

# Noncommutative Gravity in Six Dimensions

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## Abstract

A gauge theory of gravity is defined in 6 dimensional non-commutative space-time. The gauge group is the unitary group  $U(2, 2)$ , which contains the homogeneous Lorentz group,  $SO(4, 2)$ , in 6 dimensions as a subgroup. It is shown that, after the Seiberg-Witten map, in the corresponding theory the lowest order corrections are first order in the non-commutativity parameter  $\theta$ . This is in contrast with the results found in non-commutative gauge theories of gravity with the gauge group  $SO(d, 1)$ .

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# 1 Introduction

Non-commutative (NC) space-times in which the space and time coordinates do not commute with each other,

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}(x), \quad (1)$$

unlike in an ordinary space-time, have caught the imagination of physicists since Heisenberg. In 1947 Snyder published the first papers that discussed physics in such space-times [1]. Since then there has been several interesting ideas on how to define physics on manifolds with NC geometry. This interest in defining gauge and gravity theories in NC space-times has considerably intensified in recent years since the seminal paper of Seiberg and Witten [2]. Although the formulation of non-commutative gauge theories is well established [3], the Einstein's theory of gravity or other gravity theories proved to be much more difficult to be deformed into versions in NC space-times. These so called NC Gravity theories suffer from the problem of not knowing how to deform general coordinate invariance and Lorentz symmetry of gravity theories into NC space-times. There has been several different approaches over the past, both before the paper of Seiberg and Witten and the after (see [4] and references therein). One approach of Chamseddine [5] is to define gravity as a gauge theory of a large group (this is  $U(2, 2)$  in [5]) which contains the inhomogeneous Lorentz group,  $ISO(3, 1)$ , of general relativity as a subgroup. This  $U(2, 2)$  group is then broken into  $SL(2, C) \times SL(2, C)$  by imposing some constraints and this way one obtains a version of NC gravity in 4 dimensions. One other approach is to twist the diffeomorphism symmetry of general relativity [6] and write the transformation rules in the enveloping algebra. However as noted in [7] the so called twisted transformations are not bona fide physical symmetries in the sense that one cannot derive Ward identities or Noether currents from the action by using them. However, the star-gauge transformations, which are obtained by simply deforming commutative theory gauge transformation rules by insertion of star products, are bona fide physical symmetries in the sense described above. Therefore a gauge theory of gravity might be an answer to above mentioned problem. Such a theory in 4 dimensions is described already by Calmet and Kobakhidze in [8]. There in order to escape from the non-invariance issue of  $[x^\mu, x^\nu] = i\theta^{\mu\nu}$  under diffeomorphisms (since  $\theta^{\mu\nu}$  is a constant tensor) the NC version of unimodular theory of gravity [9] is advertised. One common feature of all the above theories is that the NC gravity action can be written as the usual Einstein-Hilbert action plus infinite number of corrections in a power series of  $\theta$ . The NC gravity theory of Calmet and Kobakhidze [8] is based on gauging inhomogeneous Lorentz group in 4 dimensions and as noted by Mukherjee and Saha in [10] the first order corrections in  $\theta$  in that theory vanish. Actually Mukherjee and Saha's work has more general conclusions: in any dimensions with a single time coordinate, a NC gravity theory based on gauging inhomogeneous Lorentz group,  $ISO(d, 1)$ , the first order corrections will vanish. The second

order corrections in  $\theta$  as calculated in [11] have very complicated structure. This is a similar situation in other approaches too: the simple first order correction terms in  $\theta$  vanish and complicated second order corrections survive.

Therefore cosmological implications of these NC gravity theories are hard to determine. The complexity of the lowest order correction terms in  $\theta$  to the general relativity field equations prevents one to seek exact solutions or to derive modified Friedmann equations. In this paper we are going to report a new construction of NC gravity which has lowest NC corrections to the Einstein's equations proportional to  $\theta$  and moreover these corrections are relatively less complex than the non-vanishing lowest order  $\theta^2$  corrections in other NC gravity theories. This theory will thus be better suited to be used to determine the cosmological implications of a NC gravity theory.

Since the NC gauge theories are much more well defined than the NC versions of general relativity, we are going to define the NC gravity theory as a gauge theory of gravity in the way of Utiyama[12]. However, as will be explained below, the correction terms contain the d-coefficients of the anti-commutation relations between the generators of the gauge group. If one requires the gauge algebra to be closed also in the anti-commutation relations, then the gauge group can only be a unitary group. In the formulation of general relativity by Utiyama one gauges the homogeneous Lorentz group. The unique case in which the homogeneous "special orthogonal" Lorentz group of a Lorentzian space-time is isomorphic to a special unitary group is the Lorentz group in 4 space and 2 time dimensions:

$$SO(4, 2) \cong SU(2, 2). \quad (2)$$

However, anti-commutation relations of the generators of  $su(2, 2)$  do not close in the algebra. But the generators of  $u(2, 2)$  do. Therefore the group that we are going to gauge is  $U(2, 2)$  in 6 dimensions. Gauging a group larger than the inhomogeneous Lorentz group might seem going out of Utiyama's formalism at first, however, as will be seen, the contributions coming from the  $U(1)$  part to the action and the equations of motion will all vanish. It should be pointed out that our construction is different than the one by Chamseddine. In [5] Chamseddine gauges the unitary group in 4 dimensions and immediately breaks the gauge group into  $SL(2, C) \times SL(2, C)$  by imposing some constraints. Here we describe the theory in 6 dimensions, we do not impose any constraints and also the form of the action is different. As with the diffeomorphism symmetry we are going to adopt the prescription given in [8]. That is the NC gravity theory, that we will be describing, will be a unimodular gauge theory of gravity. The NC gravity theory described in this paper can be reduced to 4 dimensions either by two time physics [13] methods as in [14] or by considering a brane-world model in codimension 2. However, since there is an extra time dimension among the extra coordinates, the codimension 2 brane-world scenarios would need to be modified accordingly.

## 2 NC Gravity Theory in 6 Dimensions

There are several different ways to write the Einstein's theory of gravity as a gauge theory [12][15][16]. One can gauge either the homogeneous or inhomogeneous Lorentz groups, or a gauge group which contains either of these. In the latter situation one then imposes some constraints to obtain the Einstein–Hilbert action. In this paper we will use the Utiyama's formulation [12] in which one gauges the homogeneous Lorentz group. We will also not impose the torsionless condition beforehand, and therefore we will let the spin connection to be an independent field until the end. Then we will see that even though the NC torsion vanishes, the commutative torsion does not. Having the spin connection as an independent field is called Hilbert–Palatini formalism [17]. The Hilbert–Palatini action in 6 dimensions can be written as

$$S_U = -\frac{1}{2\kappa_6^2} \int d^6x e e_a^m e_b^n R_{mn}{}^{ab}(\omega) \quad (3)$$

$$= -\frac{1}{96\kappa_6^2} \int d^6x \epsilon^{m_1 \dots m_6} \epsilon_{a_1 \dots a_6} e_{m_1}^{a_1} e_{m_2}^{a_2} e_{m_3}^{a_3} e_{m_4}^{a_4} R_{m_5 m_6}{}^{a_5 a_6}(\omega), \quad (4)$$

where indices from the middle of the alphabet,  $m_i = 1, \dots, 6$ , are for the curved space coordinates and indices from the beginning of the alphabet,  $a_i = 1, \dots, 6$ , are for the tangent space coordinates. Here  $\kappa_6$  is related to the 6 dimensional Newton's constant and the Planck mass by  $\kappa_6^2 = 8\pi G_6 = M_6^{-4}$ . The Riemann tensor in terms of spin connection is given by

$$R_{mn}{}^{ab}(\omega) = \partial_m \omega_n{}^{ab} - \partial_n \omega_m{}^{ab} + \omega_m{}^{ac} \omega_{nc}{}^b - \omega_n{}^{ac} \omega_{mc}{}^b. \quad (5)$$

In Utiyama formalism one takes the spin connection as the gauge field of the homogeneous Lorentz group. We would like to deform this action into a NC gravity action in the standard way. That is we are going to replace each field with its NC counterpart and also replace each product with a star product. The star product that we are going to use is the Moyal product with  $\theta^{mn}$  constant. However, as we will comment in the last section, one can use more general forms of star product.

So we write the NC gravity action as

$$S_{NC} = -\frac{1}{96\kappa_6^2} \int d^6x \epsilon^{m_1 \dots m_6} \epsilon_{a_1 \dots a_6} \hat{e}_{m_1}^{a_1} \star \hat{e}_{m_2}^{a_2} \star \hat{e}_{m_3}^{a_3} \star \hat{e}_{m_4}^{a_4} \star \hat{R}_{m_5 m_6}{}^{a_5 a_6}(\hat{\omega}), \quad (6)$$

where

$$\star = e^{\frac{i}{2} \overleftarrow{\partial}_m \theta^{mn} \overrightarrow{\partial}_n}. \quad (7)$$

Now we would like to write this theory in terms of a commutative theory to see what corrections, if any, non-commutativity of space–time brings to the field equations of general relativity. Since we are gauging only the homogeneous Lorentz group, we might as well take

$$\hat{e}_m^a = e_m^a. \quad (8)$$

However, the gauge field in the NC theory should be related to the gauge field in the corresponding commutative theory through Seiberg–Witten map, which requires the gauge orbits in NC theory and the commutative theory to be the same. The Seiberg–Witten map written for the spin connection is

$$\hat{\omega}_m^{ab}(\omega) + \delta_{\hat{\lambda}} \hat{\omega}_m^{ab}(\omega) = \hat{\omega}_m^{ab}(\omega + \delta_{\lambda}\omega). \quad (9)$$

Infinitesimal transformation of commutative field  $\omega_m^{ab}$  is given by

$$\delta_{\lambda}\omega_m^{ab} = \partial_m \lambda^{ab} + \omega_m^{ac} \lambda_c^b - \omega_m^{bc} \lambda_c^a, \quad (10)$$

For the deformed field one assumes the same form, however with every commutative theory object is exchanged with the corresponding NC theory object as follows

$$\delta_{\hat{\lambda}} \hat{\omega}_m^{ab} = \partial_m \hat{\lambda}^{ab} + \hat{\omega}_m^{ac} \star \hat{\lambda}_c^b - \hat{\lambda}^{ac} \star \hat{\omega}_{mc}^b. \quad (11)$$

In order to find a solution to eq.(9), in [3, 18] both the gauge field and the gauge transformation parameter are written in a series expansion in the deformation parameter  $\theta$  as

$$\hat{\omega}_m^{ab} = \omega_m^{ab} + \omega_m^{(1)ab}(\omega) + O(\theta^2), \quad (12)$$

$$\hat{\lambda}^{ab} = \lambda^{ab} + \lambda^{(1)ab}(\lambda, \omega) + O(\theta^2), \quad (13)$$

where  $\omega_m^{(1)ab}(\omega)$  and  $\lambda^{(1)ab}(\lambda, \omega)$  are order  $\theta$  quantities. Then a solution to eq.(9) to first order in  $\theta$  is found to be [3, 18]

$$\hat{\omega}_m^{ab} = \omega_m^{ab} - \frac{i}{4} \theta^{kl} \{\omega_k, \partial_l \omega_m + R_{lm}\}^{ab} + O(\theta^2). \quad (14)$$

The deformed Riemann tensor is given by the usual prescription,

$$\hat{R}_{mn}^{ab}(\hat{\omega}) = \partial_m \hat{\omega}_n^{ab} - \partial_n \hat{\omega}_m^{ab} + \hat{\omega}_m^{ac} \star \hat{\omega}_{nc}^b - \hat{\omega}_n^{ac} \star \hat{\omega}_{mc}^b. \quad (15)$$

Up to first order in  $\theta$  this can be easily calculated:

$$\hat{R}_{mn}^{ab} = R_{mn}^{ab} + \mathring{R}_{mn}^{ab} \quad (16)$$

$$\mathring{R}_{mn}^{ab} = \frac{i}{2} \theta^{kl} \left[ \{R_{mk}, R_{nl}\}^{ab} - \frac{1}{2} \{\omega_k, (\partial_l + \nabla_l) R_{mn}\} \right] \quad (17)$$

$$= \frac{i}{2} \theta^{kl} R_{mk}^{cd} R_{nl}^{ef} d_{cd,ef}^{ab} - \frac{i}{4} \theta^{pr} \omega_k^{cd} (\partial_l + \nabla_l) R_{mn}^{ef} d_{cd,ef}^{ab}, \quad (18)$$

where  $d_{cd,ef}^{ab}$  are the d-coefficients of the gauge group. We name  $\mathring{R}_{mn}^{ab}$  as the deformed curvature tensor. Plugging these expression into the NC gravity action, one obtains up to the first order in  $\theta$

$$S_{NC} = -\frac{1}{96\kappa_6^2} \int d^6x \epsilon^{m_1 \dots m_6} \epsilon_{a_1 \dots a_6} e_{m_1}^{a_1} e_{m_2}^{a_2} e_{m_3}^{a_3} e_{m_4}^{a_4} \left( R_{m_5 m_6}^{a_5 a_6}(\omega) + \mathring{R}_{m_5 m_6}^{a_5 a_6}(\omega) \right) \quad (19)$$

$$= -\frac{1}{2\kappa_6^2} \int d^6x e e_a^m e_b^n \left( R_{mn}^{ab}(\omega) + \mathring{R}_{mn}^{ab}(\omega) \right) \quad (20)$$

$$= -\frac{1}{2\kappa_6^2} \int d^6x e \left( R(\omega) + \mathring{R}(\omega) \right), \quad (21)$$

where we have made the definition

$$\mathring{R}(\omega) = e_a^m e_b^n \mathring{R}_{mn}{}^{ab}(\omega). \quad (22)$$

We are going to call this quantity as the deformed curvature scalar.

Next we would like to calculate the d-coefficients of the gauge group  $U(2, 2)$  and show that not all parts of the above NC correction term, i.e. deformed curvature scalar, vanish. In order to calculate the d-coefficients of anti-commutation relations of the algebra  $u(2, 2)$  we used the isomorphism  $u(2, 2) \cong Cl(2, 2) \cong o(4, 2)$ , where the  $Cl(2, 2)$  is the Clifford algebra, generated by 4 gamma matrices [18][19]. Denoting the  $so(4, 2)$  generators as  $\gamma_{ab}$  one can write their commutation and anti-commutation relations from the corresponding relations in  $u(2, 2) \cong Cl(2, 2)$ .  $\gamma_{ab}$  obey the Lorentz algebra in 6 dimensions:

$$[\gamma_{ab}, \gamma_{cd}] = 2\eta_{ad} \gamma_{bc} + 2\eta_{bc} \gamma_{ad} - 2\eta_{ac} \gamma_{bd} - 2\eta_{bd} \gamma_{ac}, \quad (23)$$

where  $\eta_{ab} = \text{sign}(+, -, -, -, +, -)$ . One can write the anti-commutation relations and calculate the d-coefficients from the corresponding expressions in  $u(2, 2) \cong Cl(2, 2)$ . We find them as

$$\{\gamma_{ab}, \gamma_{cd}\} = 2i\epsilon_{abcd}{}^{ef} \gamma_{ef} + 2(g_{ad} g_{bc} - g_{ac} g_{bd}) \mathbb{I}. \quad (24)$$

It is in this relations that one sees the need to gauge  $u(2, 2) \cong o(4, 2)$ , but not  $su(2, 2) \cong so(4, 2)$ . The  $u(1)$  part explicitly appears on the right hand side. Plugging the expression for d-coefficients into (18) we find the deformed Ricci scalar (22) as

$$\mathring{R}(\omega) = -\theta^{kl} e^{\epsilon^{prstuv}} \left( R_{pkst} R_{rluv} - \frac{1}{2} \omega_{kst} (\partial_l + D_l) R_{pruv} \right) - 2\theta^{kl} (R_{mk}{}^{ab} R^m{}_{lab}). \quad (25)$$

By using the anti-symmetry of  $\theta^{kl}$  tensor and the symmetry properties of the Riemann tensor as in [10], it can be shown that the last term vanishes. Thus, up to order  $\theta$ , the NC correction term is

$$\mathring{R}(\omega) = -\theta^{kl} e^{\epsilon^{prstuv}} \left( R_{pkst} R_{rluv} - \frac{1}{2} \omega_{kst} (\partial_l + D_l) R_{pruv} \right). \quad (26)$$

To obtain the deformed Einstein field equations we plug (26) into the action (21) and then vary it with respect to  $e_a^m$ . We obtain

$$R_{mn}(\omega) + \mathring{R}_{mn}(\omega) - \frac{1}{2} \left( R(\omega) + \mathring{R}(\omega) \right) g_{mn} = 0 \quad (27)$$

as the deformed Einstein field equations in vacuum. Here we also defined a deformed Ricci tensor as

$$\mathring{R}_{mn}(\omega) = \mathring{R}_{mr}{}^{ab}(\omega) e_b^r e_{na} \quad (28)$$

$$= -\theta^{kl} e^{\epsilon_m{}^{rstuv}} \left( R_{nkst} R_{rluv} - \frac{1}{2} \omega_{kst} (\partial_l + D_l) R_{nrsv} \right). \quad (29)$$

The field equation (27) has a very simple form and since the NC correction terms  $\mathring{R}_{mn}(\omega)$  and  $\mathring{R}(\omega)g_{mn}$  are first order in  $\theta$ , they are also not very complicated. We note that the existence of these non-trivial first order  $\theta$  corrections to equations of motion in our NC gravity theory is due to the fact that we are gauging the  $SO(4, 2)$  group, but not the  $SO(5, 1)$  group.

The equation (27) is the main result of this paper. But before commenting on this equation we would like to discuss the torsion induced due to the non-commutativity of the coordinates. As it is stated we work in the Hilbert–Palatini formalism and therefore we treat the spin connection independent of the vierbein degrees of freedom. Then we need also to vary the action (6) with respect to the NC spin connection. This variation gives us the “non-commutative” torsionless condition

$$\hat{D}_{[m} \hat{e}_{n]}^a = 0, \quad (30)$$

which is, due to (8), equivalent to

$$\hat{D}_{[m} e_{n]}^a = \partial_{[m} e_{n]}^a + \hat{\omega}_{[m}{}^{ab} e_{n]b} = 0. \quad (31)$$

Writing the spin connection of NC theory in terms of spin connection of commutative theory up to order  $\theta$  (14) we find

$$\partial_{[m} e_{n]}^a + \omega_{[m}{}^{ab} e_{n]b} - \frac{i}{4} \theta^{kl} \{ \omega_k, \partial_l \omega_{[m} + R_{l[m} \}^{ab} e_{n]b} = 0. \quad (32)$$

Then the torsion of the corresponding commutative theory is

$$T_{mn}{}^a = -2D_{[m} e_{n]}^a = -\frac{i}{2} \theta^{kl} \{ \omega_k, \partial_l \omega_{[m} + R_{l[m} \}^{ab} e_{n]b}, \quad (33)$$

where we used the convention of [19]. Therefore there is a non-vanishing torsion in the commutative theory and it is proportional to the non-commutativity tensor. In a sense the non-commuting nature of the coordinates creates this torsion.

### 3 Comments and Conclusions

We summarize the main aspects of the construction presented in this paper before commenting on the main results. We defined a gauge theory of gravity in 6 dimensional space-time with non-commuting coordinates. The reason that we formulated the gravity theory as a gauge theory is because the deformations of gauge theories is much better defined than the deformation of gravity theories. The non-invariance of the constant tensor  $\theta^{\mu\nu}$  forces one to consider the unimodular theory of gravity in NC space-time as in [8]. We followed the Utiyama’s approach in formulation of gauge theory of gravity and chose the gauge group as the homogeneous Lorentz group in 6 dimensions. This group, being  $SO(4, 2)$ , is isomorphic to the special unitary group  $SU(2, 2)$ . In NC gauge theories, after the Seiberg–Witten map,

the corrections to the gauge field (here the spin connection) or to the field strength (here the curvature tensor) contain the d-coefficients of the anti-commutation relations of Lie algebra generators and those anti-commutation relations close only in the case of unitary groups. Therefore we gauged  $U(2, 2)$  instead of  $SO(4, 2)$ . In fact the reason that we worked in 6 dimensions after all is due to the fact that only in 6 dimensional Lorentzian space-time with 4 space and 2 time dimensions the homogeneous Lorentz group is isomorphic to a unitary group. For space-times whose homogeneous Lorentz group is  $SO(d, 1)$  it is already shown [10] that the first order corrections in  $\theta$  of NC gravity to Einstein-Hilbert action vanish. In contrast in our construction those corrections do not vanish. Since in the other constructions the second order corrections in  $\theta$  of NC gravity to the Einstein-Hilbert action or to the field equations are very complicated, there is little hope to see the effects of those correction terms in a cosmological setting. However, the first order corrections in our case are relatively simple and might have some relevance on the cosmological problems.

To make contact with cosmology and to assess the effects of non-commutativity of coordinates on cosmology through NC gravity one needs to reduce the theory, that we described in 6 dimensions, into 4 dimensions. This can be done in several ways. Since this theory is described in a space-time with two time dimensions, one way is to use the techniques of two-time physics [13] as described in [14] (see also [20] for similar ideas). Having two time-like directions in the space-time might seem, at first look, unphysical. To make sense of such a theory one needs to derive a “one-time” theory from the theory with two time-like dimensions. The basic idea behind the two-time physics (see [21] for reviews of two-time physics) is that the “evolution parameter” which will be interpreted as the physical time in lower dimensional one-time theory is either a gauge choice in the higher dimensional two-time theory (in the case of particle and tensionless brane theories), or it is obtained by imposing some kinematical constraints (in the case of field theory). That is we do not interpret any of the time-like coordinates in space-time with two times as evolution parameters. The true evolution parameter, therefore the physical time, is the coordinate one obtains as a gauge choice in the subspace of two time and one extra space dimensions. Time is also a gauge choice in general relativity and in that sense the treatment of time in two-time physics is similar. In the case of particle theories, the gauge symmetry that one uses to reduce a two-time theory to a one-time theory is the  $Sp(2, R)$  symmetry of the phase space, promoted to a local symmetry. In the case of field theories defined on space-times with two times, the kinematical constraints, that one needs to impose in order to obtain a physical field theory with one time, obey the same  $Sp(2, R)$  algebra. The two-time physics is the only rigorous way to make sense of a theory defined on a space-time with two time-like coordinates. The problems with causality and unitarity do not exist in two-time physics, because the final theory has only one time as the evolution parameter with a well defined Hamiltonian. Two-time physics reduction of the present theory will be done in a future

publication. The result of that research is expected to be a non-trivial modification of the Einstein equations in 4 dimensional space–time with NC corrections in the first order in  $\theta$ .

One other way of reducing the theory to 4 dimensions is to consider a brane–world model in codimension 2 and to find the induced gravity field equations or the modified Friedmann equations on the 3–brane. Here again due to having two time–like dimensions one cannot construct a conventional codimension 2 brane–world model, but should consider embedding a 3–brane into a space–time with two times. Isometric embedding of BPS branes in space–times with two times is analyzed some time ago by L. Andrianopoli et al. in [22]. As it is commented there the minimal embedding of world–volume geometry of a genuine BPS brane of string theory in a higher dimensional space–time requires at least two extra dimensions [23] and that the higher dimensional space–time has to have at least two time–like dimensions [24]. Therefore 6 dimensional space–time with two times is the minimal choice to embed a BPS 3–brane of string theory [22]. Done either with the methods of two–time physics or by embedding a BPS 3–brane in the 6 dimensional space–time with two times, the main aim of reducing the theory to 4 dimensions would be first to determine what modifications of the NC gravity theory described in this paper will survive in 4 dimensions and then how this modifications of Einstein equations will modify the Friedmann equations and therefore the evolution of the universe. Specifically we would like to understand whether the non–commutativity of the coordinates and the NC gravity has anything to say about the still unsolved dark matter and dark energy problems. Modified Friedmann equations could be the first step in the direction of such an understanding [25].

Defining NC gravity theory in a higher dimensional space–time opens up also exciting new possibilities. Now it is possible to have just the extra dimensions non–commutative and therefore get rid of all the problems created in 4 dimensions by the non–commuting nature of the coordinates. In such a scenario, since 4 dimensional space–time would be an ordinary space–time with commuting coordinates, the breaking of Lorentz invariance in 4 dimensions could be avoided and stringent bounds [26][27] on the value of the non–commutativity parameter  $\theta$  could be lifted.

As it is commented before eq.(6) the star product need not to be the Moyal product (7). More general forms of star products, even with coordinate dependent  $\theta^{mn}$ , can be used. For example one can try a construction similar to the one described in [28] by using the Rieffel product [29]. In this case, it is again possible to restrict the non–commutativity into just the extra dimensions. Then the only non-zero components of  $\theta^{mn}$  will be  $\theta^{56} = -\theta^{65}$ . These components may be made to depend on 4D coordinates on the 3–brane and consequences of this position dependent non–commutativity can be analyzed. Works in the mentioned lines of research are still in progress. The NC gravity theories in higher dimensions have many promising avenues of research and it will be exciting to see whether they will help us to answer some of the profound questions in 4 dimensional cosmology.

## References

- [1] H. S. Snyder, Phys. Rev. **71**, 38 (1947); Phys. Rev. **72**, 68 (1947).
- [2] N. Seiberg and E. Witten, JHEP **9909**, 032 (1999) [arXiv:hep-th/9908142].
- [3] B. Jurco, L. Moller, S. Schraml, P. Schupp and J. Wess, Eur. Phys. J. C **21**, 383 (2001) [arXiv:hep-th/0104153].
- [4] R. J. Szabo, “Symmetry, gravity and noncommutativity,” arXiv:hep-th/0606233.
- [5] A. H. Chamseddine, J. Math. Phys. **44**, 2534 (2003) [arXiv:hep-th/0202137].
- [6] P. Aschieri, C. Blohmann, M. Dimitrijevic, F. Meyer, P. Schupp and J. Wess, Class. Quant. Grav. **22**, 3511 (2005) [arXiv:hep-th/0504183]; P. Aschieri, M. Dimitrijevic, F. Meyer and J. Wess, Class. Quant. Grav. **23**, 1883 (2006) [arXiv:hep-th/0510059]; F. Meyer, “Noncommutative spaces and gravity,” arXiv:hep-th/0510188.
- [7] L. Alvarez-Gaume, F. Meyer and M. A. Vazquez-Mozo, “Comments on noncommutative gravity,” arXiv:hep-th/0605113.
- [8] X. Calmet and A. Kobakhidze, Phys. Rev. D **72**, 045010 (2005) [arXiv:hep-th/0506157].
- [9] A. Einstein, Sitz. Preuss. Acad. Scis., (1919); “Do Gravitational Fields Play an Essential Role in the Structure of Elementary Particles of Matter?” in “The Principle of Relativity,” edited by A. Einstein et al. (Dover, New York, 1952); J. J. van der Bij, H. van Dam and Y. J. Ng, Physica **116A**, 307 (1982); F. Wilczek, Phys. Rept. **104**, 143 (1984); W. Buchmuller and N. Dragon, Phys. Lett. B **207**, 292 (1988); M. Henneaux and C. Teitelboim, Phys. Lett. B **222**, 195 (1989); W. G. Unruh, Phys. Rev. D **40**, 1048 (1989).
- [10] P. Mukherjee and A. Saha, “Comment on the first order noncommutative correction to gravity,” arXiv:hep-th/0605287.
- [11] X. Calmet and A. Kobakhidze, “Second order noncommutative corrections to gravity,” arXiv:hep-th/0605275.
- [12] R. Utiyama, Phys. Rev. **101**, 1597 (1956).
- [13] I. Bars, C. Deliduman and O. Andreev, Phys. Rev. D **58**, 066004 (1998) [arXiv:hep-th/9803188].
- [14] I. Bars, Phys. Rev. D **62**, 085015 (2000) [arXiv:hep-th/0002140]; Phys. Rev. D **62**, 046007 (2000) [arXiv:hep-th/0003100].

- [15] S. W. MacDowell and F. Mansouri, Phys. Rev. Lett. **38**, 739 (1977) [Erratum-ibid. **38**, 1376 (1977)].
- [16] K. S. Stelle and P. C. West, Phys. Rev. D **21**, 1466 (1980).
- [17] P. Peldan, Class. Quant. Grav. **11**, 1087 (1994) [arXiv:gr-qc/9305011].
- [18] A. H. Chamseddine, Int. J. Geom. Meth. Mod. Phys. **3**, 149 (2006) [arXiv:hep-th/0511074].
- [19] P. Van Nieuwenhuizen, Phys. Rept. **68**, 189 (1981).
- [20] C. R. Preitschopf and M. A. Vasiliev, Nucl. Phys. B **549**, 450 (1999) [arXiv:hep-th/9812113]; W. Chagas-Filho, “Gravitation with two times,” arXiv:hep-th/0604076.
- [21] I. Bars, AIP Conf. Proc. **589**, 18 (2001) [AIP Conf. Proc. **607**, 17 (2001)] [arXiv:hep-th/0106021]; Class. Quant. Grav. **18**, 3113 (2001) [arXiv:hep-th/0008164].
- [22] L. Andrianopoli, M. Derix, G. W. Gibbons, C. Herdeiro, A. Santambrogio and A. Van Proeyen, Class. Quant. Grav. **17**, 1875 (2000) [arXiv:hep-th/9912049]; “Embedding branes in flat two-time spaces,” arXiv:hep-th/0003023.
- [23] L. P. Eisenhart, *Riemannian Geometry*, Princeton University Press, Princeton, NJ, 1997.
- [24] R. Penrose, Rev. Mod. Phys. **37**, 215 (1965).
- [25] P. D. Mannheim, Prog. Part. Nucl. Phys. **56**, 340 (2006) [arXiv:astro-ph/0505266].
- [26] S. M. Carroll, J. A. Harvey, V. A. Kostelecky, C. D. Lane and T. Okamoto, Phys. Rev. Lett. **87**, 141601 (2001) [arXiv:hep-th/0105082].
- [27] T. Jacobson, S. Liberati and D. Mattingly, Phys. Rev. D **66**, 081302 (2002) [arXiv:hep-ph/0112207].
- [28] V. Gayral, J. M. Gracia-Bondia and F. Ruiz Ruiz, Nucl. Phys. B **727**, 513 (2005) [arXiv:hep-th/0504022].
- [29] M. A. Rieffel, *Deformation Quantization for Actions of  $R^d$* , Memoirs of the Amer. Math. Soc. **506**, Providence, RI, 1993.