Nonlinear stochastic model predictive control via regularized polynomial chaos expansions

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Abstract — A new method to control stochastic systems in the presence of input and state constraints is presented. The method exploits a particular receding horizon algorithm, coupled with Polynomial Chaos Expansions (PCEs). It is shown that the proposed approach achieves closed loop convergence and satisfaction of state constraints in expectation. Moreover, a non-intrusive method to compute the PCEs’ coefficients is proposed, exploiting $l_2$-norm regularization with a particular choice of weighting matrices. The method requires low computational effort, and it can be applied to general nonlinear systems without the need to manipulate the model. The approach is tested on a nonlinear electric circuit example.

I. INTRODUCTION

In recent years, there has been an increasing attention to the topic of optimal control design for systems in the presence of constraints, where some parameters or disturbances have stochastic nature, and/or the cost functions and constraints have to be satisfied in expectation or with some given probability. In this context, several control design techniques based on on-line optimization and receding horizon strategies have been proposed, commonly known as stochastic Model Predictive Control (MPC), see e.g. [1], [2], [3], [4], [5], [6], [7]. Indeed, research in stochastic MPC has been motivated either by problems that are naturally formulated with some stochastic uncertainty, or by the need to find less conservative solutions with respect to robust MPC, where information on the stochasticity of the uncertainty is disregarded and one seeks a worst-case solution (see e.g. [8]).

One of the key issues in stochastic MPC is the time propagation of the uncertainty in the model equations, i.e. the simulation of the stochastic system. While this task can be accomplished to some extent when a particular structure is present, e.g. in the case of known linear systems with stochastic additive disturbance [3], [4], or when convexity with respect to state and input variables is preserved [7], it becomes very difficult in a general framework with nonlinear systems and both uncertain parameters and disturbances. The most common method to simulate general stochastic systems is the well-known Monte Carlo (MC) approach, whose application for control design might be limited by excessive computational complexity and difficulty to handle state constraints [5].

Polynomial Chaos Expansions (PCEs, see e.g. [9], [10], [11], [12]) provide a useful tool to significantly reduce the computational effort required to simulate a stochastic system, by conceptually replacing the mapping between the uncertain variables and the states, implicitly defined by the integration of the model’s differential equations, with an explicit function which takes the form of a truncated series of polynomials. The polynomials in the PCE are orthogonal, so that the statistical moments can be computed directly from the expansion’s coefficients. PCEs have been used with good results in a number of different areas, including experimental modeling, materials, mechanics, power systems, systems biology, and control, see e.g. [13], [14], [15], [16], [17], [18].

We propose here a novel approach to design stochastic MPC laws for generally nonlinear systems, subject to random parametric uncertainty and disturbances, using polynomial chaos expansions. We consider the problem of designing a controller that induces convergence of the expected value of the state to the origin, and satisfies state constraints in expectation, subject to hard input constraints. In order to accomplish this aim, we employ PCEs with a particular receding horizon scheme. We prove that the proposed approach achieves the desired closed-loop properties.

In all the mentioned contributions dealing with the use of polynomial chaos for control design, a Galerkin projection method has been used to derive the expansions’ coefficients. This approach might require a significant and non-trivial manipulation of the system’s equations, thus potentially limiting the application of the method. While Galerkin projection methods can be straightforwardly used in our approach, as a further contribution we also propose an alternative method to compute the PCEs’ coefficients, based on $l_2$-norm regularized least square regression with a particular choice of weighting matrices. This method is related to the one proposed in [19] and typically yields good estimates of the coefficients with low computational complexity.

II. PROBLEM FORMULATION

We consider a discrete-time, time-invariant system in state-space form:

$$x(t+1) = f(x(t), u(t), \theta)$$

(1)

where $t \in \mathbb{Z}$ is the discrete-time variable, $x(t) \in \mathbb{R}^{n_x}$ is the system state, assumed to be measured, $u(t) \in \mathbb{R}^{n_u}$ is the control input, $\theta \in \mathbb{R}^{n_\theta}$ is an unknown parameter vector. Bold symbols indicate vectors of variables, e.g. $x = [x_1, \ldots, x_{n_x}]^T$, where $^T$ is the vector transpose operation. The components of vector $\theta$ are assumed to have stochastic nature: in particular, we assume $\theta$ to be a $n_\theta$-dimensional vector of independent and identically distributed (iid) random variables $\theta_i$, with known pdf $f_{\theta_i}$ such that $\theta_i \in K \subset \mathbb{L}^2(\Omega, \mathcal{F}, P), \forall i \in \{1, \ldots, n\}$. Here, $(\Omega, \mathcal{F}, P)$
is a probability space, $\Omega$ is the set of elementary events, $\mathcal{F}$ is the $\sigma$–algebra of the events and $P$ is the probability measure. The expectation (or first-order moment) of a generic random variable $\theta : \mathcal{A} \to \mathbb{R}$ is denoted as $E[\theta] = \int_\Omega \theta(\omega) dP(\omega) = \int_\mathcal{A} \theta dP_\mathcal{A}$, where $F_\mathcal{A}(k) = P(\theta < k)$ is the probability distribution function of $\theta$ over $\mathcal{A}$. $L^2(\Omega, \mathcal{F}, P)$ is the Hilbert space of all random variables $\theta$ whose $L_2$-norm, $\|\theta\|_2 = E[|\theta|^2]^{1/2}$, is finite, where $|\cdot|$ denotes the absolute value. $K$ is a subspace of $L^2(\Omega, \mathcal{F}, P)$ that contains only centered random variables (i.e. $\forall \theta \in K, E[\theta] = 0$). Finally, the pdf of $\theta$ is given by $f_\theta(\omega) = dP_\omega/dk$, and the variance (or second-order moment) of $\theta$ is indicated as $\text{Var}[\theta] = E[(\theta - E[\theta])^2] = \sigma_\theta^2$, where $\sigma_\theta$ is the standard deviation of $\theta$.

We assume that the solution of the equations (1) exists and it is unique almost surely, i.e. with probability one, for any initial condition $x(0, \theta)$ and for all $t \geq 0$.

Moreover, we also consider the presence of state-constraints of the following form:

$$g(x(t)) \leq 0,$$

(2)

where $g = [g_1, \ldots, g_m]$, $g_i : \mathbb{R}^n_\theta \to \mathbb{R}$, $j = 1, \ldots, n_g$ and the symbol $\leq$ denotes element-wise inequalities. As an example, constraints of the form (2) can be used to account for bounds on the state's system. We assume that if $E[x(t)] = 0$ then $E[g(x(t))] < 0$. For the sake of simplicity, in the following we denote $g(t) = g(x(t))$.

Finally, we consider the presence of input constraints, in the form of a convex and compact set $U \subset \mathbb{R}^n_\theta$.

The problem we consider can be formally stated as follows:

**Problem 1:** design a state-feedback control law $u(t) = \kappa_f(x(t))$, satisfying input constraints, able to asymptotically regulate the expected value of the state to the origin, while satisfying state constraints (2) in expectation:

$$\lim_{t \to \infty} E[x(t)] = 0$$

(3a)

$$\kappa_f(x(t)) \in U, \forall t \geq 0$$

(3b)

$$E[g(t)] \leq 0, \forall t \geq 0.$$  

(3c)

In order to solve Problem 1, we will employ a receding horizon strategy, typical of MPC schemes [20]. In this context, we need to predict the possible values $x(t+j)$ up to a finite number of future time steps $N$. As it is common practice in MPC (see e.g. [20] for more details), we assume that a state-feedback controller $\kappa_f(x) : \mathbb{R}^n_\theta \to \mathbb{U}$ is available, and we parameterize the control input as follows:

$$u(t) = \kappa_f(x(t)) + \overline{u}(t),$$

(4)

where the control correction $\overline{u}(t)$ is a decision variable. Note that we require the control law $\kappa_f$ to satisfy the hard input constraints described by the compact set $U$, i.e. $\kappa_f(x) \in U, \forall x \in \mathbb{R}^n_\theta$: since $U$ is convex, this can be obtained e.g. by a simple projection operator.

The system (1) with the pre-compensating controller $\kappa_f$ is equivalent to the following system:

$$x(t+1) = f_{\text{CL}}(x(t), \overline{u}(t), \theta),$$

(5)

where $f_{\text{CL}}(x(t), \overline{u}(t), \theta) = f(x(t), \kappa_f(x(t)) + \overline{u}(t), \theta),$$

We denote with $v(j|t)$ a generic variable $v(t+j)$ at time $t+j$, $j \in \mathbb{N}$, predicted at time $t$ using the system’s model (5) and starting from the initial conditions $x(t)$.

For a fixed sequence of control corrections $\overline{u}(i|t)$, $i = 0, \ldots, j - 1$, the corresponding predicted values of $x(j|t)$ and $g(j|t)$ are random variables, since they depend on the stochastic parameter $\theta$. We consider the following assumptions:

**Assumption 1:** The system (5), stochastic parameter $\theta$ and control law $\kappa_f$ are such that:

- **a.** (finiteness of variance of $x(j|t)$ and $g(j|t)$) For any values of $t \geq 0$ and $x(t)$, any $N < \infty$ and any $j = 1, \ldots, N$, $x(j|t), g(j|t) \in L^2(\Omega, \mathcal{F}, P)$,

$$\forall \theta(i|t) : \kappa_f(\theta(i|t)) + \overline{u}(i|t) \in U, i = 1, \ldots, j - 1;$$

- **b.** (pre-compensating controller) Consider the 1-step ahead propagation of the state’s and constraints’ trajectories with zero control correction, indicated as

$$x_0(t+1) = f_{\text{CL}}(x(t), 0, \theta)$$

and $g_0(t+1) = g(x_0(t+1))$. For any $t \geq 0$, if $E[x(t)] = 0$ then:

$$E[x_0(t+1)] = 0,$$

$$E[g_0(t+1)] \leq 0.$$  

The computation of $E[x(t)], E[g(t)]$ requires the simulation of the stochastic system (1). This task may be very complex and the main technique employed so far in engineering applications is the well-known Monte Carlo (MC) approach. Although MC simulations proved to be very effective in many applications, the required computational times may be prohibitive in various cases, especially when the simulation has to be embedded in a numerical optimization procedure. Polynomial Chaos Expansion techniques (see e.g. [11]) are able to significantly reduce the computational effort required by standard MC approaches. We will now recall the main features of PCEs.

**III. POLYNOMIAL CHAOS EXPANSIONS**

PCEs were first introduced by Wiener [9], who considered Gaussian random variables $\theta$. Later on, Cameron and Martin [10] showed one of the key properties of PCEs, namely their ability to uniformly approximate any random process with finite second-order moments. The polynomial chaos is an orthogonal basis of $L^2(\Omega, \mathcal{F}, P)$, hence any random variable $v(\theta) \in L^2(\Omega, \mathcal{F}, P)$ has the $L_2$-convergent expansion [10]:

$$v(\theta) = \sum_{k=0}^{\infty} a_k \Phi_{\alpha_k}(\theta),$$

(6)

where the coefficients $a_k$ are given by $a_k = \frac{E[v(\theta) \Phi_{\alpha_k}(\theta)]}{E[\Phi_{\alpha_k}(\theta)^2]}$, and $\Phi_{\alpha_k} = \Phi_{\alpha_k}(\theta_1, \ldots, \theta_n)$ is the $k$-th multivariate polynomial in the series, corresponding to the $k$-th vector of indices, or “multi-index”, $\alpha_k = [\alpha_1, \ldots, \alpha_{n,k}], \alpha_{i,k} \in \mathbb{N}$. More specifically, for a given vector of indices $\alpha_k$, we have $\Phi_{\alpha_k}(\theta) = \prod_{i=1}^{n_k} \Phi_{(\alpha_{i,k})}(\theta_i)$, where $\Phi_{(\alpha_{i,k})}(\theta_i)$ is the univariate polynomial of degree $\alpha_{i,k}$, chosen according to the Askey
scheme [12]. As an example, Hermite polynomials are used with Gaussian input random variables. Table I shows the suitable orthogonal polynomials for different kinds of input random variables. The choice of the univariate polynomials is made in order to satisfy the orthogonality property:

$$E\left[\Phi_{(\alpha_j)}\Phi_{(\alpha_i)}\right] = E\left[\Phi_{(\alpha_j)}^2\right] \delta_{ij},$$  \hspace{1cm} (7)

where $\delta_{ij} = 1$ if $i = j$ and 0 in any other case. We denote with $l_k = \sum_{i=1}^n \alpha_{i,k}$ the sum of the indices in the multi-index $\alpha_k$, and we assume that the ordering of the multivariate polynomials $\Phi_{\alpha_k}$ in (6), and of the related coefficients $a_k$, is such that $l_k \leq l_{k+1}$. For practical reasons, the series (6) is truncated by considering only the multi-indices up to a maximal total degree $\bar{l}$, i.e. $\forall \alpha_k : l_k \leq \bar{l}$. Since all the possible multi-indices $\alpha$ that sum up to $l \leq \bar{l}$ are considered, the total number $L$ of terms in the truncated expansion is:

$$L = \frac{(n + \bar{l})!}{n!\bar{l}!},$$  \hspace{1cm} (8)

and the series takes the form:

$$\hat{v}(\theta) \doteq \sum_{k=0}^{L-1} a_k \Phi_{\alpha_k}(\theta) = \Phi(\theta) a,$$

where $a \doteq [a_0, \ldots, a_{L-1}]^T$ and

$$\Phi(\theta) \doteq [\Phi_{\alpha_0}(\theta), \ldots, \Phi_{\alpha_{L-1}}(\theta)]$$

are, respectively, the vectors of the PCE’s coefficients and of the multivariate polynomials evaluated at $\theta$. We refer to the truncated expansion $\hat{v}(\theta) \approx v(\theta)$ (9) as the PCE of the random variable $v(\theta)$.

By applying the orthogonality property (7), the first and second order moments of the random variable $v(\theta)$ can be computed directly from the coefficients of its PCE, as follows:

$$E\left[v(\theta)\right] = a_0$$  \hspace{1cm} (11)

$$\text{Var}\left[v(\theta)\right] = \sum_{k=1}^{L-1} a_k^2 E\left[\Phi_{\alpha_k}(\theta)^2\right]$$  \hspace{1cm} (12)

where $a_0$ is the coefficient of the polynomial of order $l = 0$ (i.e. $\Phi_{\alpha_0} = 1$) in the PCE. As regards the practical computation of equation (12), the terms $E\left[\Phi_{\alpha_k}(\theta)^2\right], \forall k \in \{1, L-1\}$ are computed off-line once for all uses, and they can typically be obtained quite easily (see [19] for details).

As pointed out in Section II, for a given input sequence and initial state, the predicted state and constraint values, $x(i|t)$ and $g(i|t)$, are functions of the random vector $\theta$. In virtue of Assumption 1-a, these predictions have a corresponding Polynomial Chaos Expansion. In the following, we will consider the PCEs of $x(i|t)$ and $g(i|t)$, computed by applying (9) to each one of their components $x_j(i|t)$, $g_k(i|t)$, where $j = 1, \ldots, n_x$ and $k = 1, \ldots, n_p$. To this end, we introduce the symbols $a_j^{(x)}(i|t), a_j^{(g)}(i|t)$ to denote the vectors of coefficients of the PCEs of the $j$-th component of $x(i|t)$ and of the $k$-th component of $g(i|t)$, respectively. We note that the values of $E[x(i|t)]$ and $E[g(i|t)]$ can be computed by applying (11) to each component of vectors $x(i|t)$ and $g(i|t)$; in particular, we have:

$$E[x(i|t)] = \left[ a_0^{(x_1)}(i|t) \right], \ E[g(i|t)] = \left[ a_0^{(g_1)}(i|t) \right]$$

(13)

In the next Section we propose a MPC approach to solve Problem 1, which exploits PCEs.

IV. STOCHASTIC RECEDING HORIZON CONTROL WITH POLYNOMIAL CHAOS EXPANSIONS

A. Stochastic finite horizon optimal control problem with PCEs

Let $N \in \mathbb{N}$ be a prediction horizon, let $\mathcal{U} = [\mathcal{U}(0|t)^T, \ldots, \mathcal{U}(N-1|t)^T]^T$ be a vector of control corrections predicted at time $t$ up to time $t + N - 1$, let $Q = Q^T \succ 0$ and $R = R^T \succ 0$ be two weighting matrices, where $\succ 0$ denotes positive definiteness, finally let $\beta > 0$ be a scalar. $N, Q, R, \beta$ are parameters chosen by the control designer. Then, given the state $x(t)$ at the generic time $t$, we consider the following finite horizon optimal control problem (FHOCP) $\mathcal{P}(x(t))$:

$$\mathcal{P}(x(t)) : \min_{\mathcal{U}} \ J(x(t), \mathcal{U})$$

subject to

$$J(x(t), \mathcal{U}) = \sum_{i=0}^{N-1} \left( \|E[x(i|t)]\|_Q^2 + \beta \|u(i|t)\|_R^2 \right)$$

(14a)

$$u(i|t) = \kappa_f(x(i|t)) + \mathcal{U}(i|t)$$

(14b)

$$E[g(i|t)] \leq 0, \ i = 1, \ldots, N$$

(14c)

$$u(i|t) \in \mathcal{U}, \ i = 0, \ldots, N - 1$$

(14d)

$$E[x(N|t)] = 0,$$  \hspace{1cm} (14e)

where $\|v\|_M \doteq v^T M v$. We denote a (possibly local) solution to problem $\mathcal{P}(x(t))$ as $\mathcal{U}^*(t) = \mathcal{U}^*(x(t))$, and the optimal value of the cost function in (14a) as $J^*(t) = J(x(t), \mathcal{U}^*(x(t)))$.

Throughout Sections IV-A and IV-B we assume that, given a value of $x(t)$, the PCEs of variables $x(i|t), g(i|t), i = 1, \ldots, N$ are available as a function of the elements in the sequence $\mathcal{U}$, and we propose a computationally efficient method to obtain such PCEs in Section IV-C. Note that, by using equation (13), problem (14) involves just the coefficients of degree 0 of the PCEs.
Remark 1: Problem (14) is not the only stochastic control problem that can be handled by using PCEs. As an example, one can ask for satisfaction of constraints not just in expectation, but also with some guaranteed probability, e.g. by using Chebyshev’s inequality and exploiting the quadratic form (12). Moreover, one can include the variance of the predicted state values in the cost or in the constraints, again using (12). In these cases, the cost and constraints of problem (14) are still affine or quadratic in the PCEs’ coefficients. More in details, including the variances of the state components in the cost function results in the following expression:

\[ J(x(t), \overline{U}) \sum_{i=0}^{N-1} \left( \|E[x(i|t)]\|_Q^2 + \beta \|u(i|t)\|_R^2 + c^T V_x(i|t) \right) , \]

where the \( e \in \mathbb{R}^{n_x} \) is a vector of non-negative scalar weights and \( V_x(i|t) \in \mathbb{R}^{n_x} \triangleq [\text{Var}[x_1(i|t)], \ldots, \text{Var}[x_n(i|t)]]^T \) is a vector containing the variances at time \( t+i \) of each state component, predicted at time \( t \) by using the equation (12). It is clear that such cost function is still quadratic in the PCEs’ coefficients. Similarly, the inclusion of bounds on the variances of the states results in quadratic constraints with respect to the expansion’s coefficients. Generally speaking, the use of the variance in the cost and/or in the constraints will tend to penalize the dispersion of the trajectories about the expected one, eventually at the cost of worse performance in expectation. Similar considerations hold for the variances of the constraint variables \( g(i|t) \). We chose not to use such formulations in this paper only for the sake of simplicity.

Problem \( \mathcal{P}(x(t)) \) is, in general, a nonlinear program (NLP) which, under some regularity assumptions on the cost (14a) and constraints (14c)-(14e), can be solved for a local solution, e.g. by using Newton-type methods [21]. We note that, in our approach, we do not ask for a global solution to be computed in order to guarantee the desired properties. We denote as \( \mathcal{F} \in \mathbb{R}^{n_x} \) the feasibility set of problem \( \mathcal{P} \), i.e. the set of initial state values \( x \), such that problem \( \mathcal{P}(x) \) admits a solution. Finally, we adopt the convention that \( J^*(x) = +\infty \) whenever \( x \notin \mathcal{F} \).

In the following, we embed problem \( \mathcal{P}(14) \) in a particular receding horizon algorithm, in order to exploit the information on the actual system’s state that becomes available at each time step.

B. Polynomial Chaos Model Predictive Control

Let \( \epsilon > 0 \) be a scalar chosen by the control designer. Then, we propose the following receding horizon strategy, where \( \overline{U}(t) = [\overline{x}^T(0|t), \ldots, \overline{x}^T(N-1|t)]^T \) is the optimal sequence of control corrections computed by solving problem (14) (whenever it is feasible), and \( \overline{U}(t) = [\overline{u}^T(0|t), \ldots, \overline{u}^T(N-1|t)]^T \) is the actual sequence of control corrections whose first element, \( \overline{u}(0|t) \), is applied to the system (5) at time \( t \).

Algorithm 1: (Polynomial Chaos Model Predictive Control - PCMPC)

1) (initialization) let \( t = 0 \) and \( x(t) \in \mathcal{F} \). Solve problem \( \mathcal{P}(x(0)) \) to compute the values of \( J^*(0) \) and \( \overline{U}(0) \). Apply to the system the input \( u(t) = \kappa_f(x(t)) + \overline{u}(0) \), where \( \overline{u}(0) = \overline{u}(0|0) \), set \( t = 1 \) and go to 2);
2) at time \( t \), get \( x(t) \) and consider problem \( \mathcal{P}(x(t)) \). If this problem is unfeasible, set \( J^*(t) = +\infty \), otherwise solve \( \mathcal{P}(x(t)) \) to compute \( J^*(t) \) and \( \overline{U}(t) \);
3) consider the following mutually exclusive cases:
   a) if \( J^*(t) - J^*(t-1) \leq -\epsilon \), then set \( \overline{U}(t) = \overline{U}^*(t) \);
   b) else if \( J^*(t) - J^*(t-1) > -\epsilon \), then set:

\[ J^*(t) = 0 \]
\[ \overline{U}(t) = [\overline{x}^T(1|t-1), \ldots, \overline{x}^T(N-1|t-1), 0]^T, \]

(15)
4) apply to the system the input \( u(t) = \kappa_f(x(t)) + \overline{u}(0|t) \)
5) set \( t = t+1 \), go to 2)

The use of the PCMPC algorithm gives rise to a state-feedback control law \( u(t) = \kappa_f(x(t)) \), where \( \kappa_f(x(t)) = \kappa_f(x(t)) + \overline{u}(0|t) \) given in step 4) of the algorithm. The next results shows that the resulting closed loop system enjoys properties (3) for all initial states \( x(0) \in \mathcal{F} \).

Theorem 1: Consider the closed loop system obtained by applying the control law \( u(t) = \kappa_f(x(t)) \), defined by Algorithm 1, to system (1). Then, for all \( x(0) \in \mathcal{F} \) the properties (3) hold.

Proof: For the sake of brevity, just a sketch of the proof is given here. Assume case 3)-a) in the PCMPC algorithm happens for all \( t \geq 0 \). Then, it can be shown that \( \lim_{t \to \infty} J^*(t) = 0 \). Since it holds that \( \|E[x(t)]\|_2^2 \leq \frac{1}{2E[t]} J^*(t) \), this implies \( \lim_{t \to \infty} \|E[x(t)]\|_2^2 = 0 \). Vice-versa, if at some time \( t^* \) case 3)-b) occurs, then it can be noted that, from \( t^* \) on, the optimal sequence \( \overline{U}(t^* - 1) \) is applied, which by construction guarantees \( E[x(t^* + T)] = 0 \) for some \( T < N \). Then, for \( t \geq t^* + T \), according to Assumption 1 the expectation of the state is kept at zero. Similar arguments can be used to prove \( E[\|g(t)\|_2] \leq 0 \), while \( u(t) \in \mathcal{U} \) holds by construction.

C. Computation of PCEs via weighted \( \ell_2 \)-norm regularized regression

So far, we assumed that the PCEs of variables \( x(i|t), g(i|t), i = 1, \ldots, N \) were available as a function of the sequence of control corrections \( \overline{U} \), in order to solve problem \( \mathcal{P}(14) \). Clearly, the computation of the expansion’s coefficients is one of the crucial points in the use of PCEs. A class of approaches to carry out this task is known as Probabilistic Collocation Method (PCM, see e.g. [22], [23]), and basically consists in the estimation of the coefficients from a finite number of data, i.e. of \( \nu \) values of variables \( x(i|t), g(i|t) \), corresponding to \( \nu \) values of the input random variables, \( \theta(r), r \in [1, \nu] \), named “collocation points”. Here, we consider a PCM-like approach for the computation of PCEs, and we propose a new method to estimate the coefficients. The approach we propose, based on \( \ell_2 \)-norm regularization, is a variant of the one proposed in [19], where it is shown, through a series of tests, that very
good results are obtained even with a very low number of collocation points.

Let $\hat{\theta}_{(r)}$, $r \in \{1, \nu\}$ be a finite number $\nu$ of independent values of the vector of input random variables, sampled according to their distribution. Given a maximal order $l$ of the PCE and the corresponding number of terms $L$ (8), for each component $x_j(i\|t)$ and $g_k(i\|t)$ of $x(i\|t)$ and $g(i\|t)$, and for all $i = 1, \ldots, N$, we propose the following algorithm to estimate the PCEs' coefficients $a^{(x_j)}(i\|t)$, $a^{(g_k)}(i\|t)$:

**Algorithm 2:** (PCE computation via $\ell_2$-norm regularization)

1) given a sequence $\bar{U}$ of control corrections, carry out $\nu$ simulations of the system (5), each one corresponding to one of the extracted samples $\theta_{(r)}$, using $x(t)$ as initial condition;

2) collect the obtained values of the variables of interest in the vectors $\tilde{v}^{(x_j)}(i\|t) = [x_{j,1}(i\|t), \ldots, x_{j,\nu}(i\|t)]^T$ and $\tilde{g}^{(g_k)}(i\|t) = [g_k(x(1)(i\|t)), \ldots, g_k(x(\nu)(i\|t))]^T$, for all $j = 1, \ldots, n_x$, $k = 1, \ldots, n_g$, $i = 1, \ldots, N$;

3) compute the PCEs’ coefficients as follows:

$$a^{(x_j)}(i\|t) = A\tilde{v}^{(x_j)}(i\|t), \quad j = 1, \ldots, n_x, \quad i = 1, \ldots, N \tag{16}$$

$$a^{(g_k)}(i\|t) = A\tilde{v}^{(g_k)}(i\|t), \quad k = 1, \ldots, n_g, \quad i = 1, \ldots, N.$$ 

At step 3) of Algorithm 2, the matrix $A \in \mathbb{R}^L \times \mathbb{R}^\nu$ is computed as:

$$A = \left( W^TW + \gamma \Phi^T \tilde{\Lambda}^T \tilde{\Lambda} \Phi \right)^{-1} \tilde{\Lambda} \tilde{\Phi}, \tag{17}$$

where $\gamma > 0$ is a scalar and $W$, $\tilde{\Lambda}$ and $\tilde{\Phi}$ are suitable weighting matrices. In particular, the diagonal matrix $W \in \mathbb{R}^L \times \mathbb{R}^L$ is defined as:

$$W = \text{diag} \{ w(l_k) \}, \tag{18}$$

where $l_k$ is the order of the multi-index $\alpha_l$, and $w(l_k), k \in \{0, L-1\}$ is a sequence of scalar weights with the following properties:

$$w(l_k) > 0, \quad \forall k \in \{0, L-1\};$$

$$w(l_k) > w(l_j) \iff l_k > l_j, \quad \forall k, j \in \{0, L-1\};$$

$$\max_k w(l_k) = 1. \tag{19}$$

The diagonal matrix $\tilde{\Lambda} \in \mathbb{R}^\nu \times \mathbb{R}^\nu$ is defined as:

$$\tilde{\Lambda} = \text{diag} \{ \tilde{\lambda} \}, \tag{20}$$

where $\tilde{\lambda} = [f_{\theta}(\theta_{(1)}), \ldots, f_{\theta}(\theta_{(\nu)})]^T$ are the values of the pdf $f_{\theta}$, evaluated at the considered samples $\theta_{(r)}$, $r \in \{1, \nu\}$. Finally, the matrix $\tilde{\Phi} \in \mathbb{R}^\nu \times \mathbb{R}^L$ is defined as

$$\tilde{\Phi} = \begin{bmatrix} \Phi(\theta_{(1)}) \\ \vdots \\ \Phi(\theta_{(\nu)}) \end{bmatrix}, \tag{21}$$

where the vectors $\Phi(\theta_{(r)})$, $r \in \{1, \nu\}$, are computed according to (10).

Note that the matrix $A$ is computed off-line once and for all, moreover the term $\left( W^TW + \gamma \Phi^T \tilde{\Lambda}^T \tilde{\Lambda} \Phi \right)$ in (17) is invertible, thanks to the definition of matrix $W$.

**Equation (16) is the closed-form solution of the following weighted $\ell_2$-norm regularized least-squares regressions:**

$$\min_{a} \frac{1}{2} \left( \gamma \left\| \tilde{\Lambda} \left( \tilde{\Phi}^T \tilde{v} - \tilde{\Phi} a \right) \right\|_2^2 + \| W a \|_2^2 \right),$$

$$\min_{a} \frac{1}{2} \left( \gamma \left\| \tilde{\Lambda} \left( \tilde{\Phi}^T \tilde{g} - \tilde{\Phi} a \right) \right\|_2^2 + \| W a \|_2^2 \right),$$

for each $j = 1, \ldots, n_x$, $k = 1, \ldots, n_g$ and $i = 1, \ldots, N$.

In this context, the value of $\gamma$ can be tuned to tradeoff the fitting of data with the $\ell_2$-norm of the obtained solution. In practice, the estimate results to be very accurate with any sufficiently high value of $\gamma$ (e.g. $\gamma = 10^5$, depending also on the scaling of the other involved quantities). The weighting matrices $W$ and $\tilde{\Lambda}$ are chosen in order to adapt the regression to the particular properties of polynomial chaos expansions, see [19] for more details.

In practice, by using Algorithm 2 the computation of the expectations involved in the FHOCP (14) is replaced with a finite number of simulations, like it happens in MC methods. However, the use of PCEs and of the particular regularized regression (16) yields very accurate results, even with a small number $\nu$ of samples. The numerical example in the next Section highlights this aspect.

**V. NUMERICAL EXAMPLE**

Consider the electric circuit depicted in Fig. 1. The system equations are

$${i_L(t)} = \frac{1}{L} v_C(t) - \frac{R}{L} i_L(t) + \frac{1}{L} (u(t) - v_D(t)) \tag{22}$$

$${v_C(t)} = \frac{1}{C} i_L(t),$$

The resistance $R$ is assumed to be a random variable $R = R_0(1 + 0.3 \theta_1)$, where $R_0 = 3.5 \Omega$ and $\theta_1$ is a random variable with uniform distribution over $[-1, 1]$. The inductance $L$ and the capacitance $C$ are nonlinear functions of the current $i_L(t)$ and voltage $v_C(t)$, respectively:

$$L(i_L(t)) = 0.5 L_0 (1 + \exp(a_1 i_L(t)^2))$$

$$C(v_C(t)) = 0.5 C_0 (1 + \exp(a_2 v_C(t)^2)), \tag{23}$$

where $a_1 = -0.5 10^8$, $a_2 = -0.5 10^6$. Moreover, the maximal values $L_0$ and $C_0$, achieved when $v_C(t) = v_D(t) = 0$, are equal to $L_0 = L_0(1 + 0.2 \theta_2)$, $C = C_0(1 + 0.2 \theta_3)$, where $\theta_2$, $\theta_3$ are also random variables with uniform distribution over $[-1, 1]$, $\theta_1$, $\theta_2$, $\theta_3$ are assumed to be independent.

The device $D$ generates a voltage $v_D(t)$ which can be either positive or negative. In particular, $v_D(t)$ is assumed to be a stochastic input of the form $v_D(t) = a_4 \sin \left( \frac{2\pi t}{a_5} \right)$.
Fig. 2. Numerical example. Reference voltage $v_{\text{ref}}$ (dashed line), average voltage $E[v_C(t)]$ (solid), constraint $g(v_C(t))$ (dash-dot). Each gray line represents a particular realization, among the 30,000 Monte Carlo simulations used to evaluate the control performance, of the voltage $v_C(t, \theta)$ obtained in closed loop with the proposed PCMC method.

$$a_3 v_{D, \text{rand}}(t), \text{ where } a_3 = 1.10^{-2}, a_4 = 5.10^{-3}, a_5 = 1.10^{-2} \text{s.}$$

The term $\sin\left(\frac{2\pi}{a_5} t\right)$ is a known sinusoidal component, while $v_{D, \text{rand}}(t)$ is a random process with mean $\mu_D = 0$ and exponential covariance function $C_D(t_1, t_2) = \sigma_D^2 \exp(-\mu_D |t_1 - t_2|)$, with $\sigma_D = 1$ and $\mu_D = 50$. In order to model the random process $v_{D, \text{rand}}(t)$ as a function of a finite number of random input variables, we employ the Karhunen-Loeve (KL) expansion (see e.g. [11]) with 10 independent random variables $\theta_{D, 1}, \ldots, \theta_{D, 10}$ uniformly distributed in the interval $[-1, 1]$. Overall, the input random variables $\theta$ include the 3 random variables related to parametric uncertainty, $\theta_3, \theta_4, \theta_5$, plus the 10 random variables involved in the KL expansion, $\theta_{D,i}, i \in \{1, 10\}$. Thus, there are $n = 13$ input random variables in total, all uniformly distributed in the interval $[-1, 1]$.

We are interested in regulating the expected voltage $E[v_C(t)]$ to the reference value of $v_{\text{ref}} = 12 \cdot 10^{-3}$V, under input constraints $|u(t)| \leq 18 \cdot 10^{-3}$ and constraint $g(v_C(t)) = v_C(t) - 12.5 \cdot 10^{-3}$. The control objective can be cast in our problem settings by the simple change of variables $x_1 = i_L, x_2 = v_C \neq v_{\text{ref}}$. As pre-compensating controller, we use $u(t) = k_f \cdot x(t) = a_1 \sin \left(\frac{2\pi}{a_5} t\right) - \left[3.5, 1\right]^T x(t)$, given by a static feedback controller plus a term to account for the deterministic component of the disturbance $v_D(t)$, and saturated in order not to exceed the input constraints. To apply the PCMC algorithm, we discretize (22) with sampling time $T_s = 5 \cdot 10^{-4}$ s and we choose $N = 10$, $\beta = 10^4$, $\gamma = 10^5$, and $\epsilon = 10^{-5}$, $Q = R = I$, where $I$ is the identity matrix. According to Table 1, the PCES are formulated by using Legendre polynomials. We apply our method with a maximal order $l = 2$ for the PCES. This results in $L = 105$ multivariate polynomials in the expansion. We use just $\nu = 20$ collocation points to estimate the PCES coefficients with the $L_2$-norm regularized algorithm. Fig. 2 shows the closed-loop course of the mean voltage $E[v_C(t)]$, evaluated through a series of 30,000 MC simulations, as well as of some realizations, starting from the initial condition $i_L(0) = 0$, $v_C(0) = 0$. It can be noted that, as expected, the average value of the error $E[V_C(t) - v_{\text{ref}}]$ converges to zero, and the value $E[v_C(t)]$ is below the constraint of $12.5 \cdot 10^{-3}$, meaning that the average value $E[g(t)]$ is lower than zero. The input constraints $|u(t)| \leq 18 \cdot 10^{-3}$ also result to be always satisfied. All the simulations were carried out using Matlab’s function fmincon.

## References


