Adaptive fuzzy tracking control for a class of perturbed strict-feedback nonlinear time-delay systems

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Abstract

This paper is concerned with the problem of adaptive fuzzy output tracking for a class of perturbed strict-feedback nonlinear systems with time delays and unknown virtual control coefficients. Fuzzy logic systems in Mamdani type are used to approximate the unknown nonlinear functions, then the adaptive fuzzy tracking controller is designed by using the backstepping technique and Lyapunov–Krasovskii functionals. The proposed adaptive fuzzy controller guarantees that all the signals in the closed-loop system are bounded and the system output eventually converges to a small neighborhood of the desired reference signal. An advantage of the proposed control scheme lies in that the number of the online adaptive parameters is not more than the order of the original system. Finally, two examples are used to demonstrate the effectiveness of our results proposed in this paper.
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1. Introduction

The early results on adaptive control for nonlinear systems were usually obtained based on the assumption that nonlinearities in systems satisfied matching conditions [13,21]. To control the nonlinear systems with mismatched conditions, backstepping design technique has been developed in [14]. With the assumption that the nonlinear functions contain unknown parameters or are unknown with their upper bounds known, a lot of robust adaptive control results have been reported, for details see [15,5]. So far, backstepping approach has become one of the most popular design methods for a class of nonlinear systems. However, in many real plants, not only the nonlinearities in the systems are unknown but also the prior knowledge of the bounds of these nonlinearities is not available. Therefore, the existing adaptive control approaches cannot be used to control these nonlinear systems. In order to stabilize those nonlinear systems, neural networks and fuzzy logic systems are used to approximate the unknown nonlinear functions in systems, and then Lyapunov stability theory are employed to construct adaptive controllers. Recently, some adaptive neural
control methods and adaptive fuzzy control approaches have been developed for strict-feedback nonlinear systems via backstepping technique, see, for example, [17,27,24,12,11,20,3,22,28,16], and the references therein. A disadvantage of these adaptive control methods is that the proposed controllers contain a lot of adaptive parameters, so that the learning time tends to be unacceptably large when these controllers are implemented. In view of this disadvantage, authors in [25,26,4] present the novel adaptive fuzzy control schemes with much less parameters required to be updated online. In addition, a novel neural learning control scheme is also presented in [23] based on the deterministic learning mechanism, which can effectively recall and reuse the learned knowledge without any further adaptation of neural weights. However, these results in [25,26,4,23] are limited to delay-free systems.

It is well known that time delays are frequently encountered in real engineering systems. It has been shown that the existence of time delays usually becomes the source of instability and degrading performance of systems [18]. Therefore, stability analysis and controller synthesis for nonlinear time-delay systems are important both in theory and in practice [1,2]. More recently, several adaptive neural control schemes have been developed to stabilize nonlinear time-delay systems [6,10,9]. In [6,10], the adaptive neural output tracking problem has been addressed via backstepping approach for strict-feedback nonlinear systems with unknown time delays. However, these control methods proposed in [6,10,9] require a large number of neural weights to be adapted online simultaneously. This results in unacceptably large learning time.

Based on the above observation, in this paper, the problem of output tracking is revisited for nonlinear time-delay systems via adaptive fuzzy control method. Inspired by the work in [25,26,4] for nonlinear delay-free systems with strict-feedback structure, a new approach to design the adaptive fuzzy output tracking controller is proposed for a class of perturbed strict-feedback nonlinear time-delay systems. During the controller design process, fuzzy logic systems are employed to approximate unknown nonlinear functions, and then, the adaptive laws are constructed by backstepping technique. The main difficulty encountered in the controller design process is how to deal with the unknown time-delay terms in the systems. To overcome this difficulty, the novel Lyapunov–Krasovskii functionals are used to stability analysis and synthesis, and hyperbolic tangent functions are also introduced to handle the singularity problem encountered in Lyapunov synthesis. The proposed controller guarantees a good tracking performance and the boundedness of all the signals in the closed-loop system. Compared with the existing adaptive neural control methods [6,10,9], the main feature of this paper is that the number of adaptive parameters is independent of the number of rules of fuzzy logic systems and system state variables. As a result, the computational burden of the scheme is alleviated. This makes our design scheme more suitable for practical applications.

The rest of this paper is organized as follows. The problem formulation and preliminaries are given in Section 2. Section 3 presents an adaptive fuzzy tracking control scheme for a class of strict-feedback nonlinear time-delay systems by using the backstepping approach. In Section 4, two simulation examples are given to demonstrate the effectiveness of the method proposed in this paper. Finally, the conclusion is given in Section 5.

2. Problem formulation and preliminaries

2.1. Problem formulation

Consider the following unknown nonlinear time-delay dynamic system:

\[
\begin{aligned}
\dot{x}_i(t) &= f_i(\bar{x}_i(t)) + g_i(\bar{x}_i(t))x_{i+1}(t) + h_i(\bar{x}_i(t - \tau_i)) + d_i(t, x), \\ y(t) &= x_1(t), \\
\dot{x}_n(t) &= f_n(x(t)) + g_n(x(t))u(t) + h_n(x(t - \tau_n)) + d_n(t, x),
\end{aligned}
\]

(1)

where \(x = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n\), \(u \in \mathbb{R}\) and \(y \in \mathbb{R}\) denote system state vector, system control input and output, respectively. \(\bar{x}_i = [x_1, x_2, \ldots, x_i]^T\), \(i = 1, 2, \ldots, n - 1\). Functions \(f_i(\cdot), g_i(\cdot)\) and \(h_i(\cdot)\) are unknown smooth functions. \(\tau_i\) are constant unknown time delays, and \(d_i(\cdot)\) stand for external disturbance inputs, \(i = 1, 2, \ldots, n\).

**Remark 1.** As shown in [19], there are many physical processes which are governed by nonlinear differential equations in the form of (1). Since system (1) contains the external disturbance inputs, system (1) is of more general form than the one considered in [6,10,9].
The main goal of this paper is to design an adaptive fuzzy tracking controller for system (1) such that the system output $y(t)$ follows a desired reference signal $y_d(t)$, while all the signals in the closed-loop system remain bounded. To this end, define a vector function as $\tilde{y}_d = [y_d, y_d'(1), \ldots, y_d'(i)]^T$, $i = 1, 2, \ldots, n$, where $y_d'(i)$ is the $i$th time derivative of $y_d$. Then, the following assumptions are introduced.

**Assumption 1.** The desired trajectory vectors $\tilde{y}_d$ are continuous and known, and $\tilde{y}_d \in \Omega_d \subset \mathbb{R}^{i+1}$ with $\Omega_d$ being known compact sets, $i = 1, 2, \ldots, n$.

**Assumption 2.** For $1 \leq i \leq n$, there exist unknown positive smooth functions $\rho_i(\tilde{x}_i(t))$ such that $|d_i(t, x)| \leq \rho_i(\tilde{x}_i(t))$.

**Assumption 3.** For $1 \leq i \leq n$, the signs of $g_i(\tilde{x}_i(t))$ are known, and there exist unknown positive constants $b$ and $c$ such that $0 < b \leq |g_i(\tilde{x}_i(t))| \leq c < \infty$, $\forall \tilde{x}_i \in \mathbb{R}^i$. Without loss of generality, it is assumed that $g_i(\tilde{x}_i(t)) \geq b > 0$.

**Remark 2.** It should be emphasized that the constants $b$ and $c$ in Assumption 3 are introduced only for the purpose of analysis and are not used to design controllers. Therefore, the constants $b$ and $c$ can be unknown as well. In addition, unlike the assumption in [6,10,9], the upper bounds of unknown functions $g_i(\tilde{x}_i(t))$ can be unknown in this paper. This makes the design of controllers more difficult than the work in [6,10,9].

### 2.2. Function approximation with fuzzy logic systems

Throughout this paper, the following IF–THEN rules are used to develop the adaptive fuzzy controller:

- $R_i$: IF $x_1$ is $F^i_1$, and $\cdots$ and, $x_n$ is $F^i_n$ THEN $y$ is $B^i$, $i = 1, 2, \ldots, N$.

By using the singleton fuzzifier, product inference and center average defuzzifier, the fuzzy logic system can be expressed in the following form:

$$y(x) = \sum_{i=1}^{N} \Phi_i \prod_{j=1}^{n} \mu_{F^i_j}(x_j) \over \sum_{i=1}^{N} [\prod_{j=1}^{n} \mu_{F^i_j}(x_j)]$$, \hspace{1cm} (2)

where $x = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n$, $\mu_{F^i_j}(x_j)$ is the membership function of $F^i_j$, and $\Phi_i = \max_{y \in \mathbb{R}} \mu_{B^i}(y)$. Define $\phi = [\Phi_1, \Phi_2, \ldots, \Phi_N]^T$, and $\xi(x) = [\xi_1(x), \xi_2(x), \ldots, \xi_N(x)]^T$ with the fuzzy basis function $\tilde{\xi}_i(x)$ given by

$$\tilde{\xi}_i(x) = \frac{\prod_{j=1}^{n} \mu_{F^i_j}(x_j)}{\sum_{i=1}^{N} [\prod_{j=1}^{n} \mu_{F^i_j}(x_j)]}$$

Then, the fuzzy logic system (2) can be rewritten as

$$y(x) = \phi^T \xi(x)$$, \hspace{1cm} (3)

It has been proven in [16] that when the membership functions are chosen as Gaussian functions, the above fuzzy logic system is capable of uniformly approximating any continuous nonlinear function over a compact set $\Omega_x$ with any degree of accuracy. This property is shown by the following lemma.

**Lemma 1.** For any given real continuous function $f(x)$ on a compact set $\Omega_x \subset \mathbb{R}^n$ and scalar $\varepsilon > 0$, there exists a fuzzy logic system $y(x)$ in the form of (3) such that

$$\sup_{x \in \Omega_x} | f(x) - y(x) | \leq \varepsilon.$$
3. Adaptive fuzzy controller design

In this section, we will use the recursive backstepping technique to develop the adaptive fuzzy tracking control laws as follows:

\begin{align}
\dot{x}_i(t) &= -\frac{\hat{\theta}_i(t)}{2\eta_i^2} \xi_i^T(Z_i) \xi_i(Z_i) e_i(t) - k_i e_i(t), \\
\dot{\theta}_i(t) &= \frac{\gamma_i}{2\eta_i^2} \xi_i^T(Z_i) \xi_i(Z_i) e_i^2(t) - \sigma_i \hat{\theta}_i(t),
\end{align}

where \( 1 \leq i \leq n, \sigma_i, \gamma_i \) and \( \eta_i \) are design parameters with \( \sigma_i > 0 \) and \( \gamma_i > 0 \). \( \hat{\theta}_i(t) \) is the estimate of the unknown constant \( \theta_i \). \( e_i(t) = x_i(t) - x_{i-1}(t), z_0(t) \) is equal to \( y_d(t) \), the control gain \( k_i \) satisfies \( k_i > 1/b_0 \) with \( b \) being a positive design parameter, \( \xi_i(Z_i) \) is a fuzzy basis function vector with \( Z_i \) being the input vector. Note that when \( i = n \), \( x_n(t) \) is the true control input \( u(t) \).

**Remark 3.** From (5), it can be seen that the function \( S_i(t) = \gamma_i \xi_i^T(Z_i(t)) \xi_i(Z_i(t)) e_i^2(t) / 2\eta_i^2 \) is nonnegative. This implies that if \( \hat{\theta}_i(t) \leq S_i(t) / \sigma_i \), then \( \hat{\theta}_i(t) \geq 0 \). Consequently, \( \hat{\theta}_i(t) \) increases until \( \hat{\theta}_i(t) = S_i(t) / \sigma_i \). So, for any given initial condition \( \hat{\theta}_i(t_0) \geq 0 \), \( \hat{\theta}_i(t) \geq 0 \) holds for all \( t \geq t_0 \).

For simplicity, let \( \hat{e}_i \) be the upper bound of the fuzzy approximation error \( \hat{\delta}_i(Z_i) \), \( \hat{\theta}_i = \theta_i - \hat{\theta}_i \), and \( \| z \| \) denote the Euclidean norm of vector \( z \), i.e., \( \|z\|^2 = z^T z \). In addition, the time variable \( t \) is omitted in state variables except the delayed state variables \( \tilde{x}_i(t - \tau_i) \). Then, system (1) can be expressed as

\begin{equation}
\begin{aligned}
\dot{x}_i &= f_i(\tilde{x}_i) + g_i(\tilde{x}_i) x_{i+1} + h_i(\tilde{x}_i, x_i(t - \tau_i) ) + d_i(x), \\
\dot{\tilde{x}}_n &= f_n(x) + g_n(x) u + h_n(x(t - \tau_n) ) + d_n(x), \\
y &= x_1.
\end{aligned}
\end{equation}

Now, we propose the following backstepping-based design procedure.

**Step 1:** Define tracking error as \( \hat{e}_1 = x_1 - y_d \). Then, its time derivative along the first subsystem of (6) is given by

\begin{equation}
\dot{\hat{e}}_1 = f_1(x_1) + g_1(x_1) x_2 + h_1(x_1(t - \tau_1)) + d_1(x) - \hat{y}_d.
\end{equation}

Choose Lyapunov–Krasovskii functional candidate as

\[ V_{P_1} = \frac{\hat{e}_1^2}{2} + \int_{t-\tau_1}^t P_1(x_1(\tau)) \, d\tau, \]

where the unknown positive function \( P_1(x_1) \) will be specified later. Then, the time derivative of \( V_{P_1} \) is

\begin{equation}
\dot{V}_{P_1} = e_1 (f_1(x_1) + g_1(x_1) x_2 + h_1(x_1(t - \tau_1)) + d_1(x) - \hat{y}_d) + P_1(x_1) - P_1(x_1(t - \tau_1)).
\end{equation}

By Assumption 2 and the triangular inequality, the following inequality can be obtained:

\begin{equation}
\dot{V}_{P_1} \leq e_1 \left( f_1(x_1) + \frac{e_1}{2} + \frac{e_1 \rho_1^2(x_1)}{2a_{11}^2} - \hat{y}_d \right) + P_1(x_1) - P_1(x_1(t - \tau_1))
\end{equation}

\begin{equation}
+ g_1(x_1) e_1 x_2 + \frac{h_1^2(x_1(t - \tau_1))}{2} + a_{11}^2.
\end{equation}

where \( a_{11} \) is a design parameter. To cancel the unknown time-delay term in (9), \( P_1(x_1) \) is chosen as

\[ P_1(x_1) = \frac{h_1^2(x_1)}{2}. \]

As a result, the following inequality holds:

\begin{equation}
\dot{V}_{P_1} \leq e_1 \left( f_1(x_1) + \frac{e_1}{2} + \frac{h_1^2(x_1)}{2e_1} + \frac{e_1 \rho_1^2(x_1)}{2a_{11}^2} - \hat{y}_d \right) + g_1(x_1) e_1 x_2 + \frac{a_{11}^2}{2}.
\end{equation}
Notice that in (10) $h^2_1(x_1)/2e_1$ is discontinuous at $e_1 = 0$. Therefore, it cannot be approximated by the fuzzy logic system. This creates a difficulty to adaptive fuzzy control method. To overcome this difficulty, an effective approach is to introduce hyperbolic tangent function $\tanh(e_1/v_1)$ to deal with the term $h^2_1(x_1)/2e_1$ [7]. By this way, (10) becomes

$$
\dot{V}_P \leq e_1 \hat{f}_1(Z_1) + g_1(x_1)e_1x_2 + \frac{a^2_1}{2} + \left( 1 - 16 \tanh^2 \left( \frac{e_1}{v_1} \right) \right) H_1,
$$

where $v_1$ is a positive design parameter, $H_1 = h^2_1(x_1)/2$, and the function $\hat{f}_1(Z_1)$ is defined by

$$
\hat{f}_1(Z_1) = f_1(x_1) + \frac{e_1}{H_1} + \frac{16}{e_1} \tanh^2 \left( \frac{e_1}{v_1} \right) H_1 + \frac{e_1^2 \rho_2(Z_1)}{2a^2_1} - \dot{y}_d,
$$

with $Z_1 = [e_1, \tilde{y}_{d1}]^T \in \Omega_{Z_1} \subset R^3$, and $\Omega_{Z_1}$ being some known compact set in $R^3$. Note that $\lim_{e_1 \to 0} \frac{16}{e_1} \tanh^2 \left( \frac{e_1}{v_1} \right) H_1$ exists, thus, the nonlinear function $\hat{f}_1(Z_1)$ can be approximated by a fuzzy logic system $\phi^T_1 \tilde{z}_1(Z_1)$ with the input vector $Z_1 \in \Omega_{Z_1}$ such that

$$
\hat{f}_1(Z_1) = \phi^T_1 \tilde{z}_1(Z_1) + \delta_1(Z_1).
$$

Then, it follows from substituting (12) into (11) that

$$
\dot{V}_P \leq g_1(x_1)e_1x_2 + e_1 \phi^T_1 \tilde{z}_1(Z_1) + e_1 \delta_1(Z_1) + \frac{a^2_1}{2} + \left( 1 - 16 \tanh^2 \left( \frac{e_1}{v_1} \right) \right) H_1.
$$

By using

$$
e_1 \phi^T_1 \tilde{z}_1(Z_1) \leq \frac{b \theta_1}{2 \eta_1^2} \hat{z}^T_1(Z_1) \tilde{z}_1(Z_1) e_1^2 + \frac{\eta_1^2}{2}, \quad e_1 \delta_1(Z_1) \leq \frac{e_1^2}{2 \eta_1^2} + \frac{q e_1^2}{2},
$$

it can be easily verified that

$$
\dot{V}_P \leq g_1(x_1)e_1x_2 + \frac{b \theta_1}{2 \eta_1^2} \hat{z}^T_1(Z_1) \tilde{z}_1(Z_1) e_1^2 + \frac{e_1^2}{2 \eta_1^2} + d_1 + \left( 1 - 16 \tanh^2 \left( \frac{e_1}{v_1} \right) \right) H_1,
$$

where $\theta_1 = b^{-1} \| \phi_1 \|^2$ is an unknown constant, $\eta_1$ is a positive design parameter, and $d_1 = a^2_1/2 + \eta_1^2/2 + q e_1^2/2$.

In order to construct the virtual control law $\dot{x}_1$, choose a Lyapunov function candidate $V_1$ as

$$
V_1 = V_{P1} + \frac{b}{2 \gamma_1} \hat{z}_1^2.
$$

Differentiating $V_1$ and then using (14) give that

$$
\ddot{V}_1 \leq g_1(x_1)e_1x_2 + \frac{b \theta_1}{2 \eta_1^2} \hat{z}^T_1(Z_1) \tilde{z}_1(Z_1) e_1^2 + \frac{e_1^2}{2 \eta_1^2} + \frac{b \theta_1}{\gamma_1} \left( \frac{\gamma_1}{2 \eta_1^2} \hat{z}^T_1(Z_1) \tilde{z}_1(Z_1) e_1^2 - \hat{\theta}_1 \right)
$$

$$
+ d_1 + \left( 1 - 16 \tanh^2 \left( \frac{e_1}{v_1} \right) \right) H_1.
$$

Choosing the adaptive law $\hat{\theta}_1$ in (5) and defining $e_2 = x_2 - \dot{x}_1$, inequality (15) becomes

$$
\ddot{V}_1 \leq g_1(x_1)e_1e_2 + g_1(x_1)e_1x_1 + \frac{b \theta_1}{2 \eta_1^2} \hat{z}^T_1(Z_1) \tilde{z}_1(Z_1) e_1^2 + \frac{e_1^2}{2 \eta_1^2}
$$

$$
+ \frac{b \sigma_1 \hat{\theta}_1 \hat{\theta}_1}{\gamma_1} + d_1 + \left( 1 - 16 \tanh^2 \left( \frac{e_1}{v_1} \right) \right) H_1.
$$

Based on Remark 3, for any given bounded initial condition $\hat{\theta}_1(t_0) \geq 0$, we have $\hat{\theta}_1 \geq 0$. Thus, choose the virtual control law $\dot{x}_1$ in (4) to get that

$$
g_1(x_1)e_1x_1 = -g_1(x_1) \left( k_1 e_1^2 + \frac{\hat{\theta}_1}{2 \eta_1^2} \hat{z}^T_1(Z_1) \tilde{z}_1(Z_1) e_1^2 \right) \leq -b k_1 e_1^2 - \frac{\hat{\theta}_1}{2 \eta_1^2} \hat{z}^T_1(Z_1) \tilde{z}_1(Z_1) e_1^2. \tag{17}
$$
Noting
\[
\frac{\sigma_1 b \theta_1 \gamma_1}{\gamma_1} = -\frac{\sigma_1 b \theta_1^2}{\gamma_1} + \frac{\sigma_1 b \theta_1 \gamma_1}{\gamma_1} \leq -\frac{\sigma_1 b \theta_1^2}{2\gamma_1} + \frac{\sigma_1 b \theta_1^2}{2\gamma_1},
\]
and substituting (17) into (16) show that
\[
\dot{V}_1 \leq -\left( bk - \frac{1}{2g_1} \right) e_1^2 - \frac{\sigma_1 b \theta_1^2}{2\gamma_1} + g_1(x_1)e_1e_2 + C_1 + \left( 1 - 16 \tanh^2 \left( \frac{e_1}{v_1} \right) \right) H_1, \tag{18}
\]
where \( C_1 = d_1 + \sigma_1 b \theta_1^2/2\gamma_1 \). The coupling term \( g_1(x_1)e_1e_2 \) will be handled in the next step, and the last term in (18) will be considered later.

**Step i:** Similarly, for each step \( i (2 \leq i \leq n - 1) \), considering \( e_i = x_i - x_{i-1} \), the dynamics of \( e_i \)-subsystem is given by
\[
\dot{e}_i = f_i(\bar{x}_i) + g_i(\bar{x}_i)x_{i+1} + h_i(\bar{x}_i(t - \tau_i)) + d_i(x) - \dot{\bar{x}}_{i-1}, \tag{19}
\]
where
\[
\dot{\bar{x}}_{i-1} = \sum_{j=1}^{i-1} \frac{\partial \bar{x}_{i-1}}{\partial x_j} (h_j(\bar{x}_j(t - \tau_j)) + d_j(x)) + W_{i-1},
\]
with \( W_{i-1} = \sum_{j=1}^{i-1} \frac{\partial \bar{x}_{i-1}}{\partial x_j} (f_j(\bar{x}_j) + g_j(\bar{x}_j)x_{j+1}) + \sum_{j=1}^{i-1} \frac{\partial \bar{x}_{i-1}}{\partial \bar{x}_j} \dot{\bar{x}}_j + \frac{\partial \bar{x}_{i-1}}{\partial \bar{x}_{i(i-1)}} \dot{\bar{x}}_{i(i-1)}. \)

**Remark 4.** As \( x_{i-1} \) contains the variable \( \bar{x}_{i-1} \), unknown time-delay terms in the previous subsystem will appear again when \( \dot{\bar{x}}_{i-1} \) is computed. Therefore, Lyapunov–Krasovskii functionals should be designed to compensate for not only the term containing delay \( \tau_i \), but also the terms containing \( \tau_j \) for \( j < i \).

Based on such an idea, we choose the Lyapunov–Krasovskii functional as
\[
V_{P_i} = \frac{e_i^2}{2} + \sum_{j=1}^{i} \int_{t-\tau_j}^{t} P_j(\bar{x}_j(\tau)) \, d\tau,
\]
where the unknown positive function \( P_j(\bar{x}_j) \) will be specified later.

Differentiating \( V_{P_i} \) along the trajectory (19) yields that
\[
\dot{V}_{P_i} = e_i \left( f_i(\bar{x}_i) + g_i(\bar{x}_i)x_{i+1} + h_i(\bar{x}_i(t - \tau_i)) + d_i(x) - \sum_{j=1}^{i-1} \frac{\partial \bar{x}_{i-1}}{\partial x_j} (h_j(\bar{x}_j(t - \tau_j)) + d_j(x)) - W_{i-1} \right)
+ \sum_{j=1}^{i} (P_j(\bar{x}_j) - P_j(\bar{x}_j(t - \tau_j))). \tag{20}
\]
By Assumption 2 and the triangular inequality, the following inequalities can be obtained easily:
\[
e_i h_i(\bar{x}_i(t - \tau_i)) \leq \frac{e_i^2}{2} + \frac{h_i^2(\bar{x}_i(t - \tau_i))}{2}, \quad -e_i \frac{\partial \bar{x}_{i-1}}{\partial x_j} d_j(x) \leq \frac{e_i^2 \rho_j^2(\bar{x}_i)}{2a^2} \left( \frac{\partial \bar{x}_{i-1}}{\partial x_j} \right)^2 + \frac{a^2_{ij}}{2},
- e_i \frac{\partial \bar{x}_{i-1}}{\partial x_j} h_j(\bar{x}_j(t - \tau_j)) \leq \frac{e_i^2}{2} \left( \frac{\partial \bar{x}_{i-1}}{\partial x_j} \right)^2 + \frac{h_j^2(\bar{x}_j(t - \tau_j))}{2}, \quad e_i d_i(x) \leq \frac{a^2_{ij}}{2} + \frac{e_i^2 \rho_j^2(\bar{x}_i)}{2a^2_{ij}}, \tag{21}
\]
where \( a_{ij} > 0, 1 \leq j \leq i \), are design parameters.
Consequently, it follows from substituting (21) into (20) that

\[
\dot{V}_P \leq e_i \left( f_i(x_i) + e_i^2 + \sum_{j=1}^{i} \left( e_i \rho_j^2(x_j) \left( \frac{\partial \xi_{i-1}}{\partial x_j} \right)^2 + e_i \left( \frac{\partial \xi_{i-1}}{\partial x_j} \right)^2 \right) + \frac{e_i \rho_j^2(x_i)}{2a_{ij}^2} - W_{i-1} \right) + \sum_{j=1}^{i} \frac{a_{ij}^2}{2} + g_i(x_i)e_i x_{i+1} + \sum_{j=1}^{i} \frac{h_j^2(\bar{x}_j(t - \tau_j))}{2} + \sum_{j=1}^{i}(P_j(\bar{x}_j) - P_j(\bar{x}_j(t - \tau_j))).
\]

To cancel time-delay terms in (22), choose \( P_j(\bar{x}_j) = \frac{h_j^2(\bar{x}_j)}{2} \). As a result, (22) becomes

\[
\dot{V}_P \leq e_i \left( f_i(x_i) + g_{i-1}(x_{i-1})e_{i-1} + e_i^2 + \sum_{j=1}^{i-1} \left( e_i \rho_j^2(x_j) \left( \frac{\partial \xi_{i-1}}{\partial x_j} \right)^2 + e_i \left( \frac{\partial \xi_{i-1}}{\partial x_j} \right)^2 \right) \right) + \frac{e_i \rho_j^2(x_i)}{2a_{ij}^2} - W_{i-1} + \frac{H_i}{e_i} + g_i(x_i)e_i x_{i+1} - g_{i-1}(x_{i-1})e_{i-1}e_i + \sum_{j=1}^{i} \frac{a_{ij}^2}{2},
\]

with \( H_i = \sum_{j=1}^{i} \frac{h_j^2(\bar{x}_j)}{2} \).

Similar to Step 1, by introducing a function \( \tanh(e_i/v_i) \) to deal with the discontinuous term \( H_i/e_i \), (23) can be expressed as

\[
\dot{V}_P \leq e_i \hat{f}_i(Z_i) + g_i(x_i)e_i x_{i+1} - g_{i-1}(x_{i-1})e_{i-1}e_i + \sum_{j=1}^{i} \frac{a_{ij}^2}{2} + \left( 1 - 16 \tanh^2 \left( \frac{e_i}{v_i} \right) \right) H_i,
\]

where \( v_i \) is a positive design parameter, \( Z_i = [\bar{e}_i^T, \bar{y}_i^T]^T \in \Omega_{Z_i} \subset R^{2l+1} \) with \( \bar{e}_i = [e_1, e_2, \ldots, e_l]^T \) and \( \Omega_{Z_i} \) being some known compact set, and the function \( \hat{f}_i(Z_i) \) is defined by

\[
\hat{f}_i(Z_i) = f_i(x_i) + g_{i-1}(x_{i-1})e_{i-1} + e_i^2 + \sum_{j=1}^{i-1} \left( e_i \rho_j^2(x_j) \left( \frac{\partial \xi_{i-1}}{\partial x_j} \right)^2 + e_i \left( \frac{\partial \xi_{i-1}}{\partial x_j} \right)^2 \right) + \frac{e_i \rho_j^2(x_i)}{2a_{ij}^2} - W_{i-1} + \frac{16 \tanh^2 \left( \frac{e_i}{v_i} \right)}{e_i} H_i.
\]

Since \( \lim \frac{\tanh^2 \left( \frac{2}{e_i} \right)}{e_i} = 0 \) when \( e_i \to 0 \), a fuzzy logic system \( \phi_i^T \xi_i(Z_i) \) can be used to approximate the function \( \hat{f}_i(Z_i) \) such that

\[
\hat{f}_i(Z_i) = \phi_i^T \xi_i(Z_i) + \delta_i(Z_i).
\]

Consequently, by substituting (25) into (24) and then using the following inequalities:

\[
e_i \phi_i^T \xi_i(Z_i) \leq \frac{b_0}{2\eta_i^2} \xi_i^T(Z_i) \xi_i(Z_i) e_i^2 + \eta_i^2, \quad e_i \delta_i(Z_i) \leq \frac{e_i^2}{2} + \frac{q_0^2}{2},
\]

one can get

\[
\dot{V}_P \leq g_i(x_i)e_i x_{i+1} - g_{i-1}(x_{i-1})e_{i-1}e_i + \frac{b_0}{2\eta_i^2} \xi_i^T(Z_i) \xi_i(Z_i) e_i^2 + \frac{e_i^2}{2} + d_i + \left( 1 - 16 \tanh^2 \left( \frac{e_i}{v_i} \right) \right) H_i,
\]

where \( b_0 = b^{-1} \| \phi_i \|^2 \) is an unknown constant, and \( d_i = \sum_{j=1}^{i} a_{ij}^2/2 + \eta_i^2/2 + q_0^2/2 \).

Now, define a function \( V_{W_i} \) as

\[
V_{W_i} = V_P + \frac{b}{2\gamma_i} \eta_i^2.
\]
Then, the above discussion implies that its time derivative satisfies
\[
\dot{V}_{W_i} \leq g_i(\tilde{x}_i)e_i x_{i+1} - g_{i-1}(\tilde{x}_{i-1})e_{i-1}e_i + \frac{b \hat{\theta}_i}{2\eta_i} \xi_i^T(Z_i) \xi_i(Z_i)e_i^2 + \frac{e_i^2}{2q}
\]
\[
+ \frac{b \hat{\theta}_i}{\gamma_i} \left( \frac{\gamma_i}{2\eta_i} \xi_i^T(Z_i) \xi_i(Z_i)e_i^2 - \dot{\hat{\theta}}_i \right) + d_i + \left( 1 - 16 \tanh^2 \left( \frac{e_i}{v_i} \right) \right) H_i.
\]  
(28)

For any given bounded initial condition \( \hat{\theta}_i(t_0) \geq 0 \), choosing the virtual control law \( \alpha_i \) as one defined in (4), then the following inequality holds:
\[
g_i(\tilde{x}_i)e_i x_i \leq -bk_i e_i^2 - b \frac{\hat{\theta}_i}{2\eta_i} \xi_i^T(Z_i) \xi_i(Z_i)e_i^2.
\]  
(29)

Substituting (5) and (29) into (28), and using the following inequality:
\[
\sigma_i b \hat{\theta}_i \frac{\dot{\theta}_i}{\gamma_i} \leq - \sigma_i b \hat{\theta}_i^2 \frac{2\gamma_i}{2\gamma_i},
\]
we have
\[
\dot{V}_{W_i} \leq - \left( bk_i - \frac{1}{2q} \right) e_i^2 - \sigma_i b \hat{\theta}_i^2 \frac{2\gamma_i}{2\gamma_i} + g_i(\tilde{x}_i)e_i x_{i+1} - g_{i-1}(\tilde{x}_{i-1})e_{i-1}e_i + C_i + \left( 1 - 16 \tanh^2 \left( \frac{e_i}{v_i} \right) \right) H_i.
\]  
(30)

where the constant \( C_i \) is defined by \( C_i = d_i + \sigma_i b \hat{\theta}_i^2/2\gamma_i \).

Furthermore, choose a Lyapunov function candidate \( V_i = V_{i-1} + V_{W_i} \). Similar to (18), we can get
\[
\dot{V}_{l-1} \leq - \sum_{j=1}^{i-1} \left( (bk_j - \frac{1}{2q}) e_j^2 + \frac{\sigma_j b \hat{\theta}_j^2}{2\gamma_j} \right) + g_{i-1}(\tilde{x}_{i-1})e_{i-1}e_i + \sum_{j=1}^{i-1} \left( C_j + \left( 1 - 16 \tanh^2 \left( \frac{e_j}{v_j} \right) \right) H_j \right).
\]  
(31)

Combining (30) with (31) produces
\[
\dot{V}_i \leq - \sum_{j=1}^{i} \left( (bk_j - \frac{1}{2q}) e_j^2 + \frac{\sigma_j b \hat{\theta}_j^2}{2\gamma_j} \right) + g_i(\tilde{x}_i)e_i x_{i+1} + \sum_{j=1}^{i} \left( C_j + \left( 1 - 16 \tanh^2 \left( \frac{e_j}{v_j} \right) \right) H_j \right).
\]  
(32)

Step n: In this step, the true control \( u \) will be constructed. Defining \( e_n = x_n - \alpha_{n-1} \), then we have
\[
\dot{e}_n = f_n(x) + g_n(x)u + h_n(x(t - \tau_n)) + d_n(x) - \dot{\alpha}_{n-1},
\]  
(33)

where
\[
\dot{\alpha}_{n-1} = \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} (h_j(\tilde{x}_j(t - \tau_j)) + d_j(x)) + W_{n-1},
\]
with \( W_{n-1} = \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_j} (f_j(\tilde{x}_j) + g_j(\tilde{x}_j)x_{j+1}) + \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \theta_j} \hat{\theta}_j + \frac{\partial \alpha_{n-1}}{\partial \gamma_{\delta(a-1)}} \tilde{\gamma} d(a-1). \)

Similarly, choose the following Lyapunov–Krasovskii functional:
\[
V_p_n = \frac{e_n^2}{2} + \sum_{j=1}^{n} \int_{t-\tau_j}^{t} P_j(\tilde{x}_j(\tau)) \, d\tau.
\]
where the unknown positive function \( P_j(\tilde{x}_j) \) is defined by \( P_j(\tilde{x}_j) = h_j^2(\tilde{x}_j)/2 \). Its time derivative along (33) is

\[
\dot{V}_{P_n} = e_n \left( f_n(x) + g_n(x)u + h_n(x(t - \tau_n)) + d_n(x) - \sum_{j=1}^{n-1} \frac{\partial z_{n-1}}{\partial x_j} (h_j(\tilde{x}_j(t - \tau_j)) + d_j(x)) - W_{n-1} \right) \\
+ \sum_{j=1}^{n} (P_j(\tilde{x}_j) - P_j(\tilde{x}_j(t - \tau_j))).
\]  

(34)

Using a similar approach in (21), we can obtain

\[
\dot{V}_{P_n} \leq e_n \left( f_n(x) + \frac{e_n}{2} + \sum_{j=1}^{n-1} \left( \frac{e_n p_j^2(x)}{2 a_{n_j}^2} \left( \frac{\partial z_{n-1}}{\partial x_j} \right)^2 + \frac{e_n}{2} \left( \frac{\partial z_{n-1}}{\partial x_j} \right)^2 \right) + g_n(x)u - g_n-1(\tilde{x}_{n-1})e_{n-1} + \sum_{j=1}^{n} a_{n_j}^2, \right)
\]

(35)

where \( H_n = \sum_{j=1}^{n} h_j^2(\tilde{x}_j)/2 \). The function \( \tanh(\epsilon_n/\eta_n) \) is used to deal with the term \( H_n/e_n \). Then, inequality (35) becomes

\[
\dot{V}_{P_n} \leq e_n \left( f_n(Z_n) + g_n(x)u - g_n-1(\tilde{x}_{n-1})e_{n-1} + \sum_{j=1}^{n} a_{n_j}^2 + \left( 1 - 16 \tanh^2 \left( \frac{e_n}{\eta_n} \right) \right) H_n, \right)
\]

(36)

where \( \eta_n \) is a positive design parameter, \( Z_n = [e_1^T, \tilde{x}_n^T] \in \Omega_{Z_{n}} \subset \mathbb{R}^{2n+1} \) with \( e = [e_1, e_2, \ldots, e_n]^T \) and \( \Omega_{Z_{n}} \) being a known compact set, and the function \( f_n(Z_n) \) is defined by

\[
f_n(Z_n) = f_n(x) + g_n-1(\tilde{x}_{n-1})e_{n-1} + \frac{e_n}{2} + \sum_{j=1}^{n-1} \left( \frac{e_n p_j^2(x)}{2 a_{n_j}^2} \left( \frac{\partial z_{n-1}}{\partial x_j} \right)^2 + \frac{e_n}{2} \left( \frac{\partial z_{n-1}}{\partial x_j} \right)^2 \right) + \frac{e_n p_j^2(x)}{2 a_{n_j}^2} - W_{n-1} + \frac{H_n}{e_n} 16 \tanh^2 \left( \frac{e_n}{\eta_n} \right). \]

(37)

Similar to (25), a fuzzy logic system \( \phi_n^T \xi_n(Z_n) \) is used to approximate \( f_n(Z_n) \) such that

\[
\hat{f}_n(Z_n) = \phi_n^T \xi_n(Z_n) + \delta_n(Z_n).
\]

(38)

By substituting (37) into (36), and applying the triangular inequality, one can get

\[
\dot{V}_{P_n} \leq g_n(x)u - g_n-1(\tilde{x}_{n-1})e_{n-1} + \frac{b \theta_n}{2 \eta_n^2} \xi_n(Z_n) \xi_n(Z_n) e_n^2 + \frac{e_n^2}{2 q} + d_{n} + \left( 1 - 16 \tanh^2 \left( \frac{e_n}{\eta_n} \right) \right) H_n,
\]

(39)

where \( \theta_n = b^{-1} \| \phi_n \|^2 \) and \( d_{n} = \sum_{j=1}^{n} a_{n_j}^2/2 + \eta_n^2/2 + q e_n^2/2 \).

Furthermore, define a function \( V_{W_n} \) as

\[
V_{W_n} = V_{P_n} + \frac{b}{2 \eta_n^2} \tilde{\eta}^2.
\]

(40)

By taking the true control law \( u \) in (4) and using Remark 3, we have

\[
g_n(x)u \leq -bk_n e_n^2 - \frac{\hat{\theta}_n}{2 \eta_n^2} \xi_n(Z_n) \xi_n(Z_n) e_n^2.
\]

(41)
Moreover, using
\[ \frac{\sigma_n b \hat{\theta}_n \hat{\theta}_n}{\gamma_n} = -\frac{\sigma_n b \hat{\theta}_n^2}{\gamma_n} + b \sigma_n \hat{\theta}_n \hat{\theta}_n \leq -\frac{\sigma_n b \hat{\theta}_n^2}{2\gamma_n} + \frac{\sigma_n b \theta_n^2}{2\gamma_n}, \]
and substituting (40) into (39) show that
\[ \dot{V}_W \leq - \left( bk_n - \frac{1}{2q} \right) e_n^2 - \frac{\sigma_n b \hat{\theta}_n^2}{2\gamma_n} - g_{n-1}(\bar{x}_{n-1}) e_{n-1} e_n + C_n + \left( 1 - 16 \tanh^2 \left( \frac{e_n}{v_n} \right) \right) H_n, \]
where \( C_n = d_n + b \sigma_n \theta_n^2/2\gamma_n \).

Now, choose a Lyapunov function candidate \( V_n = V_{n-1} + V_W \). Similar to (31), the following inequality holds:
\[ \dot{V}_{n-1} \leq - \sum_{j=1}^{n-1} \left( bk_j \right) e_j^2 + \frac{b \sigma_j \bar{\theta}_j^2}{2\gamma_j} + g_{n-1}(\bar{x}_{n-1}) e_{n-1} e_n + \sum_{j=1}^{n-1} \left( C_j + \left( 1 - 16 \tanh^2 \left( \frac{e_j}{v_j} \right) \right) H_j \right). \]
(42)

Thus, from (41) and (42), it can clearly be seen that the time derivative of \( V_n \) satisfies
\[ \dot{V}_n \leq - \sum_{j=1}^{n} \left( \bar{k}_j e_j^2 + \frac{b \sigma_j \bar{\theta}_j^2}{2\gamma_j} \right) + C_n \sum_{j=1}^{n} \left( 1 - 16 \tanh^2 \left( \frac{e_j}{v_j} \right) \right) H_j, \]
(43)
where \( \bar{k}_j = bk_j - 1/2q > 1/2q \) with \( k_j > 1/b \), and \( C_n = \sum_{j=1}^{n} C_j + C_j = d_j + b \sigma_j \theta_j^2/2\gamma_j \). The last term in (43) will be considered later.

At the present stage, we are in the position to give our main result in the following theorem.

**Theorem 1.** Consider the closed-loop system consisting of system (1) under Assumptions 1–3, the control laws (4) and the adaptive laws (5). Suppose that the packaged uncertain functions \( f_i(Z_i), i = 1, 2, \ldots, n \), can be approximated by fuzzy logic systems in the sense that approximation errors are bounded. Then, the closed-loop system has the following properties for bounded initial conditions with \( \hat{\theta}_j(\tau_0) \geq 0, i = 1, 2, \ldots, n \), (i) all the signals in the closed-loop system are bounded; (ii) the error signal \( e \), in the mean square sense, eventually converges to the following set:
\[ \Omega_s := \{ e \in R^n | e_{rs} \leq \mu_e \}, \]
where \( e = [e_1, e_2, \ldots, e_n]^T, e_{rs} = \frac{1}{T} \int_0^T \| e(\tau) \|^2 d\tau, \) and \( \mu_e \) will be given later. \( \Omega_s \) can be made as small as desired by appropriately choosing design parameters.

Before the proof of this theorem, we first introduce the following lemma.

**Lemma 2** (Ge and Tee [7]). For \( 1 \leq j \leq n \), consider the set \( \Omega_{v_j} \) given by \( \Omega_{v_j} := \{ e_j | e_j | < 0.2554 v_j \} \). Then, for \( e_j \notin \Omega_{v_j} \), the inequality \( 1 - 16 \tanh^2 (e_j/v_j) \leq 0 \) is satisfied.

**Proof.** (i) Boundedness of all the signals in the closed-loop system can be proven by the following three cases.

**Case 1:** For \( j = 1, 2, \ldots, n, e_j \in \Omega_{v_j} \). In this case, \( |e_j| < 0.2554 v_j \) with \( v_j \) being the positive design parameter, therefore, \( e_j \) is bounded. Moreover, according to the choice of the adaptive parameter \( \hat{\theta}_j \), it is obvious that \( \hat{\theta}_j \) is bounded for any bounded \( e_j \). Further, \( \hat{\theta}_j \) is bounded as \( \theta_j \) is a constant. According to Assumption 1, \( y_d, \hat{y}_d, \ldots, y_d^{(n)} \) are bounded. Since \( e_1 = x_1 - y_d \) and \( y_d \) are bounded, thus \( x_1 \) is also bounded. Using (4) with \( i = 1 \), and noting that \( e_1, \hat{\theta}_1 \) and \( z_1(Z_1) \) are all bounded, we can conclude that \( x_1 \) is bounded. Consequently, it follows from \( x_2 = e_2 + x_1 \) that \( x_2 \) is bounded. Following the same way, \( x_{j-1}, u \) and \( x_j, j = 3, \ldots, n, \) can be proven to be bounded. Thus, all the signals in the closed-loop system are bounded in Case 1.

**Case 2:** All \( e_j \notin \Omega_{v_j} \). In this case, from the definition of \( H_j \), we know that \( H_j \) is nonnegative. It follows from Lemma 2 that
\[ \left( 1 - 16 \tanh^2 \left( \frac{e_j}{v_j} \right) \right) H_j \leq 0. \]
(45)
By using (45), (43) becomes
\[ \dot{V}_n \leq - \sum_{j=1}^{n} \left( k_j e_j^2 + \frac{b \sigma_j \tilde{\theta}_j^2}{2 \gamma_j} \right) + C, \]  
(46)
where \( C \) is a constant and \( \tilde{k}_j \) satisfies \( k_j > 1/2q > 0 \). As shown in [23], (46) shows that all the signals in the closed-loop system are bounded.

Case 3: Some \( e_i \in \Omega_{v_i} \), while some \( e_j \notin \Omega_{v_j} \).

For those \( e_i \in \Omega_{v_i} \), define \( \Sigma_i \) as the subsystem consisting of \( e_i \in \Omega_{v_i} \). Similar to Case 1, \( e_i, \tilde{\theta}_i, \) and \( \tilde{\theta}_i \) can be proven to be bounded for \( i \in \Sigma_i \). For those \( e_j \notin \Omega_{v_j} \), define \( \Sigma_j \) as the subsystem consisting of \( e_j \notin \Omega_{v_j} \) and choose the Lyapunov function candidate as
\[ V_{\Sigma_j} = \sum_{j \in \Sigma_j} \left( V_{p_j} + \frac{b \tilde{\theta}_j^2}{2 \gamma_j} \right). \]

By using (30) and (45), we have
\[ \dot{V}_{\Sigma_j} \leq - \sum_{j \in \Sigma_j} \left( \left( k_j e_j^2 + \frac{b \sigma_j \tilde{\theta}_j^2}{2 \gamma_j} \right) - C_j \right) + \sum_{j \in \Sigma_j} \left[ g_j(\tilde{x}_j)e_j e_{j+1} - g_{j-1}(\tilde{x}_{j-1})e_j e_{j-1} \right]. \]  
(47)

The last term of (47) can be expressed as
\[ \sum_{j \in \Sigma_j} \left[ g_j(\tilde{x}_j)e_j e_{j+1} - g_{j-1}(\tilde{x}_{j-1})e_j e_{j-1} \right] = \sum_{j \in \Sigma_j} g_j(\tilde{x}_j)e_j e_{j+1} - \sum_{j \in \Sigma_j} g_{j-1}(\tilde{x}_{j-1})e_j e_{j-1} \]
\[ + \sum_{j \in \Sigma_j} g_j(\tilde{x}_j)e_j e_{j+1} - \sum_{j \in \Sigma_j} g_{j-1}(\tilde{x}_{j-1})e_j e_{j-1} \]  
(48)
As shown in [7], the first two terms in (48) can be canceled during backstepping. Therefore, from Assumption 3 and Lemma 2, it can be shown that
\[ \sum_{j \in \Sigma_j} \left[ g_j(\tilde{x}_j)e_j e_{j+1} - g_{j-1}(\tilde{x}_{j-1})e_j e_{j-1} \right] \leq \sum_{j \in \Sigma_j} \left( \frac{e_j^2}{4q} + gc^2 e_j^2 \right) + \sum_{j \in \Sigma_j} \left( \frac{e_{j-1}^2}{4q} + gc^2 e_{j-1}^2 \right) \]
\[ \leq \sum_{j \in \Sigma_j} \frac{e_j^2}{2q} + \sum_{j \in \Sigma_j} (gc^2(0.2554v_{j-1})^2 + gc^2(0.2554v_{j+1})^2). \]

Based on the above inequality, (47) can be rewritten as
\[ \dot{V}_{\Sigma_j} \leq - \sum_{j \in \Sigma_j} \left( \left( k_j e_j^2 + \frac{b \sigma_j \tilde{\theta}_j^2}{2 \gamma_j} \right) - C_{\Sigma_j} \right) + \sum_{j \in \Sigma_j} \left[ g_j(\tilde{x}_j)e_j e_{j+1} - g_{j-1}(\tilde{x}_{j-1})e_j e_{j-1} \right], \]  
(49)
where the positive constant \( C_{\Sigma_j} \) is given by
\[ C_{\Sigma_j} = \sum_{j \in \Sigma_j} C_j + \sum_{j \in \Sigma_j} \left( gc^2(0.2554v_{j-1})^2 + gc^2(0.2554v_{j+1})^2 \right), \]  
(50)
whose size depends on design parameters. Since \( \tilde{k}_j > 1/2q \), we have \( k_j - 1/2q > 0 \). Similar to Case 2, it can be shown that \( e_j, \tilde{\theta}_j \) and \( \tilde{\theta}_j \) are bounded for \( j \in \Sigma_j \). Now, we consider the boundedness of all the signals in the whole
closed-loop system. According to the above analysis, we know that all \( e_i, \hat{\theta}_i, \) and \( \hat{\theta}_i \) are bounded, \( i = 1, 2, \ldots, n. \) From Assumption 1, we have that \( y_d, \dot{y}_d, \ldots, y_d^{(n)} \) are bounded. Following the similar discussion in Case 1, we can conclude that all the signals in the closed-loop system are bounded under this case.

In light of the discussion for Cases 1–3, we can conclude that all the signals in the closed-loop system are bounded. This ends the proof of (i).

(ii) In this part, we will prove the error signal \( e \), in the mean square sense, eventually converges to a compact set \( \Omega_e \). Similar to the proof of (i), the proof is divided into the following three cases.

Case 1: For all \( e_j \in \Omega_{v_j}, \) we obtain that \( |e_j| < 0.2554 v_j, \) \( j = 1, 2, \ldots, n. \) Let \( v = [v_1, v_2, \ldots, v_n]^T \) and note the definition of \( e_{rs}. \) It is easy to get that

\[
e_{rs} = \frac{1}{t} \int_0^t \| e(\tau) \|^2 \, d\tau < (0.2554)^2 \| v \|^2.
\]

Case 2: For all \( e_j \notin \Omega_{v_j}, \) From (46) and the definition of \( \hat{k}_j, \) we have

\[
\dot{V}_n \leq - \sum_{j=1}^n \left( bk_j - \frac{1}{2q} \right) e_j^2 + C.
\]

Integrating (52) from 0 to \( t \) shows that

\[
\frac{1}{t} (V_n(t) - V_n(0)) \leq - \frac{1}{t} \sum_{j=1}^n \left( bk_j - \frac{1}{2q} \right) \int_0^t e_j^2(\tau) \, d\tau + C.
\]

Noting that \( bk_j > 1/\bar{q} > 0 \) and defining \( k = \min\{bk_1, bk_2, \ldots, bk_n\}, \) we have

\[
e_{rs} = \frac{1}{t} \int_0^t \| e(\tau) \|^2 \, d\tau \leq \frac{V_n(0)/t + C}{k - 1/2\bar{q}}.
\]

Case 3: Some \( e_i \in \Omega_{v_i}, \) while some \( e_j \notin \Omega_{v_j}. \) According to Case 3 in the proof of (i), and considering the subsystem \( \Sigma_I \) consisting of \( e_i \in \Omega_{v_i}, \) we can get

\[
e_{rs}\big|_{\Sigma_I} = \frac{1}{t} \int_0^t \| e_{\Sigma_I} \|^2 \, d\tau < (0.2554)^2 \| v_{\Sigma_I} \|^2.
\]

where \( \| e_{\Sigma_I} \|^2 = \sum_{j \in \Sigma_I} e_j^2 \) and \( \| v_{\Sigma_I} \|^2 = \sum_{i \in \Sigma_I} v_i^2. \) For the subsystem \( \Sigma_J \) consisting of \( e_j \notin \Omega_{v_j}, \) it follows from (49) and the definition of \( \hat{k}_j \) that

\[
\dot{V}_{\Sigma_J} \leq - \sum_{j \in \Sigma_J} \left( bk_j - \frac{1}{\bar{q}} \right) e_j^2 + C_{\Sigma_J},
\]

where \( C_{\Sigma_J} \) is defined by (50). Applying the similar procedures as (53) and (54), we have

\[
e_{rs}\big|_{\Sigma_J} = \frac{1}{t} \int_0^t \| e_{\Sigma_J} \|^2 \, d\tau \leq \frac{V_{\Sigma_J}(0)/t + C_{\Sigma_J}}{k - 1/\bar{q}},
\]

where \( \| e_{\Sigma_J} \|^2 = \sum_{j \in \Sigma_J} e_j^2, \) and \( k \) is defined in (54). Therefore, from (55) and (56), it is obtained that

\[
e_{rs} = \frac{1}{t} \int_0^t \| e(\tau) \|^2 \, d\tau \leq (0.2554)^2 \| v_{\Sigma_J} \|^2 + \frac{V_{\Sigma_J}(0)/t + C_{\Sigma_J}}{k - 1/\bar{q}}.
\]

Finally, from Cases 1–3, we can conclude that

\[
e_{rs} \leq \max \left\{ (0.2554)^2 \| v \|^2, \frac{V_n(0)/t + C}{k - 1/2\bar{q}}, (0.2554)^2 \| v_{\Sigma_J} \|^2 + \frac{V_{\Sigma_J}(0)/t + C_{\Sigma_J}}{k - 1/\bar{q}} \right\},
\]

which means that \( e \) eventually converges to the following set:

\[\Omega_s := \{ e \in \mathbb{R}^n \, | \, e_{rs} \leq \mu_s \}.\]
where

\[ \mu_s = \max \left\{ \frac{(0.2554)^2 \|v\|^2}{k - 1/2q}, \frac{C}{k - 1/2q}, \frac{(0.2554)^2 \|v_S\|^2}{k - 1/2q} \right\}. \]

This completes the proof of Theorem 1. \(\square\)

**Remark 5.** Recently, the similar results have been proposed in [25] for SISO nonlinear delay-free systems, and [4] for MIMO nonlinear delay-free systems. Unlike [25,4], this paper mainly addresses the tracking control problem for nonlinear time-delay systems with the strict-feedback structure. Although the adaptive fuzzy controller has been constructed based on the same design idea and backstepping technique, the existence of nonlinear time-delay terms \(h_i(x_i(t - \tau_i))\) makes controller design more difficult than the case without time delays. To compensate for the effect of delay terms, both the novel Lyapunov–Krasovskii functionals and the hyperbolic tangent functions are employed. As a result, the stability analysis of the closed-loop system in this paper is not only more complex than ones in [25,4] but also different from them. In addition, when removing the time-delay terms from the system, our result will include the one in [25] as a special case.

### 4. Simulation examples

In this section, two examples will be used to illustrate the effectiveness of the scheme presented in this paper.

**Example 1.** Consider a controlled Brusselator model with external disturbances in [26] as follows:

\[
\begin{align*}
\dot{x}_1 &= C - (D + 1)x_1 + x_1^2x_2 + d_1(t, \bar{x}_2), \\
\dot{x}_2 &= Dx_1 - x_1^2x_2 + (2 + \cos(x_1))u + d_2(t, \bar{x}_2), \\
y &= x_1,
\end{align*}
\] (57)

where \(x_1\) and \(x_2\) denote the concentrations of the reaction intermediates, \(C, D > 0\) are parameters which describe the supply of “reservoir” chemicals. \(u\) is the control input. \(d_1(t, \bar{x}_2)\) and \(d_2(t, \bar{x}_2)\) are the external disturbance terms, which come from the modeling errors and other types of unknown nonlinearities in the practical chemical reactions. As stated in [8], it is assumed that \(x_1 \neq 0\). The Brusselator model is one of the most popular nonlinear oscillatory models of chemical kinetics. As a practical chemical reaction, the existence of time delays is inevitable in the Brusselator model. Therefore, we add some time-delay terms in (57) to get that

\[
\begin{align*}
\dot{x}_1 &= C - (D + 1)x_1 + x_1^2x_2 + d_1(t, \bar{x}_2) + h_1(x_1(t - \tau_1)), \\
\dot{x}_2 &= Dx_1 - x_1^2x_2 + (2 + \cos(x_1))u + d_2(t, \bar{x}_2) + h_2(\bar{x}_2(t - \tau_2)), \\
y &= x_1,
\end{align*}
\] (58)

where \(h_1(x_1(t - \tau_1))\) and \(h_2(\bar{x}_2(t - \tau_2))\) are the unknown time-delay functions. In the simulation, we choose \(C = 1, D = 3, d_1 = 0.7x_1^2 \cos(1.5t), d_2 = 0.5(x_1^2 + x_2^2) \sin^3(t), h_1(x_1) = 2x_1^2, h_2(\bar{x}_2) = 0.2x_2 \sin(x_2), \tau_1 = 1s, \text{ and } \tau_2 = 2s. \) The control objective is to design an adaptive fuzzy controller such that all the signals in the closed-loop system remain bounded and the system output \(y\) follows the given reference signal \(y_d = 3 + \sin(0.5t) + 0.5 \sin(1.5t).\) To this end, we define fuzzy membership functions as follows:

\[
\begin{align*}
\mu_{F_1} &= \exp \left[ -0.5(x + 1.5)^2 \right] / 4, & \mu_{F_2} &= \exp \left[ -0.5(x + 1)^2 \right] / 4, \\
\mu_{F_3} &= \exp \left[ -0.5(x + 0.5)^2 \right] / 4, & \mu_{F_4} &= \exp \left[ -0.5x^2 \right] / 4, \\
\mu_{F_5} &= \exp \left[ -0.5(x - 0.5)^2 \right] / 4, & \mu_{F_6} &= \exp \left[ -0.5(x - 1)^2 \right] / 4, \\
\mu_{F_7} &= \exp \left[ -0.5(x - 1.5)^2 \right] / 4.
\end{align*}
\] (59)
According to Theorem 1, construct the virtual control law, the true control law and the adaptive laws as follows:

\[
x_1 = -\frac{\dot{\theta}_1}{2\eta_1} \xi_1^T(Z_1)\xi_1(Z_1)e_1 - k_1e_1,
\]

\[
u = -\frac{\dot{\theta}_2}{2\eta_2} \xi_2^T(Z_2)\xi_2(Z_2)e_2 - k_2e_2,
\]

\[
\dot{\hat{\theta}}_i = \frac{\gamma_i}{2\eta_i} \xi_i^T(Z_i)\xi_i(Z_i)e_i^2 - \alpha_i\hat{\theta}_i, \quad i = 1, 2,
\]
where \( e_1 = x_1 - y_d, e_2 = x_2 - x_1, Z_1 = [e_1, y_d, \dot{y}_d]^T, Z_2 = [e_1, e_2, y_d, \dot{y}_d, \ddot{y}_d]^T, \) and the design parameters are chosen as \( k_1 = 10, k_2 = 11, \eta_1 = 0.25, \eta_2 = 0.5, \gamma_1 = 50, \gamma_2 = 80, \sigma_1 = 0.1 \) and \( \sigma_2 = 0.25. \) The simulation is run under the initial conditions \([x_1(0), x_2(0), \hat{\theta}_1(0), \hat{\theta}_2(0)]^T = [2.5, 1, 0, 0]^T.\) And the simulation results are shown in Figs. 1–4. Fig. 1 shows the system output \( y \) and the reference signal \( y_d. \) Fig. 2 shows the response of state variable \( x_2. \) Fig. 3 displays the control input signal \( u, \) and Fig. 4 shows the boundedness of adaptive parameters \( \hat{\theta}_1 \) and \( \hat{\theta}_2. \) From the simulation results, it can clearly be seen that the proposed controller guarantees the boundedness of all the signals in the closed-loop system, and also achieves the good tracking performance.
Example 2. To further show the effectiveness of our results, we apply the adaptive fuzzy tracking controller (60)–(62) to the following system which is considered in [6]

\[
\begin{aligned}
\dot{x}_1 &= (1 + x_1^2)x_2 + x_1 e^{-0.5x_1} + 2x_1^2(t - \tau_1), \\
\dot{x}_2 &= (3 + \cos(x_1x_2))u + x_1 x_2^2 + 0.2x_2(t - \tau_2) \sin(x_2(t - \tau_2)), \\
y &= x_1,
\end{aligned}
\]

(63)

where \(x_1\) and \(x_2\) denote state variables, \(u\) is the system control input and \(y\) is the system output. In this example, the fuzzy membership functions (59) are used. The reference signal is \(y_d = 0.5(\sin(t) + \sin(0.5t))\), which is the same as in [6]. Apply the controller (60)–(62) with \(k_1 = 25, k_2 = 25, \eta_1 = \eta_2 = 2, \gamma_1 = 1000, \gamma_2 = 2000, \sigma_1 = 0.2, \) and
The simulation is carried out under the initial conditions $[x_1(0), x_2(0), \hat{\theta}_1(0), \hat{\theta}_2(0)]^T = [0, 0, 0, 0]^T$ and $\tau_1 = \tau_2 = 2\nu$. The simulation results are shown by Figs. 5–8. Apparently, the simulation results show that under the action of the suggested controller, a good tracking performance has been achieved.

**Remark 6.** It is evident that the tracking performance depends on the design parameters. Theoretically, a good tracking performance can be achieved by choosing $k_i$ large enough or $\eta_i$ small enough. Then, how to choose the optimal parameters, such as $k_i$, $\eta_i$, $\gamma_i$ and so on, to get the optimal tracking performance is still an open problem. In the presented simulations, the design parameters are set using a trial-and-error method.

It should be pointed out that the existing adaptive fuzzy control approaches cannot be used to control systems (58) and (63) because of the existence of nonlinear delayed functions. However, the adaptive neural control approaches

\[ \sigma_2 = 0.15. \]
suggested in [6,10] can be applied to design the tracking controllers. Because all the ideal weights are regarded as the estimated parameters, when the controllers in [6,10] are implemented, there will be a large number of adaptive parameters which are required to be tuned online simultaneously. This considerably increases the computational burden. For instance, in the simulation studies of [6], 270 adaptive parameters are required to be tuned online to control system (63). Unlike [6,10], in this paper, a key technique is introduced to reduce the number of adaptive parameters, that is, the unknown constant $\theta_i = \frac{1}{\|\phi_i\|^2}$ is used as an estimated parameter, which results in only one adaptive parameter $\hat{\theta}_i$ for each virtual control law $x_i$. Therefore, for the second-order nonlinear time-delay systems (58) and (63), only two adaptive parameters $\hat{\theta}_1$ and $\hat{\theta}_2$ are required to be updated online.

5. Conclusion

In this paper, the adaptive fuzzy control methods proposed in [25,26,4] have been extended to a class of strict-feedback nonlinear systems with unknown time delays. By appropriately choosing Lyapunov–Krasovskii functionals and hyperbolic tangent functions, an adaptive fuzzy tracking control scheme has been presented for a class of perturbed strict-feedback nonlinear time-delay systems. The proposed adaptive fuzzy tracking controller guarantees the boundedness of all the signals in the closed-loop system, while the tracking error eventually converges to a small neighborhood of the origin. Moreover, the suggested adaptive fuzzy controller contains less adaptive parameters. This makes our design scheme easier to be implemented in practical applications. Simulation results have been given to illustrate the effectiveness of the proposed scheme.

References


