Mode Scheduling under Dwell Time Constraints in Switched-Mode Systems

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Abstract—This paper presents an algorithmic technique for optimal mode scheduling in switched-mode dynamical systems under the constraint of a minimum dwell time. The technique is based on an optimal control formulation of the problem and on narrowing the optimality gap associated with the maximum principle. It consists of the following four successive steps: (i) solving the relaxed optimal control problem; (ii) approximating the relaxed solution by a switched-mode control; (iii) modifying the switched-mode schedule to incorporate the dwell-time constraints; and (iv) improving on the solution obtained in (iii) by further modifying the switching times of the modes. The main innovations of the paper are in the incorporation of the dwell-time constraints in step (iii), and in combining all four steps in a coherent framework. Despite the growing literature on switched-mode optimal control there hardly have emerged general-purpose algorithms which address dwell-time constraints. This paper presents an initial study of our proposed approach and demonstrates its efficacy via simulation examples while deferring theoretical studies to future research.

I. INTRODUCTION

Autonomous switched-mode dynamical systems typically have the form

\[ \dot{x} = f(x, v) \]  

where \( x := x(t) \in \mathbb{R}^n \) is the state of the system, \( v := v(t) \in V \) where \( V \) is a given finite set, and \( f : \mathbb{R}^n \times V \to \mathbb{R}^n \) is the dynamic response function of the system. This paper considers the optimal control of such systems where it is desirable to minimize a cost-performance functional \( J \) having the form

\[ J = \int_0^T L(x) \, dt, \]  

with \( L : \mathbb{R}^n \to \mathbb{R} \) being a cost function of the state variable. The initial state \( x_0 := x(0) \in \mathbb{R}^n \) and the final time \( T > 0 \) are given and fixed. We point out that every point \( v \in V \) represents a mode of the system associated with the dynamic response function, namely \( f(\cdot, v) \), and we will refer to \( v \in V \) as the system’s mode. In this setting, a control function \( v(t), t \in [0, T] \) is a time-varying signal of modes and can be viewed as the system’s mode schedule.

The aforementioned optimal control problem can then be understood as the problem of determining the mode schedule which minimizes \( J \).

As a matter of notation, we will refer to \( v \in V \) as a point in \( V \), while a control signal \( v(t) : [0, T] \to V \) will be denoted by \( v \). Thus, \( \mathbf{v} = \{v(t) : t \in [0, T]\} \), and \( v(t) \in V \forall t \in [0, T] \).

Such optimal control problems have arisen in various application areas such as robotics, power electronics, communications, and automotive powertrain control; see [20] for a survey. Consequently the development of computational techniques for solving these problems has been the focus of research in the past twenty years. The resultant approaches include first- and second-order techniques [19], [14], zoning algorithms based on the geometric properties of the underlying systems [8], [15], relaxed control techniques [6], [9], [16], and methods based on needle variations [3], [5], [11], [17]. A comprehensive recent survey can be found in [20].

This paper is different from the published literature on optimal mode scheduling in that it incorporates constraints on minimum mode dwell times. These constraints specify a minimum length for each interval where a single mode is active. A given mode schedule, or alternatively a control \( \mathbf{v} \), can be described via its sequence of modes \( \{v_i\} \), \( i = 1, \ldots, N + 1 \) for some \( N \geq 1 \), and the switching times between them. Let us denote the switching time between the mode \( v_i \) and mode \( v_{i+1} \) by \( \tau_i \), and define the switching-times vector by \( \tau = (\tau_1, \ldots, \tau_N) \in \mathbb{R}^N \). We further define \( \tau_0 := 0 \) and \( \tau_{N+1} := T \) for notational convenience. In this case we say that the control \( \mathbf{v} \) (and its associated schedule) has \( N + 1 \) modes and \( N \) switching times, and we note that a particular mode may appear multiple times in the sequence \( \{v_i\} \), i.e., we allow \( v_i = v_j \) for \( i \neq j \). We say that a control \( \mathbf{v} \) is admissible if \( N \) is finite. The dwell time constraint is expressed via the inequality

\[ \tau_i - \tau_{i-1} \geq \delta, \quad i = 1, \ldots, N + 1, \]  

for a given \( \delta > 0 \) referred to as the minimum dwell time. The problem that we consider is to minimize \( J \) defined in Equation (2) subject to the dynamics in Equation (1) and the minimum dwell time constraint defined in (3).

The computational technique that we propose is comprised of four steps, as described in the abstract. It is underscored by the idea of reducing the optimality gap, related to the maximum principle, at each step. Therefore, although step (i) can be accomplished by any technique for solving the relaxed optimal control problem, we have used a highly efficient algorithm that directly minimizes the Hamiltonian at each step [12]. In Step (ii) we use pulse-width modulation to approximate the solution point of (i) by a switched mode schedule. Solving the relaxed problem and then using PWM to approximate it has proven more computationally efficient than finding the optimal switched schedule directly. Step (iii)
applies a similar idea to (i) to compute a mode schedule compliant with the minimum dwell-time constraint, and Step (iv) also applies a variant of the same idea to optimize on just the switching times. The remainder of the paper is organized as follows. Section II describes the algorithmic procedure, Section III presents simulation examples, and Section IV concludes the paper.

II. ALGORITHM FOR THE OPTIMAL SCHEDULING PROBLEM

Consider a switched control signal \( v \) and a time \( s \in [0, T) \) that is not a switching time of \( v \). Let \( \xi \) be a positive number such that the mode of \( v \) is constant in the interval \([s, s + \xi]\). Now consider replacing the control \( v(t) \) by a mode \( w \in V \) at every \( t \in [s, s + \xi] \), and denote by \( \tilde{J} (\xi) \) the cost functional \( J \) resulting from this swap as a function of \( \xi \). The one-sided derivative \( \frac{d\tilde{J}}{d\xi}(0) \) is called the needle variation [15]. Since taking the derivative in this manner amounts to examining the change in cost when inserting the mode \( w \) at time \( s \), we call this derivative the insertion gradient, and we denote it by \( D_{\nu, s, w} \). Then (see, e.g., [10]) the insertion gradient has the following form,

\[
D_{\nu, s, w} = p(s)^T \left( f(x(s), w) - f(x(s), v(s)) \right),
\]

where \( p(\cdot) \in R^n \) is the costate (adjoint) variable of the system, defined via the equation

\[
\dot{p} = - \left( \frac{\partial f}{\partial x}(x, v) \right)^T p - \left( \frac{dL}{dx}(x) \right)^T
\]

with the boundary condition \( p(T) = 0 \).

The Hamiltonian of this system with the control \( v \) is

\[
H(x, v, s) = p(s)^T f(x(s), v(s)) + L(x(s)).
\]

The maximum principle is satisfied if, for every \( s \in [0, T] \), \( v(s) \) minimizes the Hamiltonian \( H(x(s), w, p(s)) \) over all \( w \in V \). If \( v \) is not a local minimum we can use the insertion gradient to quantify the extent to which the maximum principle fails to be satisfied. To this end, define, for every \( s \in [0, T] \),

\[
D_{\nu, s} = \min_{w \in V} D_{\nu, s, w},
\]

and define

\[
\theta(v) = \int_0^T D_{\nu, s} ds.
\]

Observe that, by (4), \( D_{\nu, s, v(s)} = 0 \) and hence, and by (6) and (7), \( D_{\nu, s} \leq 0 \) and \( \theta(v) \leq 0 \) as well. Furthermore, \( \theta(v) = 0 \) if and only if the control \( v \) satisfies the maximum principle at all \( s \in [0, T] \) (potentially except in, at most, a set of Lebesgue measure zero). More generally, the term \( \theta(v) \) qualifies as an \textit{optimality function}, namely a measure of the extent to which the control \( v \) does not satisfy the maximum principle. Reference [13] developed a theory of optimization over infinite-dimensional spaces (including optimal control problems) in which the concept of optimality functions plays a central role. Accordingly, convergence of algorithms is characterized by the requirement that \( \lim_{k \to \infty} \theta(v_k) = 0 \), where \( \{v_k\}_{k=1}^\infty \) is a sequence of controls an algorithm would compute. The development in the sequel follows this principle. These four steps of the proposed algorithm are now described in detail.

A. The relaxed problem

The theory of relaxed controls was developed to ensure the existence of solutions to optimal control problems via convexification of the vector field defined by the system’s dynamic response function [18]. In the case of switched-mode problems it captures, in a natural way, situations where the infimum of \( J \) (defined in (2)) is obtained at a sliding-mode control, which is not a switched-mode control per se.

In the setting of our optimal control problem defined by (1) and (2), the relaxed control problem has the following form. Suppose that the set \( V \) has \( m \) elements, \( V = \{v_j : j = 1, \ldots, m\} \), and define the set \( M \subset R^m \) as

\[
M = \{\mu = (\mu_1, \ldots, \mu_m)^T \in R^m\}
\]

such that

\[
\sum_{j=1}^m \mu_j \geq 0 \forall j \in \{1, \ldots, m\} \text{ and } \sum_{j=1}^m \mu_j = 1
\]

For every \( \mu \in M \), define \( v_\mu := \sum_{j=1}^m \mu_j v_j \). Observe that \( v_\mu \) is a convex combination of the elements of \( V \), and \( v_\mu \in V \) if and only if \( \mu = e_j \), namely the \( j \)-th element of the canonical basis for \( R^m \). \( \mu \in M \) is called a relaxed control variable. A relaxed control signal is a measurable function \( \mu(t) : [0, T] \to M \), henceforth referred to as a relaxed control and denoted by \( \mu : (\mu_1, \ldots, \mu_m)^T \). Note that the components of \( \mu \) satisfy Equation (9) at each point in time. We say that a relaxed control is \textit{admissible} if \( \mu(\cdot) \) has a finite number of discontinuities, has bounded variation on each interval where it is continuous, and is left-continuous for all \( t \in [0, T] \). We henceforth implicitly assume that every mentioned relaxed control is admissible.

In analogy with Equation (1), the dynamics of the relaxed system are defined by

\[
\dot{x} = \sum_{j=1}^m \mu_j f(x, v_j), \quad x(0) = x_0,
\]

where \( \mu = \mu(t) = (\mu_1(t), \ldots, \mu_m(t))^T \), \( t \in [0, T] \). The relaxed optimal control problem is to minimize the functional \( J \), defined in (2), as a function of \( \mu \).

Given a relaxed control \( \mu \), the costate (adjoint) variable for the relaxed problem is defined by the equation

\[
\dot{p} = - \sum_{j=1}^m \mu_j \left( \frac{\partial f}{\partial x}(x, v_j) \right)^T p - \left( \frac{dL}{dx}(x) \right)^T
\]

with the boundary condition \( p(T) = 0 \), and the Hamiltonian has the form

\[
H(x, \mu, p) = \sum_{j=1}^m \mu_j p^T f(x, v_j) + L(x).
\]

Let \( x \) and \( p \) be the state and costate variables associated with the relaxed control \( \mu \) via Equations (10) and (11), and let \( v := \nu(t), t \in [0, T] \) be a relaxed control that minimizes the Hamiltonian \( H(x, \cdot, p) \) over \( M \) at every \( t \in [0, T] \).

By Equation (12), \( \nu(t) \) is always equal to \( e_j \) for some \( j \in \{1, \ldots, m\} \), and hence \( \nu_j(t) = v_j \) at each point in time.
Thus, for every \( t \in [0, T] \), the Hamiltonian can be minimized by comparing its value when \( v(t) \) is equal to each \( v_j \), which amounts to comparing \( m \) numbers. Obviously the maximum principle for the relaxed problem requires that \( \mu(t) = v(t) \) for all \( t \in [0, T] \), and in general, the term

\[
\theta(\mu) := \int_0^T (H(x, \nu, p) - H(x, \mu, p)) dt
\]

(13)
is a natural choice for an optimality function. We note here that if \( \mu(t) = e_j(t) \) for some \( j(\cdot) = 1, \ldots, m \) then the relaxed control is associated with a switched-mode control, and \( \theta(\mu) = \theta(v_\mu) \). This means that the optimality function in Equation (13) extends to the space of relaxed controls the optimality function for switched controls defined in Equation (7).

The algorithm that we use to solve the relaxed control problem moves from a given relaxed control \( \mu \) in the direction \( \nu - \mu \). This means that, for a properly chosen step size \( \lambda \in [0, 1] \), the relaxed control obtained from \( \mu \), denoted by \( \mu_{\text{next}} \), is

\[
\mu_{\text{next}} = \mu + \lambda(\nu - \mu).
\]

(14)

Considering \( J \) as a function of \( \lambda \), variational arguments yield that

\[
\frac{dJ}{d\lambda} (\mu + \lambda(\nu - \mu)) \bigg|_{\lambda=0} = \theta(\mu),
\]

(15)

and therefore, \( \nu - \mu \) is a descent direction unless \( \mu \) is a local optimum.

Observe that the descent direction in (14) and the optimality function in (13) do not correspond to the steepest descent, but rather to a form of gradient projection. We found this direction advantageous for various reasons including the ease of its computation; see [14] for further discussion.

The step size that we use in our algorithm follows the Armijo procedure [2], [13], and it has the following form: given constants \( \alpha \in (0, 1) \) and \( \beta \in (0, 1) \), the step size \( \lambda \) in (14) is defined via

\[
\lambda := \max \{ \beta^k, k = 0, 1, \ldots : J(\mu + \beta^k(\nu - \mu)) - J(\mu) \leq \alpha \beta^k \theta(\mu) \}.
\]

(16)

In summary, the algorithm for solving the relaxed optimal control problem has the following form.

**Algorithm 1:**

Given a relaxed control \( \mu \), compute the next relaxed control, \( \mu_{\text{next}} \), as follows.

**Step 1:** Compute \( x(\cdot) \) and \( p(\cdot) \) according to Equations (10) and (11), respectively.

**Step 2:** For every \( t \in [0, T] \), compute

\[
\hat{j}_\mu(t) := \arg \min_{j \in \{1, \ldots, m\}} \{H(x(t), v_j, p(t))\}.
\]

(17)

Set \( \nu(t) = e_{\hat{j}_\mu(t)}(t) \)

**Step 3:** Compute \( \theta(\mu) \) by Equation (13).

**Step 4:** Compute

\[
k_\mu := \min \{ k = 0, 1, \ldots : J(\mu + \beta^k(\nu - \mu)) - J(\mu) \leq \alpha \beta^k \theta(\mu) \}.
\]

(18)

Set \( \lambda_\mu := \beta^{k_\mu} \).

**Step 5:** Set \( \mu_{\text{next}} = \mu + \lambda_\mu(\nu - \mu) \).

Now a recursive algorithm would compute a sequence \( \mu_k, k = 1, 2, \ldots \), such that \( \mu_{k+1} = \mu_{k, \text{next}} \).

Algorithms comprised of gradient-descent directions with the Armijo step size have been analyzed extensively in [13], and are shown to be globally stable and convergent to local minima. Furthermore, they often converge quite rapidly from an initial starting point (relaxed control, in our case) towards a local minimum. They have been applied in [12] to a particular optimal control problem with notable success.

**B. Computing a switched-mode control from a solution of the relaxed problem**

Consider two relaxed controls, \( \mu_1 \) and \( \mu_2 \), and let \( x_1 \) and \( x_2 \) denote their respective state trajectories defined by (10). If \( \sup \{|x_2(t) - x_1(t)| : t \in [0, T]\} = 0 \) is “small” then (see (2)) \( \|J(\mu_2) - J(\mu_1)\| \) would be small as well, and this justifies the use of the term \( \|x_2 - x_1\|_{L^\infty} \) as a measure of proximity of \( \mu_2 \) to \( \mu_1 \).

The chattering lemma [7] guarantees that every admissible relaxed control can be approximated by a switched-mode control to within any degree of precision in the above sense, namely the respective state trajectories of the two controls can be made arbitrarily small in the \( \|x\|_{L^\infty} \) norm. A natural way to do this approximation is via the following pulse-width modulation (PWM) procedure. Consider an admissible relaxed control \( \mu \). At every time \( t \in [0, T] \), \( \mu(t) = (\mu_1, \ldots, \mu_m)^T \) corresponds to a convex combination of the elements of \( V \) via \( v_\mu(t) = \sum_{j=1}^m \mu_j v_j \). Let us partition the interval \([0, T]\) into \( N \) consecutive subintervals (cycles) denoted by \( C_\ell, \ell = 1, \ldots, N \). Of course it is reasonable for all but perhaps the last of them to have the same length, but this not necessary. In the forthcoming we use the notation \(|I|\) for the length of an interval \( I \). Algorithm 2, below, formalizes the process of approximating a relaxed control, \( \mu \), by a switched control, \( v \).
Algorithm 2:
For every \( \ell = 1, \ldots, N \), do:

**Step 1:** Compute
\[
\xi_\ell = \frac{1}{|C_\ell|} \int_{C_\ell} \mu(t) dt \tag{19}
\]

note that \( \xi_\ell \in \mathbb{R}^m \) and denote its \( j \)th coordinate by \( \xi_{\ell,j} \).

**Step 2:** Divide (partition) each \( C_\ell \) into \( m \) intervals, \( I_{\ell,j}, j = 1, \ldots, m \), such that \( |I_{\ell,j}| = |C_\ell|/m \).

**Step 3:** Set \( v(t) = v_j \forall t \in I_{\ell,j} \).

It is evident that the switched-mode control \( v \) computed by Algorithm 2 can approximate \( \mu \) to within any degree of precision if \( \max \{|C_\ell| : \ell = 1, \ldots, N\} \) is small enough.\(^1\)

#### C. Incorporating dwell-time constraints

This subsection modifies a switched-mode schedule so as to make it comply with minimum dwell-time constraints. The imposition of such constraints reduces the feasible set of admissible schedules and hence modifies, potentially drastically, the algorithmic approaches to the optimization problem. To our knowledge, no systematic techniques have been proposed for handling dwell-time constraints. In this section we present an approach to this problem.

Let a fixed \( \delta > 0 \) be the lower bound on the dwell times. Given a switched-mode control \( v \), the algorithm presented below has the following structure: scanning from time \( t = 0 \) onwards, it first searches for a pair of consecutive mode-switching times which are spaced less than \( \delta \) seconds apart, namely times \( \tau_i \) and \( \tau_{i+1} \) such that \( \tau_{i+1} - \tau_i < \delta \). Then it sets \( \tau_{i+1} \) to \( \tau_{i+1} = \tau_i + \delta \). Next, it selects a mode for the interval \( [\tau_i, \tau_{i+1}] \) by computing the element \( w^*_i \in V \) that minimizes the term
\[
A_{i,v} := \int_{\tau_i}^{\tau_{i+1}} D_{v,s,w} ds \tag{20}
\]

from among all \( w \in V \). Notice that this extends the concept behind \( D_{v,s} \) as defined in (6), considering that modes must be inserted only on time-intervals of no less than \( \delta \) seconds. In a way, this choice of \( w^*_i \) can be viewed as an attempt to narrow the optimality gap over such intervals (rather than at each point individually), and this consideration is based on the principle underlying the definition of the optimality function \( \theta(v) \).

The algorithm repeats these steps until all of the modes in a given schedule satisfy the dwell-time constraints. Formally, given a switched-mode control \( v \), it computes the compliant control \( w \) as follows. Suppose that \( v \) has \( N \) switching times, \( \tau_1 < \cdots < \tau_N \), and define \( \tau_0 := 0 \) and \( \tau_{N+1} := T \).

\(^1\)The meaning of the term “approximate” of one relaxed control by another was defined at the start of this subsection to mean a small \( L^\infty \)-norm of the difference between the respective state trajectories of the two relaxed controls.

Algorithm 3:

**Step 0:** Set \( w(t) = v(t) \forall t \in [0, T] \).

**Step 1:** If \( \tau_{i+1} - \tau_i \geq \delta \) for every \( i = 0, \ldots, N \), then stop and exit. The resulting control is \( w \).

**Step 2:** Let \( j := \min \{i : \tau_{j+1} - \tau_j < \delta\} \). If \( \tau_j + \delta < T \) then go to Step 3, and on the other hand, if \( \tau_j + \delta \geq T \), go to Step 4.

**Step 3:** Set \( \tau_{j+1} = \tau_j + \delta \), compute \( w^*_j \) (see (20)), and set \( w(t) = w^*_j \forall t \in [\tau_j, \tau_{j+1}] \). Go to Step 1. 

**Step 4:** Set \( w(t) = w(\tau_j^-) \forall t \in [\tau_j, T] \), and go to Step 1.

It is evident that the control \( w \) produced by Algorithm 3 satisfies the dwell-time constraints.

#### D. Timing optimization under dwell-time constraints

The algorithm presented in the last subsection computes mode-schedules that are compliant with dwell-time constraints, but it does not optimize with respect to their switching times. This subsection proposes an algorithm that further reduces the value of \( J \) by modifying the switching times of \( w \). Thus, given the sequence of modes \( \{v_i\}_{i=1}^{N+1} \) in \( w \), let us consider \( J \) as a function of the switching-times vector \( \bar{\tau} := (\tau_1, \ldots, \tau_N)^\top \in \mathbb{R}^N \). The objective of the algorithm presented below is to minimize \( J(\bar{\tau}) \) subject to the constraints that
\[
\tau_i - \tau_{i+1} + \delta \leq 0, \quad \forall i = 0, \ldots, N, \tag{21}
\]

where, as before, we define \( \tau_0 = 0 \) and \( \tau_{N+1} = T \).

Although this problem can be solved via standard nonlinear programming methods, the special form of its feasible set, as defined by (21), considerably simplifies the way the constraints can be handled. We use this point in a gradient-descent technique where the direction from a given point \( \bar{\tau} \) is the projection of the vector \(-\nabla J(\bar{\tau})\) onto the set of feasible directions from \( \bar{\tau} \). Gradient projections of this kind are further discussed in [5].

The following algorithm reduces \( J \) by moving along the above-mentioned gradient-projection directions and using Armijo step sizes. Observe that it utilizes the fact that the feasible set is a polygon by moving along its boundary if needed, like the simplex method in linear programming.

Given constants \( \alpha \in (0, 1) \) and \( \beta \in (0, 1) \), the algorithm computes, from a point \( \bar{\tau} = (\tau_1, \ldots, \tau_N)^\top \), the next point, \( \bar{\tau}_{next} \) in the following way.

Algorithm 4:

**Step 1:** Compute \( \nabla J(\bar{\tau}) \).

**Step 2:** Compute \( h(\bar{\tau}) = (\tilde{h}_1(\bar{\tau}), \ldots, \tilde{h}_N(\bar{\tau}))^\top \), defined as the projection of \(-\nabla J(\bar{\tau})\) onto the set of feasible directions at \( \bar{\tau} \).

**Step 3:** Compute a set of numbers \( \gamma_j \geq 0, j = 1, \ldots, N \), such that every \( \gamma_j \) is maximal with respect to the condition defined by the following two constraints:

(i)
\[
\tau_{j-1} + \gamma_{j-1} \tilde{h}_{j-1}(\bar{\tau}) - \delta \leq \tau_j + \gamma_j \tilde{h}_j(\bar{\tau}) \leq \tau_{j+1} + \gamma_{j+1} \tilde{h}_{j+1}(\bar{\tau}) + \delta, \tag{22}
\]
and (ii) $\gamma_j \leq 1$. For every $j = 1, \ldots, N$, set $h_j = \gamma_j \tilde{h}_j$.  

Step 4: Compute

$$k_j := \max \{ k = 0, 1, \ldots : J(\bar{\tau} + \beta^k h(\bar{\tau})) - J(\bar{\tau}) \leq -\alpha \beta^k (h(\bar{\tau}), \nabla J(\bar{\tau})) \}. \quad (23)$$

Set $\lambda(\bar{\tau}) = \beta^{k_j}$.

Step 5: Set $\bar{\tau}_{\text{next}} = \bar{\tau} + \lambda(\bar{\tau}) h(\bar{\tau})$.

### III. Simulation Results

In this section we present numerical results demonstrating the effectiveness of the proposed method and each of its intermediate steps. The considered system is comprised of the DC-AC power-converter circuit shown in Figure 1, where the objective is to have the DC voltage source supply an AC current to the resistive load by controlling the state of the switch. The switch in the circuit connects either to the voltage source $V_s$ or to ground, corresponding to $v = 1$ and $v = 0$, respectively, so that $V = \{0, 1\}$. The state of the system is the inductor current, denoted $x(t) := i_L(t)$.

The initial state of the system is $x(0) = 1$. Kirchoff's Voltage Law says that the system evolves according to

$$\dot{x} = -\frac{R}{L} x + \frac{V}{L} v. \quad (24)$$

Here the component values used were $R = 1 \Omega$, $L = 0.5 H$, and $V_s = 2 V$. The objective is to have the inductor current match the reference signal $r(t) = 1 + 0.3 \sin(\pi t)$, by minimizing the performance functional $J$ defined by

$$J = \int_0^T (r(t) - x(t))^2 dt, \quad (25)$$

where the final time is $T = 10$. We point out that this is the same circuit used in [1] as a DC-AC power converter. The proposed optimization method was carried out for this system with numerical integrations being done using Euler's method with a stepsize of $dt = 0.01$.

The computation of such a set $\{\gamma_1, \ldots, \gamma_N\}$, which is not unique, is trivial, and it ensures that $h(\bar{\tau})$ is a feasible direction from $\bar{\tau}$.

### A. Generating the Relaxed Control

Algorithm 1 was run for this system with $\alpha = 0.5$ and $\beta = 0.5$, and was run for 50 iterations. The cost was reduced from $4.784 \times 10^{-3}$ to $8.492 \times 10^{-5}$, with the cost dropping to 0.02368 after just 2 iterations. 50 iterations of the algorithm required just 7.99 seconds of CPU time to run.

The optimal relaxed control generated by the algorithm is plotted in Figure 2. The control signal is not quite the sinusoid we expect. The reason is that there is a considerable insensitivity of $x$ and $J$ to variations in the control, and characterizing convergence of the algorithm in the weak topology, we certainly can claim rapid convergence.

### B. Converting to a Switched Schedule

The output of Algorithm 1 serves as the input to Algorithm 2 in order to convert it to a switched schedule. To do this, we set $C = 0.06$ which gave a cost of 0.00296, a small increase over the cost of the relaxed control. The optimal switched mode schedule is shown in Figure 3. The control signals shown in Figures 2 and 3 are very different, though, in accordance with the chattering lemma, their corresponding state trajectories and therefore costs are quite similar.

### C. Switched Scheduling with Dwell Time Constraints

The switched schedule generated by Algorithm 2 was next fed into Algorithm 3 in order to incorporate dwell time constraints. The dwell time chosen here was $\delta = 0.25$ which, in conjunction with taking $T = 10$, allows a maximum of 40 switches over the system's time horizon. The mode schedule generated by Algorithm 3 is shown in Figure 4. The cost after this step is 0.3387 and while this is a non-negligible increase in cost over the output of Algorithm 2, it is hardly surprising given the restrictive nature of the dwell time constraints being used.

### D. Timing Optimization with Dwell Time Constraints

Algorithm 4 was run on the output of Algorithm 3 with $\alpha = 0.3$ and $\beta = 0.3$. The cost after this step was 0.2433, a noticeable decrease in the cost after the preceding step. The resulting control signal is shown in Figure 5.

### IV. Conclusion

A four-part procedure was presented for optimal mode scheduling in switched-mode systems when a dwell time is imposed. This algorithm solves a relaxed version of the mode scheduling problem, converts the relaxed schedule to a switched schedule, modifies the switched schedule to incorporate dwell time constraints, and then optimizes the timing of the switches using a routine that respects dwell time constraints. Numerical results were provided to demonstrate the effectiveness of the approach.

### References


Fig. 2. The relaxed control generated by Algorithm 1.

Fig. 3. The switched schedule generated by Algorithm 2.

Fig. 4. The switched schedule with dwell time constraints generated by Algorithm 3.

Fig. 5. The switched schedule with dwell time constraints generated by Algorithm 4.


