6-DOF isotropic parallel manipulators with three PPSR or PRPS chains

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Abstract—A 6-DOF parallel manipulator with three PPSR chains is commonly used as a minimanipulator that requires accuracy and dexterity. The desired properties can be significantly improved if the manipulator is close to an isotropic configuration. This paper presents methods for developing 6-DOF isotropic parallel manipulators with three PPSR or PRPS chains. Many isotropic designs can be directly obtained from isotropy generators that consist of six straight lines satisfying the isotropy conditions. Some almost isotropic designs with desired link parameters are also developed using isotropy generators and/or optimization methods.

Keywords: isotropy, parallel manipulator, PPSR, PRPS

I. Introduction

6-DOF Steward-Gough parallel manipulators have been studied by many researchers and have been employed in various applications. The manipulators possess high accuracy, rigidity, capacity, and load-to-weight ratio, but their workspaces are smaller than those of serial manipulators. A larger workspace can be expected if a parallel manipulator comprises fewer limbs that are driven by two or more actuators. Kohli et al. [1] presented manipulators with three kinematic chains and each chain is driven by a 2-DOF base-mounted rotary-linear actuator. Tsai and Tahmasebi [2] introduced a new class of 6-DOF parallel minimanipulators for providing high resolution and high stiffness for fine position and force control in a hybrid serial-parallel manipulator system. Majid et al [3] analyzed the workspace of a 6-DOF parallel manipulator with three PPSR chains and showed that its workspace is larger than that of a Stewart platform. For the design with three PPSR chains, only a fraction of an actuator joint rate/force is employed to generate the twist/wrench of the mobile platform, so they provide high resolution for fine position and force control. However, high resolution also indicates that a manipulator might be close to an inverse kinematic configuration where computation errors are generally very large in instantaneous and kinetostatic analysis. An isotropic design can get around this drawback because we can expect better computation accuracy, and a manipulator can be controlled equally well in all possible directions at an isotropic configuration.

This paper proposes methods for developing isotropic designs with three PPSR or PRPS chains driven by six linear actuators. Two special isotropy generators whose six straight lines intersect at three different points are first proposed. The generators and some additional isotropy conditions are then utilized to determine the directions of all prismatic axes. Many isotropic designs can be easily developed using the generators and the related data of the prismatic axes. Furthermore, we present two almost isotropic designs with desired link parameters. The design with PRPS chains is developed from an isotropic manipulator and then modified using an optimization method. The first two joints of the design constitute a cylindrical pair with the axis in a horizontal plane, and the third prismatic axis is perpendicular to the cylindrical axis. Because the designs with 3-PPSR chains are commonly used as minimanipulators, the other almost isotropic design with PPSR chains is developed using a general optimization method in which one of the parameters can be employed to determine the resolution (in position or force control) of the obtained manipulators.

II. Isotropy generators and isotropy conditions

A 6-DOF isotropic Steward-Gough parallel manipulator can be easily obtained by placing the tool center point at the origin of the reference frame of an isotropy generator, and the two spherical joints (of each chain) on the corresponding straight line of the isotropy generator [4]. This approach, however, cannot be directly employed to develop the manipulators with only three kinematic chains. This section proposes isotropy generators and isotropy conditions for developing 6-DOF isotropic manipulators with three PRPS or PPSR chains.

6-DOF manipulators with three PRPS or PPSR chains are shown in Fig. 1. Each chain of the manipulators has one passive revolute joint and one passive spherical joint, so we can find two zero-pitch screws (related to the two actuated screws) that are reciprocal to the screws associated with the two passive joints. The two straight lines corresponding to the two reciprocal screws must intersect at the center point of the spherical joint. Therefore, only some special isotropy generators with three pairs of straight lines intersect at three different points can be used for developing isotropic manipulators. The generators with orthogonal or intersecting axes can be found in the literature [5], and two of the generators with
desired properties are shown in Fig. 2. The origin of the reference frame attached on the first generator is at the geometric center of the tetrahedron, while the origin of the second generator is at the geometric center of the equilateral triangle $B_iB_jB_k$. In this paper, the coordinate frame with the origin at the geometric center of triangle $B_iB_jB_k$ and the x-axis in the direction of $B_iB_k$ is used as the reference frame for both generators. The related data of the two generators are provided in Table 1, where GCP denotes the geometric center point, $S = OB_i$, and unit vector $e_i$ represents the direction of the straight line $L_i$ on the generators. Note that there are three straight lines through point $B_i$ of the first generator. In this paper, we associate $L_{2i-1}$ and $L_{2i}$ with point $B_i$ for $i = 1, 2, 3$.

For the manipulators with three PRPS or PPSR chains, the wrench acting on the tool center point (denoted by TCP) can be related to the actuator force $\tau_i$ by

$$[F] = [s_1, s_2, s_3, s_4, s_5, s_6] \cdot [u_{1i}^2, u_{2i}^2, u_{3i}^2, u_{4i}^2, u_{5i}^2, u_{6i}^2]$$

$$= H \tau_i$$

(1)

where $F$ and $C$ represent, respectively, the resultant force and couple, $u_i$ denotes the direction of the $i^{th}$ prismatic joint, and $a_i$ denotes the position vector from the TCP to spherical joint $A_i$. Unit vectors $s_{2m-1}$ and $s_{2m}$ satisfy $s_{2m-1} \cdot u_{2m} = 0$ and $s_{2m} \cdot u_{2m-1} = 0$ for $m = 1, 2, 3$. The $j^{th}$ column of matrix $H$ represents the Plücker coordinates of the unit screw that is reciprocal to all the screws on a chain except the $j^{th}$ actuated screw. Let $N_j$ denote the straight line associated with the $j^{th}$ reciprocal screw. Then $N_{2k-1}$ and $N_{2k}$ go through spherical joint $A_k$ and intersect the passive revolute axis on chain $k$ for $k = 1, 2, 3$. An isotropic design can be obtained if the following conditions are satisfied:

1. $[s_i \cdot u_i]$ yields the same non-zero value for $i = 1, 2, ..., 6$.
2. The revolute axis on chain $k$ is in a plane that is through $A_k$ and spanned by $s_{2k-1}$ and $s_{2k}$.
3. $L_j = N_j$ for $j = 1, 2, ..., 6$ and GCP = TCP of any isotropy generator in Fig. 2.

In the third condition, $L_i = N_i$ indicates that $B_i = A_i$ for $i = 1, 2, 3$, and $e_j = s_j$ for $j = 1, 2, ..., 6$. Many isotropic designs can be developed from the proposed conditions.

III. Isotropic designs

Unit vectors $u_{2k-1}$ and $u_{2k}$ that represent the direction of the two prismatic joints on chain $k$ can be expressed as functions of four variables. Since $e_i = s_i$ at an isotropic configuration, the variables must satisfy the following constraint equations:

$$e_{2k-1} \cdot u_{2k} = 0$$

$$e_{2k-1} \cdot u_{2k-1} = 0$$

(2)

where $e_{2k-1}$ and $e_{2k}$ are the two corresponding unit vectors of a given isotropy generator. Therefore, we can only specify one desired link parameter for the two prismatic joints.

To simplify kinematic analysis, obtain a larger workspace, or reduce link interactions, we prefer manipulators with some special link parameters, such as $\alpha = 0, \alpha = 0$ or $90^\circ$. Let $u_i = [c\alpha, s\beta, c\alpha, s\beta, c\beta]$. Then some desired link parameters can be obtained if the following conditions are satisfied:

1. $\cos \beta_{2k-1} = 0$ or $\cos \beta_{2k} = 0$ for $k = 1, 2, 3$. The direction of at least one prismatic joint of each chain is parallel to a horizontal plane, so we can expect a higher payload capacity.
2. $u_{2k-1} \cdot u_{2k} = 0$ for $k = 1, 2, 3$. In this case, three pairs of perpendicular prismatic axes can be obtained.

A manipulator is close to an inverse kinematic singular configuration (where a manipulator will lose one or more degrees of freedom) if any inner product $e_i \cdot u_i$ approaches zero. Therefore, a larger singularity-free workspace can be obtained for a larger $|e_i \cdot u_i|$ for $i = 1, 2, ..., 6$. The isotropic design with the optimum inner product of $e_i$ and $u_i$ can be obtained by letting $e_{2k-1} \cdot u_{2k-1} = e_{2k} \cdot u_{2k} = c = 1$ and solving Eq. (2). If no real solution is obtained, then repeat reducing $c$ and solving the equation until the equation yields real solutions. The obtained solutions for the desired link parameters are given in Table 2, where $z = [0 \ 0 \ 1]$ represents the direction of the $z$-axis, $w_k = u_{2k-1} \times u_{2k}$ denotes the normal vector of the plane spanned by the two prismatic axes on the $k^{th}$ chain. A manipulator has better resolution in position or force control with a smaller $|e_i \cdot u_i|$, but the manipulator is also relatively close to inverse kinematic singularities. In general, we prefer a 3-PRPS manipulator with $|z \cdot u_{2k-1}| = 0$ and $|u_{2k-1} \cdot u_{2k}| = 0$, and a 3-PPSR manipulator with $|z \cdot w_k| = 1$ and $|u_{2k-1} \cdot u_{2k}| = 0$. 
How to develop a 3-PRPS isotropic design with optimum $|e_i \cdot u_i|$ from the fifth solution in Table 2 is illustrated in Fig. 3, where $u_1$ and $u_2$ are in a plane spanned by $e_1$ and $e_2$ with $e_i \cdot u_i = e_i \cdot u_2 = \cos 30^\circ$. First, draw a straight line $M_1$ through point $A_1$ and in the opposite direction of $u_1$. Next, obtain an arbitrary point $Q_1$ on the straight line and draw another straight line $M_1$ through point $Q_1$ and in the direction of $u_1$. Straight line $M_1$ is used as the first prismatic axis and the second revolute axis (the two joints constitute a 2-DOF cylindrical joint), and $M_2$ is used as the third prismatic axis. Figure 4 shows a 3-PPSR isotropic design developed from the same solution. Points $Q_1$ and $Q_2$ are on the two straight lines, $L_1$ and $L_2$, through point $A_1$ and in the direction of $e_1$ and $e_2$ respectively. The straight line $L'$ through $Q_1$ and $Q_2$ is used as the revolute axis, and $u_1$ and $u_2$ represent the directions of the two prismatic joints. The link connecting the spherical joint $A_1$ and the revolute axis is on the straight line which is through point $A_1$ and perpendicular to $L'$. The remaining two chains of the two manipulators can be generated by similar procedures. The related data of the isotropic designs with one specified link parameter can be used as the initial values to develop almost isotropic designs with more desired link parameters.

IV. Objective function for almost isotropic designs

If $|s_i \cdot u_i|$ yields the same non-zero value for $i = 1, 2, \ldots, 6$, then the closeness to an isotropic configuration can be evaluated by the previously proposed normalized isotropy measure [6]:

$$
\mu = \left( \frac{\sigma_{a1} \cdot \sigma_{p3}}{\sigma_{a1} \cdot \sigma_{pi}} \cdot \Phi \right)^{\frac{1}{2}}
$$

with

$$
\Phi = \sqrt{1 - \left( \frac{\sigma_{a1} + \sigma_{p1}}{2} \right)^2}
$$

where $(\sigma_{p3}, \sigma_{pi})$ and $(\sigma_{a1}, \sigma_{a1})$ denote pairs of the smallest and largest singular values of the two $3 \times 6$ submatrices (denoted by $H_p$ and $H_a$) consisting of the first three rows and the last three rows of matrix $H$ in Eq. (1) respectively, and $\sigma_a$ and $\sigma_p$ denote respectively the smallest and largest singular values of $\hat{H}_p \hat{H}_a^*$ in which $\hat{H}_p$ and $\hat{H}_a$ are two matrices with orthonormal row vectors that span the same row spaces of $H_p$ and $H_a$ respectively. The measure yields the optimum value of one at isotropic configurations, so $\mu$ can be used as the objective function in an optimization method for developing almost isotropic designs. The measure is independent of the size of a manipulator and the physical unit of $H$, but it is not zero at singular configurations (except for some special cases where the row vectors of $H_a$ or $H_p$ are linear dependent, or the row vectors of $H_a$ and $H_p$ span the same subspace). Therefore, it is possible that $\mu$ may yield a relatively large value even at a singular configuration.

At an isotropic configuration, $\sigma_{p1} = \sigma_{pi} = \sqrt{2}$ and $\sigma_{a1} = \sigma_{a1} = \sigma_\nu$, where $\sigma_\nu$ depends on the size of manipulators and physical units, and different values of $\sigma_\nu$ may be obtained if a manipulator can reach multiple isotropic positions. In this paper, we assume that $\sigma_\nu > \sqrt{2}$ (even if $\sigma_\nu$ is less than $\sqrt{2}$, we can change the physical unit to make $\sigma_\nu > \sqrt{2}$). Let $\sigma_{\mu_{\min}}$ denote the minimum singular value of matrix $H$. Then we will use the following index as the objective function for developing almost isotropic designs:

$$
\mu_{\nu} = \left( \frac{\sigma_{\mu_{\min}} \cdot \sigma_{a1} \cdot \sigma_{p3} \cdot \sigma_{p1}}{\sqrt{2} \cdot \sigma_{p1} \cdot \sigma_{a1} \cdot \Phi} \right)^{\frac{1}{2}}
$$

One drawback of the index is that the physical unit of $\sigma_{\mu_{\min}}$ is not defined, and $\mu_{\nu}$ is not invariant to the size of a manipulator or changes of physical units. However, the index yields one at isotropic configurations and zero at singular configurations.

V. A 3-PRPS almost isotropic design

This section presents an almost 3-PRPS isotropic design and compares its kinematic properties with those of an isotropic design. The almost isotropic design is similar to the 3-PRPS design shown in Fig. 1a. From Fig. 3, if we let $e'_2 = u_2$ and $e'_1 = e_1$, then vector $e'_1$ is in a horizontal plane and $e'_2$ is perpendicular to $e'_1$ (note that unit vector $e'_i$ represents the direction of a straight line on an almost isotropy generator). Let $s_i = e'_i$. Then Eq. (2) yields $u_i = s_i$. Therefore $e'_i = u_i = s_i$ (i = 1, 2, ..., 6) for the almost isotropic design. With $S = 50$, and the TCP at the GCP of the tetrahedron generator, the evaluation of function $\mu_{\nu}$ gives 0.849. To search for a new generator with a better $\mu_{\nu}$, we express the elements of matrix $H$ as functions of two variables: the z-coordinate of the new
TCP and a rotation angle \( \theta \) of \( s_{2k} \) about \( s_{2k-1} \) for \( k = 1, 2, 3 \). Let the coordinates of the new TCP be \([0, 0, z]\) and \( s_{2k} = \text{rot}(s_{2k-1}, \theta)s_{2k-1} \). The optimization of objective function \( \mu \) gives \( \mu = 0.9028 \), \( z = 8.22 \), and \( \theta = -6.16^\circ \). The related data of the new generator are provided in Table 3. With \( u_i = e_i \), we can obtain the desired almost isotropic design using the same procedure for developing the isotropic design in Fig. 3. Next, we will compare the singularity-free joint spaces and global isotropy of the isotropic design and the almost isotropic design.

Inner product \( s_i \cdot u_i = 0.866 \) (for \( i = 1, 2, ..., 6 \)) at any configuration for the isotropic design, and \( s_i \cdot u_i = 1 \) for the almost isotropic design. Therefore, we can rearrange Eq. (1) as

\[
\begin{bmatrix} F' \\ C' \end{bmatrix} = (s_i \cdot u_i)H_T = H' \tau
\]

The minimum singular value of \( H' \) is used to determine singularity-free actuator ranges of the two designs. The actuator ranges are obtained by a grid-scanning technique. The initial actuator ranges are first determined as \( \rho_{i m a x} = \rho_{i m} + \Delta \rho_i \) and \( \rho_{i m m} = \rho_{i m} - \Delta \rho_i \) for \( i = 1, 2, ..., 6 \) with \( \rho_{i m} \) denoting the \( i \)th actuated joint displacement at the initial assembly configuration. Next, we divide the joint space into many grids and solve direct kinematics at each grid. We record the grids where the minimum singular value of \( H' \) are less than the specified value \( \varepsilon \). After the scanning process, we modify \( \Delta \rho_i \) and \( \Delta \rho_i \) to obtain the singularity-free joint space. From Fig. 3 with \( S = 50, \| A_{o}, Q_i \| = 40 \) and \( \varepsilon = 0.05 \), we obtain \( \Delta \rho_i = 15 \) and \( \Delta \rho_i = 3 \) for the isotropic design, and \( \Delta \rho_i = 10 \) and \( \Delta \rho_i = 4 \) for the almost isotropic design. Therefore, the isotropic design has larger singularity-free joint space.

If matrix \( H' \) (instead of \( H \)) is used to evaluate \( \mu_i \) and \( s_i \cdot u_i = s_i \cdot u_i = .... = s_i \cdot u_s = \text{constant} \), then \( \sqrt{2} \) in Eq. (4) should be replaced by \( |s_i \cdot u_i|^{1/2} \), but we also obtain the minimum singular value of \( |s_i \cdot u_i| \sigma_{m m} \) (where \( \sigma_{m m} \) denote the minimum singular value of \( H \)). Therefore, either \( H \) or \( H' \) can be used to evaluate \( \mu_i \) because the two matrices yield the same result. The global isotropy is compared in terms of the mean value of \( \mu_i \) evaluated at the same number of grid points that are homogeneously distributed within the intersection of the two joint spaces: \( \Delta \rho_i = 10 \) and \( \Delta \rho_i = 3 \). The results give 0.8382 and 0.8369 for the isotropic design and the almost isotropic design respectively. It shows that the two designs have nearly identical global isotropy.

VI. A 3-PPSR almost isotropic design

It is difficult to develop the almost isotropic design with PPSR chains shown in Fig. 1b using a similar approach because we cannot find a solution from Table 2 with \( z = w_k \) at one and \( \| u_{2k-1} \cdot u_k \| \) close to zero. Therefore, we directly employ an optimization method to develop the desired manipulator. Besides the z-coordinate of the TCP, the design variables for the directions of \( s_i \) and the directions of the prismatic joints are defined as

\[
s_i = \begin{bmatrix} \gamma_i \rho_i s_i \\ \gamma_i \rho_i s_i \\ \eta_i \end{bmatrix}
\]

for \( i = 1, 2, ..., 6 \) (6) and

\[
u_{2k} = \begin{bmatrix} \cos \theta_k \\ \sin \theta_k \\ 0 \end{bmatrix}, u_{2k} = \begin{bmatrix} -\sin \theta_k \\ \cos \theta_k \\ 0 \end{bmatrix} \text{ for } k = 1, 2, 3
\]

The number of variables can be significantly reduced by solving the following constraint equations:

\[
s_{2j-1} \cdot u_{2j} = s_{2j} \cdot u_{2j-1} = 0 \text{ for } j = 1, 2, 3
\]

and

\[
s_i \cdot u_i = s_i \cdot u_i = s_i \cdot u_3 = s_i \cdot u_4 = s_i \cdot u_5 = s_s \cdot u_s = \lambda
\]

where \( 0 < \lambda \leq 1 \). Higher resolution in position or force control can be obtained for a smaller \( \lambda \). First, solving \( s_i \cdot u_i = \lambda \) and \( s_i \cdot u_s = 0 \) yields

\[
\begin{bmatrix} \gamma_i \rho_i s_i \\ \gamma_i \rho_i s_i \\ \eta_i \end{bmatrix} = \lambda \begin{bmatrix} \gamma_i \rho_i \\ \gamma_i \rho_i \end{bmatrix}
\]

(10)

Next, \( \eta_i \) of unit vector \( s_i \) can be expressed as

\[
\eta_i = \pm \sqrt{1-(\gamma_i \rho_i)^2} \quad (\gamma_i \rho_i)^2
\]

Substituting Eq. (10) into (11) gives

\[
\eta_i = \pm \sqrt{1-\lambda^2}
\]

Either positive or negative sign of \( \eta_i \) can be used. In this paper, we will use positive \( \eta_i \) for further analysis. After solving for the remaining \( s_i \) by a similar procedure, we obtain
\[
\mathbf{s}_{2k-1} = \begin{bmatrix}
\lambda \cos \theta_k \\
\lambda \sin \theta_k \\
\sqrt{1-\lambda^2}
\end{bmatrix}
\quad \text{and} \quad
\mathbf{s}_k = \begin{bmatrix}
\lambda \cos \theta_k \\
\lambda \sin \theta_k \\
\sqrt{1-\lambda^2}
\end{bmatrix}
\quad \text{for } k = 1, 2, 3
\] (13)

Therefore, elements of matrix \( \mathbf{H} \) can be expressed as functions of \( z, \theta_1, \theta_2, \theta_3 \) and \( \lambda \). We can provide \( \lambda \) and then optimize \( \mu_\lambda \) to obtain the design with desired resolution. In this paper, we use \( \lambda \) as a variable. Using an optimization software, we obtain \( \mu_\lambda = 0.8247 \) with \( \theta_1 = 344.98^\circ, \theta_2 = 104.97^\circ, \theta_3 = 225^\circ, z = 30.6529 \) and \( \lambda = 0.7555 \). Vectors \( \mathbf{s}_i \) and \( \mathbf{u}_i \) for \( i = 1, 2, \ldots, 6 \) can be obtained by back substitutions. With \( \mathbf{e}_i' = \mathbf{s}_i \), the obtained results are given in Table 4. There are six free parameters that can be used to specify the three revolute joint axis, so many almost isotropic designs can be obtained following the same procedure (with \( \mathbf{e}_i \) replaced by \( \mathbf{e}_i' \)) in developing the 3-PPSR isotropic design in Fig. 4.

VII. Conclusion

Methods for developing 3-PRPS and 3-PPSR isotropic or almost isotropic manipulators have been proposed in this work. Two special isotropy generators and some related data were presented that can be directly used for developing isotropic designs or used as the initial values for developing almost isotropic designs with desired link parameters. The related data of a 3-PRPS isotropic design were used as the initial values to develop an almost 3-PRPS isotropic design with simple link parameters, and a general optimization method was proposed for developing almost isotropic 3-PPSR designs whose prismatic axes lie in a horizontal plane, and the two prismatic axes on each chain are perpendicular. Many isotropic or almost isotropic designs can be developed because the proposed method provides several free parameters that can be used to specify the revolute joint axes. The methods can also be used for developing the isotropic manipulators with hybrid kinematic chains.

Acknowledgments

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References


**Table 1. Related data of two isotropy generators**

| Generator | Sol. | \( \boldsymbol{e}_i \cdot \boldsymbol{u}_1 \) | \( \left| \boldsymbol{u}_{2a-1} \cdot \boldsymbol{u}_{2a} \right| \) | \( \left| \boldsymbol{z} \cdot \boldsymbol{w}_k \right| \) |
|-----------|------|---------------------------------|-----------------|-----------------|
| I         | 1    | 0.29                           | 0.94            | 0.88            |
|           | 2    | 0.29                           | 0.83            | 0.52            |
|           | 3    | 0.71                           | 0               | 0.58            |
|           | 4    | 0.71                           | 0               | 0.2             |
|           | 5    | 0.86                           | 0.50            | 0.49            |
|           | 6    | 0.82                           | 0               | 0.71            |
|           | 7    | 0.94                           | 0.32            | 0.81            |

**Table 2. Special solutions for isotropic designs**

\[
\text{TCP} = (0.0, 0.0, 0.61305),
\]

\[
A_1 = \begin{pmatrix}
\frac{-\sqrt{3}}{2} S \\
\frac{-1}{2} S \\
0
\end{pmatrix},
A_2 = \begin{pmatrix}
\frac{\sqrt{3}}{2} S \\
\frac{-1}{2} S \\
0
\end{pmatrix},
\]

\[
A_3 = (0.0, S, 0.0)
\]

\[\boldsymbol{e}_1' = (0.7297, -0.1958, 0.6551),\]

\[\boldsymbol{e}_2' = (0.1958, 0.7297, 0.6551),\]

\[\boldsymbol{e}_3' = (-0.1951, 0.7299, 0.6551),\]

\[\boldsymbol{e}_4' = (-0.7299, -0.1951, 0.6551),\]

\[\boldsymbol{e}_5' = (-0.5342, -0.5342, 0.6551),\]

\[\boldsymbol{e}_6' = (0.5342, -0.5342, 0.6551)\]

**Table 3. Related data of an almost isotropy generator for 3-PRPS designs**

\[
\text{TCP} = (0.0, 0.0, 0.16445),
\]

\[
A_1 = \begin{pmatrix}
\frac{-\sqrt{3}}{2} S \\
\frac{-1}{2} S \\
0
\end{pmatrix},
A_2 = \begin{pmatrix}
\frac{\sqrt{3}}{2} S \\
\frac{-1}{2} S \\
0
\end{pmatrix},
\]

\[\boldsymbol{e}_1' = (1.0, 0.0, 0.0),\]

\[\boldsymbol{e}_2' = (0.0, 0.4326, 0.9016),\]

\[\boldsymbol{e}_3' = (-0.5, 0.8660, 0.0),\]

\[\boldsymbol{e}_4' = (-0.3746, -0.2164, 0.9016),\]

\[\boldsymbol{e}_5' = (-0.5, -0.8660, 0.0),\]

\[\boldsymbol{e}_6' = (0.3746, -0.2164, 0.9016)\]

**Table 4. Related data of an almost isotropy generator for 3-PPSR designs**

\[
\text{TCP} = (0.0, 0.0, 0.0),
\]

\[
A_1 = \begin{pmatrix}
\frac{-\sqrt{3}}{2} S \\
\frac{-1}{2} S \\
0
\end{pmatrix},
A_2 = \begin{pmatrix}
\frac{\sqrt{3}}{2} S \\
\frac{-1}{2} S \\
0
\end{pmatrix},
\]

\[\boldsymbol{e}_1' = (0.7297, -0.1958, 0.6551),\]

\[\boldsymbol{e}_2' = (0.1958, 0.7297, 0.6551),\]

\[\boldsymbol{e}_3' = (-0.1951, 0.7299, 0.6551),\]

\[\boldsymbol{e}_4' = (-0.7299, -0.1951, 0.6551),\]

\[\boldsymbol{e}_5' = (-0.5342, -0.5342, 0.6551),\]

\[\boldsymbol{e}_6' = (0.5342, -0.5342, 0.6551)\]