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Remarks on Liouville theory with boundary

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Abstract: The bootstrap for Liouville theory with conformally invariant boundary conditions will be discussed. After reviewing some results on one- and boundary two-point functions we discuss some analogue of the Cardy condition linking these data. This allows to determine the spectrum of the theory on the strip, and illustrates in what respects the bootstrap for noncompact conformal field theories with boundary is richer than in RCFT. We briefly indicate some connections with $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ that should help completing the bootstrap.

Keywords: CFT with boundary, D-branes, Liouville theory.

1. Motivation

D-branes on compact spaces can be studied in the "stringy regime" ($\alpha' \sim \mathcal{O}(1)$) by means of conformal field theory in the presence of boundaries [1][2]. The treatment of D-branes on noncompact spaces requires consideration of CFT with continuous spectrum of Virasoro representations (noncompact CFT). Liouville theory may be considered as a prototypical example of such CFT. It seems to be the natural starting point for the development of techniques for the exact study of the class of CFT that describe D-branes on noncompact backgrounds. Moreover, physically interesting examples such as the $SL(2)/U(1)$ black hole or AdS_3 are closely related to Liouville theory from the technical point of view.

2. Liouville theory w/o boundary

We will very briefly assemble a few facts concerning Liouville theory with periodic boundary conditions that will be referred to later.

The classical 2D field theory is defined on $\mathbb{R}\times S^1$ by the Lagrangian

$$\mathcal{L} \;=\; \frac{1}{4\pi}(\partial_a\phi)^2 \;+\; \mu e^{2b\phi}.$$

The spectrum is believed [3] to be of the following form:

$$\mathcal{H} \;=\; \int_{\mathbb{S}} d\alpha\; \mathcal{V}_\alpha \otimes \mathcal{V}_\alpha, \qquad \mathbb{S} = \frac{Q}{2} + i\mathbb{R}^+$$

where $\mathcal{V}_\alpha$: hwr of Virasoro algebra (generators L_n, $\bar{L}_n$), highest weight $\Delta_\alpha = \alpha(Q-\alpha)$, $Q = b + \frac{1}{b}$.

Conformal symmetry allows one to map the cylinder to the complex plane via $z = e^w$. Basic objects for the understanding of the theory are the local primary fields $V_\alpha(z,\bar{z})$,

$$\begin{aligned}[L_n, V_\alpha(z,\bar{z})] &= z^n(z\partial_z + \Delta_\alpha(n+1))V_\alpha(z,\bar{z})\\ [L_n, V_\alpha(z,\bar{z})] &= \bar{z}^n(\bar{z}\partial_{\bar{z}} + \Delta_\alpha(n+1))V_\alpha(z,\bar{z}),\end{aligned}$$

Thanks to conformal symmetry they turn out to be fully characterized by their three point functions

$$D(\alpha_3,\alpha_2,\alpha_1) = \langle 0|V_{\alpha_3}(\infty)V_{\alpha_2}(1)V_{\alpha_1}(0)|0\rangle. \tag{2.1}$$

A formula for $D(\alpha_3,\alpha_2,\alpha_1)$ has been proposed by Dorn, Otto and AL.B., A.B. Zamolodchikov [4, 5]. The data $(\mathbb{S}, D)$ can be seen to contain the full information about Liouville theory on $\mathbb{R}\times S^1$. In particular, it is possible to reconstruct the Liouville field itself, to show that it satisfies a proper generalization of the classical equation of motion, and to verify the canonical commutation relations [6].

Let us note that the proposals concerning $(\mathbb{S}, D)$ are not rigorously proven so far. However, there exists ample evidence to their support, cf. e.g. [4, 5, 7, 6] and references therein.

3. Liouville theory with boundary

One needs to study the quantization of the classical 2D field theory defined by the action

$$\mathcal{S} \;=\; \int\limits_{\mathrm{S}} d^2w \,\Big(\frac{1}{4\pi}(\partial_a\phi)^2 \;+\; \mu e^{2b\phi}\Big) \;+\; \int\limits_{\mathbb{R}} d\tau \,\Big(\rho_1 e^{b\phi}\big|_{\sigma=\pi} \;+\; \rho_2 e^{b\phi}\big|_{\sigma=0}\Big),$$

where S: strip $\mathbb{R}\times[0,\pi]$.

Conformal symmetry is expected to organize $\mathcal{H}$ as

$$\mathcal{H} \;=\; \int\limits_{\mathbb{S}^{\mathrm{B}}} d\alpha \;\mathcal{V}_\alpha, \tag{3.1}$$

where $\mathcal{V}_\alpha$: highest weight representation of the Virasoro algebra (generators L_n) with highest weight $\Delta_\alpha = \alpha(Q-\alpha)$. The set $\mathbb{S}^{\mathrm{B}}$ to be integrated over will in general depend on the values ρ_2, ρ_1 labelling the boundary conditions.

In discussions of the euclidean theory it is often convenient to map the strip to the upper half plane by means of the conformal transformation $w \to z = e^w$.

One is mainly interested in primary fields $\mathcal{O}_\alpha(z)$. One needs to consider two types of such fields:

- **bulk** fields $V_\alpha(z)$, defined on $\{z\in\mathbb{C};\Im(z)>0\}$, which transform as

$$\begin{aligned}[L_n,\mathcal{O}_\alpha(z)] =& z^n(z\partial_z+\Delta_\alpha(n+1))\mathcal{O}_\alpha(z)\\ &+\bar{z}^n(\bar{z}\partial_{\bar{z}}+\Delta_\alpha(n+1))\mathcal{O}_\alpha(z),\end{aligned}$$

- **boundary** fields $\Psi^{\beta}_{\rho_2\rho_1}(x)$, $x\in\mathbb{R}$:

$$[L_n,\Psi^{\beta}_{\rho_2\rho_1}(x)] = x^n(x\partial_x+\Delta_\beta(n+1))\Psi^{\beta}_{\rho_2\rho_1}(x).$$

4. Bootstrap

4.1 Data

The conformal Ward identitites together with factorization of correlation functions lead to an unambigous construction of any correlation function in terms of a set of elementary amplitudes (structure functions): This set contains in addition to the three point function D of the theory *without* boundary (2.1) the following data:

- **One point function**:

$$\langle V_\alpha(x)\rangle \;=\; A(\alpha|\rho)\;\,|z-\bar{z}|^{-2\Delta_\alpha}$$

- **Boundary two point function**:

$$\begin{aligned}\langle\Psi^{\beta}_{\rho_1\rho_2}(x)\Psi^{\beta'}_{\rho_2\rho_1}(y)\rangle \;=&\; |x-y|^{-2\Delta_\beta}\\ &\times N_0\big(\delta_{Q-\beta,\beta'}+\delta_{\beta,\beta'}B(\beta|\rho_1,\rho_2)\big)\end{aligned}$$

- **Bulk-boundary two point function**:

$$\langle V_\alpha(z)\Psi^{\beta}_{\rho\rho}(x)\rangle \;= \;=\frac{A(\alpha,\beta|\rho)}{|z-\bar{z}|^{2\Delta_\alpha-\Delta_\beta}\,|z-x|^{2\Delta_\beta}}$$

- **Boundary three point function**:

$$\langle\Psi^{\beta_3}_{\rho_1\rho_3}(x_3)\Psi^{\beta_2}_{\rho_3\rho_2}(x_2)\Psi^{\beta_1}_{\rho_2\rho_1}(x_1)\rangle = =\frac{C\big[{}^{\beta_3\,\beta_2\,\beta_1}_{\rho_3\,\rho_2\,\rho_1}\big]}{|x_{12}|^{-\Delta_{12}}\,|x_{23}|^{-\Delta_{23}}\,|x_{31}|^{-\Delta_{31}}},$$

where $x_{ij}=x_i-x_j$, $\Delta_{ij}=\Delta_k-\Delta_i-\Delta_j$.

Two comments concerning the form of the boundary two point function seem to be in order: First note that the spectrum $\mathbb{S}^{\mathrm{B}}$ will later be found to have both continuous and discrete parts in general, $\mathbb{S}^{\mathrm{B}}=\mathbb{S}\cup\mathbb{D}$. The symbol $\delta_{\beta,\beta'}$ will accordingly be interpreted as a Kronecker-delta in the case that $\beta,\beta'\in\mathbb{D}$, and as delta-distribution in the case that $\beta,\beta'\in\mathbb{S}$. We have furthermore left undetermined a normalization factor N_0 since we will later find relations between the normalizations of bulk- and boundary operators.

4.2 Consistency conditions

These data are restricted by consistency conditions similar to those found by Cardy and Lewellen in the case of RCFT [8, 9, 10]: Let us first note the two conditions

$$\begin{aligned}\int\limits_{\mathbb{F}^{\mathrm{B}}_{21}} d\beta_s F_{\beta_s\beta_t}\big[{}^{\beta_3\,\beta_2}_{\beta_4\,\beta_1}\big]C\big[{}^{\beta_4\,\beta_3\,\beta_s}_{\rho_4\,\rho_3\,\rho_1}\big]\,C\big[{}^{\beta_s\,\beta_2\,\beta_1}_{\rho_3\,\rho_2\,\rho_1}\big]&\\ = C\big[{}^{\beta_4\,\beta_t\,\beta_1}_{\rho_4\,\rho_2\,\rho_1}\big]\,C\big[{}^{\beta_t\,\beta_3\,\beta_2}_{\rho_4\,\rho_3\,\rho_2}\big]&,\end{aligned} \tag{4.1}$$

expressing associativity of the OPE of boundary operators, and

$$\int_{\mathbb{F}_{21}} \frac{d\alpha}{B(\alpha)}\, D(\alpha,\alpha_2,\alpha_1)A(\alpha|\rho)F_{\alpha\beta}\left[\begin{smallmatrix}\alpha_1 & \alpha_1\\ \alpha_2 & \alpha_2\end{smallmatrix}\right] = \\ = \frac{A(\alpha_2,\beta|\rho)A(\alpha_1,\beta|\rho)}{B(\beta|\rho,\rho)}. \tag{4.2}$$

The data appearing in (4.1)(4.2) that have not yet been introduced are the following: $B(\alpha)$ and $D(\alpha_3,\alpha_2,\alpha_1)$ are two-and three point functions in Liouville theory with periodic boundary conditions [4][5][1], $F_{\beta\gamma}[..]$ are the fusion coefficients that describe the relation between s-channel and t-channel conformal blocks [7], the set $\mathbb{F}_{21}$ is the set of α labelling the primary fields V_α that appear in the OPE of two bulk fields V_{α_2} and V_{α_1} [6] and $\mathbb{F}_{21}^{\rm B}$ is similarly the set of β labelling the boundary fields $\Psi^{\beta}_{\rho_3\rho_1}$ that appear in the OPE of $\Psi^{\beta_2}_{\rho_3\rho_2}$ and $\Psi^{\beta_1}_{\rho_2\rho_1}$. We consider $(D,F;\mathbb{F}_{21})$ to be known from [4, 5, 7, 6] and intend to determine $(A,B,C;\mathbb{F}_{21}^{\rm B})$ as solution of the consistency conditions linking these two sets of data.

In addition to the conditions written above there are only two further conditions to consider [10]: One comes from correlation functions of the form $\langle V\Psi\Psi\rangle$ and will not be considered here. The final condition is an analogue of what is known as the Cardy condition. It will be the main focus of this note. However, in order to even formulate it, we will have to go through some preparations.

5. Known structure functions

5.1 One point function

The one point function $A(\alpha|\rho)$ has been determined in [11][12]:

$$A\big(\tfrac{Q}{2}+iP|\rho\big) = A_0\big(\pi\mu\gamma(b^2)\big)^{-\frac{iP}{b}}\frac{\cos(4\pi sP)}{iP}\cdot \\ \cdot\Gamma(1+2ibP)\Gamma(1+2ib^{-1}P),$$

where s parametrizes the boundary conditions via

$$\cosh\big(2\pi bs\big) = \frac{\rho}{\sqrt{\mu}}\sqrt{\sin(\pi b^2)}.$$

[1] The two point function is recovered from the expression for D by sending $\alpha_3\to 0$, cf. e.g. [6]

A couple of remarks are in order:

1.) As opposed to [11], we have included a prefactor A_0 independent of α, ρ. The Cardy condition that we intend to discuss will imply a relation between A_0 and the normalization factor N_0 that appears in the two point function. Moreover, condition (4.2) further constrains the choice of A_0. We therefore expect all these quantities to be fixed once the normalization of the bulk operators V_α is fixed. The formula for D that was proposed in [5] implies the following normalization:

$$\langle V_{\frac{Q}{2}-iP'}(\infty)V_{\frac{Q}{2}+iP}(0)\rangle_{\rm bulk} = 2\pi\delta(P'-P)$$

for $P,P'\in\mathbb{R}^+$.

2.) ρ determines s only up to $s\to s+inb^{-1}$. But $A(\alpha|s)$ is **not** periodic under $s\to s+inb^{-1}$. We will therefore consider s as true parameter for the boundary conditions. This is related to the existence of nontrivial quantum corrections to the effective action such as the appearance of a "dual" boundary interaction $\tilde\rho e^{\phi/b}$.

Let us furthermore note that for real values of ρ one finds two regimes:

a) $\rho\sqrt{\sin(\pi b^2)} > \sqrt{\mu} \quad\Rightarrow\quad s\in\mathbb{R}$,

b) $\rho\sqrt{\sin(\pi b^2)} < \sqrt{\mu} \quad\Rightarrow\quad s\in i\mathbb{R}$

3.) One may consider the one-point function as defining a boundary "state" ${}_B\langle s|$ as is done in RCFT with boundary [8]:

$${}_{\rm B}\langle s| \;=\; \frac{1}{2\pi}\int_{\mathbb{S}} d\alpha\; A(\alpha|s)\;{}_{\rm I}\langle\alpha|$$

where ${}_{\rm I}\langle\alpha|$: Ishibashi state constructed from the bulk-primary state $\langle P|$. In the present case one may note, however, that the non-normalizability of ${}_{\rm B}\langle s|$ is worse than usual due to the pole that $A(\alpha|s)$ has at $2\alpha=Q$. It will still be a useful object when considered as a distribution, analogous to the interpretation of the so-called microscopic states in [6].

5.2 Boundary two point function

The following expression was found in [11]:

$$B\big(\tfrac{Q}{2}+iP|s_2,s_1\big) \;= \big(\pi\mu\gamma(b^2)b^{2-2b^2}\big)^{-\frac{iP}{b}}\,\frac{\Gamma_b(+2iP)}{\Gamma_b(-2iP)}\cdot \frac{S_b\big(\frac{Q}{2}-i(P+s_1+s_2)\big)S_b\big(\frac{Q}{2}-i(P-s_1-s_2)\big)}{S_b\big(\frac{Q}{2}+i(P+s_1-s_2)\big)S_b\big(\frac{Q}{2}+i(P+s_2-s_1)\big)}$$

Integral representations defining the special functions G_b and S_b can be found in [11]. Both of them are closely related to the Barnes Double Gamma function [13, 14], and S_b was independently introduced under the name of "quantum dilogarithm" by L. Faddeev, cf. e.g. [15] for related applications, properties and references.

Let us note that

$$\boxed{\big|B\big(\tfrac{Q}{2}+iP|s_2,s_1\big)\big|^2 \;=\; 1}$$

in the following three cases:

1. $s_1, s_2 \in \mathbb{R}$,
2. $s_1, s_2 \in i\mathbb{R}$ and
3. $(s_1)^* = -s_2$.

5.3 Remark on uniqueness

The above two results where obtained by considering special cases of the consistency conditions (4.1) and (4.2) in which these equations reduce to finite difference equations with coefficients that can be calculated by other means. A proper discussion of uniqueness of solutions to these difference equations is beyond the scope of the present note, but we would like to mention that at present a strict proof of uniqueness requires assumptions concerning the analyticity of the structure functions w.r.t. α, β: Although invoking the $b \to b^{-1}$-duality as in [16] yields uniqueness on any line parallel to the real axis in the α-plane, one needs further input to fix the dependence on the imaginary part of α, such as e.g. analyticity in certain regions of the α-plane. However, we do not believe that this casts serious doubts on the correctness of the results obtained by these methods: On the one hand it is quite clear that alternative solutions would be of a rather bizarre kind, and on the other hand one may observe that analyticity w.r.t. α in certain reqions of the complex plane is important for the physical consistency of Liouville theory, as will be discussed in the example of the bulk three point function D in [6].

6. Remarks on canonical quantization on the strip

In canonical quantization one would like to find a Hilbert space $\mathcal{H}$ such that the algebra of fields is generated by operators $\phi(\sigma,\tau)$ and $\Pi(\sigma,\tau) = (2\pi)^{-1}\partial_\tau\phi(\sigma,\tau)$ that satisfy

$$[\Pi(\sigma,\tau),\phi(\sigma',\tau)] = -i\delta(\sigma-\sigma').$$

Time evolution should be generated by a Hamiltonian of the form

$$H = \int_0^\pi d\sigma \;:\Big(\frac{1}{4\pi}\big((\partial_t\Phi)^2+(\partial_\sigma\Phi)^2\big) \;+\; \mu e^{2b\phi}\Big): \quad + :\Big(\rho_l e^{b\phi}\big|_{\sigma=\pi} \;+\; \rho_r e^{b\phi}\big|_{\sigma=0}\Big): .$$

The dots are supposed to symbolize all normal orderings and quantum corrections necessary to make H well-defined [2].

Assume that the Liouville zero mode $q \equiv \int_0^\pi d\sigma\phi(\sigma)$ can be diagonalized:

$$\mathcal{H} \;=\; \int_{\mathbb{R}}^{\oplus} dq\;\mathcal{H}_q.$$

The zero mode $\Pi_0 \equiv \int_0^\pi d\sigma\Pi(\sigma)$ then acts as $\frac{1}{\pi i}\partial_q$ and

$$H = -\partial_q^2 + \ldots$$

The *exponential decay* of the interaction terms for $q \to -\infty$ leads one to expect that

$$\mathcal{H}_q \underset{q\to-\infty}{\sim} \mathcal{F},$$

where $\mathcal{F}$: Fock-space generated by the non-zero modes a_n acting on Fock-vaccuum Ω. One may therefore characterize generalized eigenfunctions of H by their asymptotic behavior for $q \to -\infty$:

$$\Psi_E(q) \underset{q\to-\infty}{\sim} e^{iPq}F_N^+ + e^{-iPq}F_N^-,$$

[2]This may include addition of "dual" interactions like the boundary interaction $\tilde{\rho}e^{\phi/b}$.

where $F_N^{\pm} \in \mathcal{F}$, N: level, $E = \frac{Q^2}{4} + P^2 + N$.

Exponential blow-up of interaction terms for $q \to +\infty$ one the other hand leads one to expect *reflection*, i.e.

$$F_N^- \; = \; \mathcal{R}(P|s_2,s_1)F_n^+.$$

Conformal symmetry determines $\mathcal{R}(P|s_2,s_1)$ up to a scalar multiple $R(P|s_2,s_1)$. R describes the asymptotic behavior the wave-functions for primary states $|P,s_2,s_1\rangle$ which satisfy $L_n|P,s_2,s_1\rangle = 0$ for $n>0$, $L_0|P,s_2,s_1\rangle = (P^2+\frac{Q^2}{4})|P,s_2,s_1\rangle$:

$$\Psi_P(q) \underset{q\to-\infty}{\sim} \big(e^{iPq} + R(P|s_2,s_1)e^{-iPq}\big)\Omega.$$

One may observe that the reflection amplitude $R(P|s_2,s_1)$ also describes the relation between $\Psi_P(q)$ and its analytic continuation $\Psi_{-P}(q)$:

$$\Psi_P(q) \; = \; R(P|s_2,s_1)\Psi_{-P}(q).$$

$R(P|s_2,s_1)$ is clearly a fundamental quantity for describing the dynamics of Liouville theory on the strip. It is therefore satisfactory to observe that *state-operator correspondence* leads to the following relation between $R(P|s_2,s_1)$ and the two point function $B(\beta|s_2,s_1)$:

$$R(P|s_2,s_1) \; = \; B\big(\tfrac{Q}{2}+iP|s_2,s_1\big).$$

7. Partition function on the strip

The naive definition $\mathrm{Tr}_{\mathcal{H}^B}\, q^{H^B_{s_2,s_1}}$ is divergent. It is better to consider **relative** partition functions such as

$$\mathcal{Z}^{\mathrm{B}}_{s_2,s_1}(q) = \mathrm{Tr}_{\mathcal{H}^{\mathrm{B}}}\Big(q^{H^{\mathrm{B}}_{s_2,s_1}} - q^{H^{\mathrm{B}}_{s_0,s_0}}\Big),$$

where s_0 parametrizes some fixed reference boundary condition. This object provides information on the dependence of the spectrum w.r.t. the boundary conditions. In view of (3.1) one expects $\mathcal{Z}^{\mathrm{B}}$ to have the general form

$$\mathcal{Z}^{\mathrm{B}}_{s_2,s_1}(q) = \int_{\mathbb{S}^{\mathrm{B}}} d\alpha\;\chi_\alpha(q)\;N(\alpha|s_2,s_1), \qquad (7.1)$$

where the Virasoro character $\chi_\alpha(q)$ is given by

$$\begin{aligned}\chi_\alpha(q) =& \eta^{-1}(q)q^{-(\alpha-\frac{Q}{2})^2}\\ =& q^{\frac{1-c}{24}+\Delta_\alpha}\prod_{k=1}^{\infty}(1-q^k).\end{aligned}$$

Let us now present two guesses concerning the ingredients of (7.1) that will later be confirmed independently:

First note that in the case $\rho_2 > 0$,$\rho_1 > 0$ one would expect the boundary contributions to the Hamiltonian to be positive. In this case one does not expect any bound states:

$$\mathbb{S}^{\mathrm{B}} = \mathbb{S} \equiv \frac{Q}{2} + i\mathbb{R}^+. \qquad (7.2)$$

Second, Al.B. Zamolodchikov has proposed [17] that the relation between spectral density N and reflection amplitude R that is well-known for quantum mechanical problems (see e.g. [18], and [19] for a simple heuristic argument) will still work in the present situation:

$$N(\beta|s_2,s_1) \; = \; \frac{1}{2\pi i N_0}\frac{\partial}{\partial P}\log\frac{R(P|s_2,s_1)}{R(P|s_0,s_0)}, \qquad (7.3)$$

where β and P are related by $\beta = \frac{Q}{2} + iP$.

8. Cardy condition

Consider Liouville theory on an annulus. There are two ways to describe the same amplitude:

- as **partition function** on the strip (periodic imaginary time)

$$\mathcal{Z}^{\mathrm{B}}_{s_2,s_1}(q) = \int_{\mathbb{S}^B} d\beta\;\chi_\beta(q)\;N(\beta|s_2,s_1).$$

- as **transition amplitude** in the theory with periodic boundary conditions.:

$$\begin{aligned}&{}_B\langle s_2|\tilde{q}^{H-\frac{c}{24}}|s_1\rangle_B - {}_B\langle s_0|\tilde{q}^{H-\frac{c}{24}}|s_0\rangle_B =\\ &= \int_{\mathbb{S}}\frac{d\alpha}{2\pi}\Big(\big(A(\alpha|s_2)\big)^*A(\alpha|s_1)\\ &\qquad - \big(A(\alpha|s_0)\big)^*A(\alpha|s_0)\Big)\chi_\alpha(\tilde{q}).\end{aligned}$$

The Cardy condition will then be the relation

$$\boxed{\begin{aligned}\mathcal{Z}^{\mathrm{B}}_{s_2,s_1}(q) =& {}_B\langle s_2|\tilde{q}^{H-\frac{c}{24}}|s_1\rangle_B\\ &- {}_B\langle s_0|\tilde{q}^{H-\frac{c}{24}}|s_0\rangle_B,\end{aligned}} \qquad (8.1)$$

where $\tilde{q} = \exp(-2\pi i/\tau)$ if $q = \exp(2\pi i\tau)$.

8.1 Case a)

In this case the validity of some analogue of the Cardy condition (8.1) was first verified by Al.B. Zamolodchikov [17][3]. One may start with the right hand side of (8.1). The characters $\chi_\alpha(\tilde{q})$ can be expressed as sum over charaters $\chi_\alpha(q)$ by means of

$$\chi_\alpha(\tilde{q}) = \int_{\mathbb{S}} d\beta \; S(\alpha,\beta)\, \chi_\beta(q), \qquad (8.2)$$

where

$$S(\alpha,\beta) \;=\; 2\sqrt{2}\cos\bigl(4\pi(\alpha-\tfrac{Q}{2})(\alpha'-\tfrac{Q}{2})\bigr)$$

In the presently considered case there is no problem to exchange the orders of integrations over α and β:

$$_B\langle s_2|\tilde{q}^{H-\frac{c}{24}}|s_1\rangle_B \;= \int_{\mathbb{S}} d\beta \left\{\int_{\mathbb{S}} d\alpha M(\alpha,\beta|s_2,s_1)\right\}\chi_\beta(q)$$

where

$$M(\alpha,\beta|s_2,s_1) = \Bigl(\bigl(A(\alpha|s_2)\bigr)^* A(\alpha|s_1) - \bigl(A(\alpha|s_0)\bigr)^* A(\alpha|s_0)\Bigr)\, S(\alpha,\beta)$$

The integral in curly brackets can now easlily be identified with the integral representation for $N(\beta|s_2,s_1)$ that follows from the expression [11], equation (3.18) for $B(\beta|s_2,s_1)$ if A_0 and N_0 are related by

$$\frac{1}{\pi N_0} \;=\; \sqrt{2}|A_0|^2.$$

8.2 Case b)

Exchange of orders of integration produces a divergent result. Instead one may write the modular transformation of characters in the form

$$\chi_\alpha(\tilde{q}) = \sqrt{2}\int_{r+i\mathbb{R}} d\beta \;\, e^{4\pi i(\alpha-\frac{Q}{2})(\alpha'-\frac{Q}{2})}\;\chi_\beta(q),$$

where the contour of integration was shifted into the half plane by an amount that will be chosen such that

$$r > \max\bigl(|\sigma_+|,|\sigma_-|,\tfrac{Q}{2}\bigr) - \tfrac{Q}{2},$$

[3] From the information available to the author it seems that a different regularization was used in [17].

where $\sigma_\pm \equiv i(s_2 \pm s_1) \in \mathbb{R}$. Now it is possible to exchange orders of integrations over α and β:

$$_B\langle s_2|\tilde{q}^{H-\frac{c}{24}}|s_1\rangle_B - {}_B\langle s_0|\tilde{q}^{H-\frac{c}{24}}|s_0\rangle_B = = \int_{r+i\mathbb{R}} d\beta\; N(\beta|s_2,s_1)\;\chi_\beta(q)$$

The contour of integration may be deformed into $\frac{Q}{2}+i\mathbb{R}$ plus a finite sum of circles around poles of N. By using the fact that $S_b(x)$ has poles for $x=-nb-mb^{-1}$, $n,m\in\mathbb{Z}^{\geq 0}$ [14, 15] one may now read off the spectrum on the strip:

$$\boxed{\mathcal{H}^{\rm B} \;=\; \int_{\mathbb{S}} d\alpha\;\mathcal{V}_\alpha \;\;\oplus\;\; \bigoplus_{\alpha\in\mathbb{D}_{s_2s_1}} \mathcal{V}_\alpha,}$$

where

$$\mathbb{D}_{s_2s_1} = \;\bigl\{\beta\in\mathbb{C}\mid \beta = Q-|\sigma_s|+nb+m\tfrac{1}{b} < \tfrac{Q}{2}, \text{ where } n,m\in\mathbb{Z}^{\geq 0}, s=+,-\bigr\}.$$

By means of this calculation we have determined the spectrum on the strip from the spectrum of Liouville theory on the cylinder! Let us note that:

1.) The spectrum is unitary only if $|\sigma_\pm| < Q$. Otherwise one finds nonunitary representations in the spectrum.

2.) As long as both $|\sigma_\pm| < Q/2$ one still has $\mathbb{D}_{s_2s_1} = \emptyset$.

3.) It is interesting, but puzzling to observe that there is a third case where one obtains a spectrum that is compatible with hermiticity of the Hamiltonian, namely $(s_1)^* = -s_2$. As noted earlier, one also finds unitarity of the reflection amplitude in this case.

9. Outlook: Connection to noncompact quantum group

In order to carry out the program to determine the structure functions as solutions of the consistency conditions they satisfy, it would be certainly quite useful if one could establish connections between the structure functions and the characterisic data of (a generalization of) a modular functor, such as e.g. the 6j- or Racah-Wigner symbols of a quantum group and the associated modular transformation coefficients S. In the case of rational conformal field theories connections of this kind have been exploited in [20][21] to derive expressions for the structure functions in terms of these data and to prove validity of the consistency equations on the basis of identities that the defining data of the modular functor have to satisfy.

From this point of view the following two partial results look quite encouraging: We have found (details will be given elsewhere) that

$$\boxed{C\left[\begin{smallmatrix}\beta_3 & \beta_2 & \beta_1\\ s_3 & s_2 & s_1\end{smallmatrix}\right] \;=\; M(\beta_3,\beta_2,\beta_1)\left\{\begin{smallmatrix}\sigma_1 & \beta_1\\ \beta_2 & \sigma_3\end{smallmatrix}\Big|\begin{smallmatrix}\sigma_2\\ \beta_3\end{smallmatrix}\right\}'_b,} \tag{9.1}$$

where $\sigma_i \equiv is_i$, $\{\dots\}'_b$ are b-Racah-Wigner symbols associated to a continuous series of representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$. They differ from those constructed in [22] by a change of normalization of the Clebsch-Gordan coefficients:

$$\left\{\begin{smallmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{smallmatrix}\Big|\begin{smallmatrix}\alpha_s\\ \alpha_t\end{smallmatrix}\right\}'_b = \frac{S_b(\alpha_t+\alpha_4-\alpha_1)S_b(\alpha_3+\alpha_t-\alpha_2)}{S_b(\alpha_4+\alpha_3-\alpha_s)S_b(\alpha_s+\alpha_2-\alpha_1)}\left\{\begin{smallmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{smallmatrix}\Big|\begin{smallmatrix}\alpha_s\\ \alpha_t\end{smallmatrix}\right\}_b.$$

The expression (9.1) for the boundary three point function satisfies the associativity condition (4.1) due to the relation between fusion coefficients F and b-Racah-Wigner symbols [7], as well as the pentagon equation satisfied by the latter [22]. Condition (4.1) clearly leaves the freedom to change the normalization of the boundary fields. This freedom can be fixed by considering special cases of (4.1) where one of $\beta_4,\dots,\beta_1$ is chosen to correspond to the degenerate representation $\mathcal{V}_{-b}$. One then obtains *linear* finite difference equations for the boundary three point function with coefficients given in terms of know fusion coefficients and the special three point function c_- calculated in [11][4].

It seems moreover suggestive to observe that the formula for the one point function can for $\alpha \in \mathbb{S}$ be rewritten in a form that resembles the Cardy formula [8] for the one-point function:

$$\boxed{A(\alpha|s) \;=\; e^{i\delta(\alpha)}\frac{S(\alpha;s)}{\sqrt{\mu(\alpha)}},} \tag{9.2}$$

[4]The evaluation of the integral for c_- that was not presented in [11] may be circumvented by using old results of Gervais and Neveu [23]

where $e^{2i\delta(\alpha)} \equiv R(\alpha)$ is the bulk reflection amplitude [5], and $\mu(\alpha)$ turns out to be the Plancherel measure of the quantum group dual to the category of representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ that was considered in [7][22]. One clearly sees in what respects the bootstrap in our noncompact case is richer than in RCFT (reflection amplitude, loss of direct relation between "quantum dimension" and modular transformation coefficients). On the other hand one may well consider (9.2) to be the most natural generalization of the Cardy formula for the one point function that one may reasonably hope for.

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