six profiles propagate the decisiveness of any subgroup for a pair of alternatives over all the possible pairs of alternatives, and they are minimal and sufficient for deducing a dictatorship under Arrow's axioms without **U**. There are also another six profiles where Q_k is (r_2, r_1), which corresponds to $P_k = (r_1, r_2)$, $k = 1, \ldots, 6$.

3.2 Profile Elimination

The profile elimination procedure implemented in Prolog used first by Indo (2010) can provide domains and rules finer than those conditions introduced in Section 2.3. The next section explains the computational proof for the impossibility results and its modification to implement the profile elimination.

In order to generate a possible domain and a social choice rule more effectively, it is beneficial to eliminate the profiles from M = $\{P_1, \ldots, P_6\} \cup \{Q_1, \ldots, Q_6\}$. Indeed, Indo (2010) demonstrated that after the removal of (P_k, Q_j) such that $|(k - j) \bmod 6| \leq 1$, the remaining domain that consists of 34 profiles has 18 Arrovian aggregation rules that can be reduced to essentially six unanimous and constant rules (this result can be verified by using `test1`/0 in the Prolog program). Those rules are maximally robust in the sense of Dasgupta and Maskin (2008).

4 COMPUTATIONAL APPROACH

This section introduces the Prolog application for the computational version of axiomatic social choice theory. Because the basic technique adopted in this paper is essentially the same as that used in previous studies, we omit the detail. The source code is available from http://www.xkindo.net/sclp/pl/icaart2014.pl.

The program has been tested for version 6.4.1 running 64 bit Windows 7 (and it may run for any PC that has installed SWI-Prolog version 5.6.52 or later).

4.1 Social Choice Theory in Logic Programming

A Prolog program consists of clauses. A generic discrete functional form `f( Function, Domain, Axiom )` is represented as follows:

```
f([ ], [ ], _).
f([ X - Y | F ], [ X | D ], Axiom):-
  f(F, D, Axiom),
  Goal =.. [ Axiom, X, Y, F],
  Goal.
```

The first clause is the null function defined on an empty domain. The second clause assigns recursively a logically possible value without violating any axiom given by the modeler under the values `F` that have already been assigned to the subdomain `D`. Note that, abusing the notation, we write "`X-Y`" to indicate an assignment of a value `Y` in the region to a value `X` in the domain. Predicate `=../2` in the second clause stands for a "term to list conversion."

The axioms of SWFs and SCFs can be written as follows:

```
swf_axiom( X, Y, F):- rc( _, Y),
  pareto( X - Y), iia( X - Y, F).
scf_axiom( X, Y, F):- x( Y),
  \+ manipulable( _, X - Y, F).
```

The SWFs and SCFs defined on some domain *D* can be written as follows:

```
swf( F, D):- f( F, D, swf_axiom),
  \+ dictatorial_swf( _, F).
scf( F, D):- f( F, D, scf_axiom),
  non_imposed(F),
  \+ dictatorial_scf(_,F).
```

4.2 Profile Elimination

For the sake of later use, the rankings represented by a predicate `rc`/2 are as follows:

```
rc( 1, [a, c, b]).
rc( 2, [a, b, c]).
rc( 3, [b, a, c]).
rc( 4, [b, c, a]).
rc( 5, [c, b, a]).
rc( 6, [c, a, b]).
```

The possible profiles (`pp`/1) and unrestricted domain (`all_pp`/1) are written as follows:

```
pp( [P, Q]):- rc( _, P), rc(_, Q).
all_pp(U):- findall( O, pp(O), U)..
```

Note that these six rankings should be numbered in the specified sequence (modulo 6). The super-Arrovian domain P_k and Q_j (k, j = 1, …, 6) described in Section 3.1 can be generated by pairing (and by exchanging) indices such that (k, j) = (1, 5), (2, 6), (3, 1), (4, 2), (5, 3), and (6, 4).

The simple recursive program described in the previous section is also useful for finding a way in which to escape from the impossibility results. A candidate domain consists of the remaining profiles in the domain after a subset C $\subseteq$ M has been removed from the unrestricted domain U, where D = U − C. In the Prolog program, `select_n`/3 (user-defined) for generating subsets and `subtract`/3 (builtin) for deleting list elements are used.

5 POSSIBILITY RESULTS

Tables 1–3 summarize the experimental results of the profile elimination (these results are reproducible by the automated proofs `test2`/0 and `test3`/0 in the Prolog program).

5.1 Possible SWFs

The top row in Table 1 (labeled 2) indicates the cases that no Arrow-type aggregation rule exists except for dictatorships. In particular, Cell (2, 12) corresponds to the case of Arrow's impossibility theorem. Moreover, Cell (2, 6) valued 2 implies that even when one super-Arrovian domain has been completely eliminated, the other remaining super-Arrovian domain is sufficient to prove a dictatorship. Further, Cell (3, 10) and Cell (20, 0) both indicate the 18 rules mentioned in Section 3.2. Note that there are 4096 domains in total. The bottom row coincides with binomial coefficients $_{12}C_{column}$.

Interestingly, Cell (k, 6) (k >2) suggests that any exchange between P and Q can reverse impossibility if the status quo domain is a complete removal of one of the super-Arrovian domains. We term this the *exchange property*.

5.2 Possible SCFs

With regard to Table 2, the total number of domains is the same as shown in Table 1. Similar to the SWF cases in Table 1, the top row labeled 2 also indicates the impossibility results that no non-imposed

Table 1: Arrow-type preference aggregation rules (SWFs) generated by profile elimination in super-Arrovian domains. The rows represent the counts of SWFs that include two dictatorships, the columns represent the remaining numbers of the super-Arrovian profiles, and the cells contain the counts of restricted domains on which an SWF exists.

swf	0	1	2	3	4	5	6	7	8	9	10	11	12	total
2							2	12	48	76	48	12	1	199
3								60	156	108	18			342
4							54	228	225	36				543
5						12	170	348	60					590
6						60	390	120	6					576
7						228	252	24						504
8					48	348	50							446
9					156	120	6							282
10					225	24								249
11				76	60									136
12				108	6									114
13				36										36
14			48											48
15			18											18
17		12												12
20	1													1
total	1	12	66	220	495	792	924	792	495	220	66	12	1	4096

Table 2: Non-imposed strategy-proof voting procedures (SCFs) generated by profile elimination in super-Arrovian domains. The rows represent the counts of SCFs that include two dictatorships, the columns represent the remaining numbers of the super-Arrovian profiles, and the cells contain the counts of restricted domains on which an SCF exists.

sp	0	1	2	3	4	5	6	7	8	9	10	11	12	total
2							2	12	30	64	48	12	1	169
3									114	120	18			252
4								144	255	36				435
5							62	300	90					452
6						12	150	252	6					420
7							294	72						366
8						120	242	12						374
9						132	78							210
10					18	192	72							282
11					36	48								84
12					57	108	18							183
13					30	48	6							84
14				4	36	72								112
15					36	12								48
16					69	24								93
17				12	36									48
18					72									72
19					12	24								36
20				36	12									48
21				12										12
22				36	36									72
23				12										12
25					30									30
26				12										12
28				24	3									27
29					6									6
30					6									6
31				24										24
34				12										12
35				12										12
37			12											12
38			6	12										18
40			6	12										18
41			12											12
46			6											6
48			12											12
50			6											6
74			6											6
88		12												12
196	1													1
total	1	12	66	220	495	792	924	792	495	220	66	12	1	4096

strategy-proof voting procedure exists except for the dictatorship of each person. Cell (2, 12) valued 1 corresponds to the case of the Gibbard–Satterthwaite theorem. Similar to Table 1, Cell (2, 6) valued 2 suggests that the super-Arrovian domain is sufficient to prove a dictatorship. Cell (*k*, 6) also satisfies the exchange property. Cell (3, 10) valued 18 corresponds to the 18 maximal domains.

Table 3: Parallel possibility results of SWFs and SCFs. The rows represent the numbers of SCFs that include two dictatorships, the columns represent the numbers of SWFs, and the cells contain the counts of restricted domains.

	swf																
scf	2	3	4	5	6	7	8	9	10	11	12	13	14	15	17	20	total
2	169																169
3	24	228															252
4	6	84	345														435
5		6	144	302													452
6		24	24	168	204												420
7				36	192	138											366
8			24	36	78	168	68										374
9				12	12	60	96	30									210
10				36	36	18	120	60	12								282
11						24	12	24	24								84
12					30	36	30	24	48	12	3						183
13			6			24	24		12	18							84
14						24	48		30	10							112
15							12	24		12							48
16					12		18	48	12		3						93
17								12	24			12					48
18									60	12							72
19					12	12			12								36
20								6	6	24	12						48
21											12						12
22							12	24			24	12					72
23												12					12
25								24	6								30
26											12						12
28									3	24							27
29								6									6
30							6										6
31											24						24
34											12						12
35											12						12
37													12				12
38										12				6			18
40										12			6				18
41														12			12
46													6				6
48													12				12
50													6				6
74													6				6
88															12		12
196																1	1
total	199	342	543	590	576	504	446	282	249	136	114	36	48	18	12	1	4096

5.3 Parallel Possibility Results

In Table 3, the two distributions seem to be positively correlated, but the precise interrelation is unclear. The top-left corner (2, 2) has a value of 169, which indicates that the parallel impossibility results have occurred in the 169 domains. Cell (3, 2) implies that 24 domains have a strategy-proof voting procedure but no non-dictatorial Arrow-type preference aggregation. In Cell (196, 20), the two super-Arrovian domains have been eliminated; the remaining 24 profiles deduce 20 SWFs and 196 SCFs.

6 CONCLUSIONS

This paper presented a complete distribution map of all Arrow-type aggregation rules without the unrestricted domain axiom for the two-individual three-alternative linear ordering case in parallel with non-imposed strategy-proof voting procedures by using a Prolog program that systematically removes the arbitral subset of the super-Arrovian domains.

We can summarize the presented observations into the following three parallel possibility results.

Result 1. *The impossibility result no longer occurs if more than half of the 12 profiles have been* *eliminated (see the top row in Table 1 and Table 2: Cell (2, j) = 0 if j < 6).*

In addition, the exchange property described in the previous section is satisfied.

Result 2. *The possibility may occur if more than two of the 12 profiles are eliminated appropriately (see the second and subsequent rows in Table 1 and Table 2: If Cell (k, j) > 0 & k > 2, then j>2).*

The minimal number of eliminations sufficient to deduce a possibility is two. Indeed, these are the 18 profile pairs (see Cell (3, 10) in Table 1 and Table 2 as well as `test1` in the author's Prolog program).

Result 3. *(i) There are 169 domains where Arrow-type aggregation (SWF) and non-dictatorial non-imposed strategy-proof voting (SCF) are both empty (i.e., Cell (2, 2) = 169 in Table 3). (ii) There are also 30 domains where SCF exists but SWF is empty (i.e., Cell (3, 2) = Cell (4, 2) = 0 in Table 3). (iii) There is no domain where SWF exists but SCF is empty (i.e., Cell (2, j) = 0 if j > 2 in Table 3). (iv) In the other domains, SWF and SCF are both non-empty.*

Additionally, if we substitute Maskin monotonicity and unanimity for strategy-proofness and non-imposition, then Table 2 is the same as shown in Table 1 (see `test4`). Muller and Satterthwaite (1977) proved the equivalence for the unrestricted domain. Lastly, the *n*-person and *m*-alternative case possibility result can be proven by assuming that *n* – 2 individuals are dummy and everyone is indifferent for m – 3 alternatives, but further study in this regard is needed.

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