To Vladimir Igorevich Arnold on the occasion of his 70th birthday.

Elimination theory and Newton polytopes.

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1 Introduction.

Let $N \subset \mathbb{C}^n$ be an affine algebraic variety, and let $\pi : \mathbb{C}^n \to \mathbb{C}^m$ be a projection. The goal of elimination theory is to describe the defining equations of $\pi(N)$ in terms of the defining equations of N. We shall study defining equations of projections in the context of Newton polytopes: suppose that the variety $N \subset (\mathbb{C} \setminus 0)^n$ is defined by equations $f_1 = \ldots = f_k = 0$ with given Newton polytopes and generic coefficients, and the projection $\pi(N) \subset (\mathbb{C} \setminus 0)^m$ is given by one equation $g = 0$. Under this assumption, we shall describe the Newton polytope and the leading coefficients of the Laurent polynomial g in terms of the Newton polytope and the leading coefficients of the Laurent polynomials $f_1, \ldots, f_k$ (by the leading coefficients we mean the coefficients of monomials from the boundary of the Newton polytope).

In this section, we define the equation g of a projection of a complete intersection $f_1 = \ldots = f_k = 0$ (Definition 1.3) and describe its Newton polytope in terms of the Newton polytopes $\Delta_1, \ldots, \Delta_k$ of the equations $f_1, \ldots, f_k$ (Theorem 1.7). Section 2 contains some facts about the geometry of this polytope. In particular, this polytope is an increasing function of polytopes $\Delta_1, \ldots, \Delta_k$ (Theorem 2.3) and equals the mixed fiber polytope of $\Delta_1, \ldots, \Delta_k$ up to a shift and dilatation (Theorem 2.12). The existence and other basic properties of mixed fiber polytopes (Definition 2.11) are proved in Section 3. Sections 4 and 5 are concerned with computation of leading coefficients of g. For example, one can use Theorems 4.6 and 5.13 to compute explicitly the coefficients of monomials, which correspond to the vertices of the Newton polytope of g, provided that the polytopes $\Delta_1, \ldots, \Delta_k$ satisfy some condition of general position (Definition 5.2). In Section 6, we present some other versions of elimination theory in the context of Newton polytopes, such as elimination theory for rational and analytic functions.

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Elimination theory and Newton polytopes. Many important problems, related to Newton polytopes and tropical geometry, turn out to be special cases of this version of elimination theory. Let us give some examples of such problems.

1) To compute the number of common roots of polynomial equations with given Newton polytopes and generic coefficients. The answer is given by Kouchnirenko-Bernstein's formula (see [B] or Theorem 1.6 below).

2) To compute the product of common roots of polynomial equations with given Newton polytopes and generic coefficients. If the Newton polytopes satisfy some conditions of general position, then the answer is given by Khovanskii's product formula (see [Kh2] or Theorems 5.11 and 5.13 below).

3) To compute the sum of values of a polynomial over the common roots of polynomial equations with given Newton polytopes and generic coefficients. If the Newton polytopes satisfy some conditions of general position, then the answer is given by Gelfond-Khovanskii's formula (see [GKh] or Theorem 5.8 below).

4) Implicitization theory: to compute the Newton polytope and the leading coefficients of the defining equation of a hypersurface, parameterized by a polynomial mapping $(\mathbb{C}\setminus 0)^n \to (\mathbb{C}\setminus 0)^{n+1}$ with given Newton polytopes and generic coefficients of the components. The Newton polytope was described by Sturmfels, Tevelev and Yu (see [STY]).

5) To describe the Newton polytope and the leading coefficients of a multidimensional resultant. The Newton polytope and the absolute values of leading coefficients were computed by Sturmfels (see [S]).

6) To prove the existence of mixed fiber polytopes (Definition 2.11). Existence of mixed fiber polytopes was predicted in [McD] and proved in [McM].

Problems 1-3 in the context of elimination theory. To put problems 1-3 in the context of elimination theory, consider a Laurent monomial as a projection $\pi : (\mathbb{C}\setminus 0)^n \to (\mathbb{C}\setminus 0)$. Then the defining equation of the projection of a 0-dimensional complete intersection $\{f_1 = \ldots = f_n = 0\} = \{z_1, \ldots, z_N\}$ is a polynomial $g(t) = \prod_i \big(t - \pi(z_i)\big)$ in one variable.

Lemma **1.1.** *1) The length of the (1-dimensional) Newton polytope of g equals the number of common roots of $f_1, \ldots, f_n$.*

2) The constant term of g (which is a leading coefficient in our terminology) equals the product of the values of the monomial $-\pi$ over all common roots of $f_1, \ldots, f_n$.
3) Let S_m be the polynomial of m variables, such that $S_m(\sum_i x_i, \sum_i x_i^2, \ldots, \sum_i x_i^m)$ equals the m-th elementary symmetric function of independent variables x_i. Then the coefficient of the monomial t^m in the polynomial g equals $(-1)^{N-m} S_m(p_1, \ldots, p_m)$, where p_m is the sum of the values of the monomial π^m over all common roots of $f_1, \ldots, f_n$.

All these facts are obvious, and we omit the proof. We generalize this lemma to projections of complete intersections of an arbitrary dimension: see Theorem 1.7 and Section 4.

Lemma 1.1 implies that Kouchnirenko-Bernstein's formula (Theorem 1.6), Khovanskii's product formula (Theorems 5.11 and 5.13) and Gelfand-Khovanskii's formula (Theorem 5.8) can be seen as explicit formulas for the Newton polytope of g, the leading coefficients of g, and all coefficients of g respectively, provided that the Newton polytopes of $f_1, \ldots, f_n$ satisfy some condition of general position. We generalize these observations to projections of complete intersections of an arbitrary dimension: see Section 5.

Problems 4-6 in the context of elimination theory. One can consider implicitization theory as a special case of elimination theory. Indeed, consider a mapping $g = (g_0, \ldots, g_k) : (\mathbb{C} \setminus 0)^n \to (\mathbb{C} \setminus 0)^{k+1}$ and a k-dimensional complete intersection $F = \{f_1 = \ldots = f_{n-k} = 0\} \subset (\mathbb{C} \setminus 0)^n$ with $g_0, \ldots, g_k, f_1, \ldots, f_{n-k}$ being Laurent polynomials on $(\mathbb{C} \setminus 0)^n$. Let π be the standard projection $(\mathbb{C} \setminus 0)^n \times (\mathbb{C} \setminus 0)^{k+1} \to (\mathbb{C} \setminus 0)^{k+1}$, and let $y_0, \ldots, y_k$ be the standard coordinates on $(\mathbb{C} \setminus 0)^{k+1}$. Then the defining equation of the image $g(F) \subset (\mathbb{C} \setminus 0)^{k+1}$ equals the defining equation of the projection $\pi(\{g_0 - y_0 = \ldots = g_k - y_k = f_1 = \ldots = f_{n-k} = 0\})$.

A multidimensional resultant is the "universal" special case of elimination theory, which is clear from the following version of the definition of a resultant. Consider polynomials $g_i(x_1, \ldots, x_k) = \sum\limits_{b \in B_i} c_{b,i} x^b$, $i = 0, \ldots, k$, $B_i \subset \mathbb{Z}^k$ as polynomials f_i in variables $c_{b,i}$ and x_j with all coefficients equal to 1. Let π be the projection of the domain of the polynomials $(f_0, \ldots, f_k)$ along the domain of the polynomials $(g_0, \ldots, g_k)$. Then the defining equation of the projection $\pi(\{f_0 = \ldots = f_k = 0\})$ is called the $(B_0, \ldots, B_k)$-*resultant*. Note that this definition of the multidimensional resultant is somewhat different from the classical one if we understand the defining equation of a projection in the sense of Definition 1.3, since it is not always square free. We consider the square free version of Definition 1.3 in Section 6 (see Theorem 6.2).

Elimination theory, implicitization theory and the theory of multidimensional resultants are equivalent in the sense that they can be formulated in terms of each other. Thus, the contents of this paper can be written in terms of resultants or implcitization theory. When written in these terms, Theorem 1.7 turns into the descriptions of Newton polytopes from [S] and [STY], while the facts from Sections 4 and 5 give some new information about the leading coefficients. For example, one can use Theorems 4.6 and 5.13 to compare the signs of the leading coefficients of a multidimensional resultant (see [EKh]).

Theory of mixed fiber polytopes turns out to be the Newton-polyhedral counterpart of elimination theory in the following sense. Define the *composite polytope* of polytopes $\Delta_1, \ldots \Delta_k$ as the Newton polytope of a projection of a complete intersection $f_1 = \ldots = f_k = 0$, provided that the Newton polytope of f_i is Δ_i and the coefficients of $f_1, \ldots, f_k$ are in general position. Then Theorems 1.7 and 2.12 imply that the composite polytope satisfies the definition of the mixed fiber polytope up to a shift and dilatation, which proves the existence of mixed fiber polytopes. We omit the details and prefer to give an independent elementary proof of the existence of mixed fiber polytopes in Section 3 to make our paper self-contained (the proof from [McM] is based on the paper [McM2], which has not been published by now). Note that composite polytopes are more convenient then mixed fiber polytopes in some sense; for example, they are monotonous (Theorem 2.3).

The composite polynomial. Denote the Zariski closure of a set M by $\overline{M}$. For an algebraic mapping $f : M \to (\mathbb{C} \setminus 0)^n$ of an irreducible algebraic variety M, denote the number of points in the preimage $f^{(-1)}(x)$ of a generic point $x \in f(M)$ by $m(f)$, provided that this number is finite, and let $m(f)$ be 0 otherwise. Define a *cycle* $N = \sum_i a_i N_i$ in $(\mathbb{C} \setminus 0)^n$ as a formal linear combination of irreducible algebraic varieties $N_i \subset (\mathbb{C} \setminus 0)^n$ of the same dimension with integer coefficients a_i.

Definition **1.2.** Let $\pi : (\mathbb{C} \setminus 0)^n \mapsto (\mathbb{C} \setminus 0)^{n-k}$ be an epimorphism of complex tori. For a cycle $N = \sum_i a_i N_i$ in $(\mathbb{C} \setminus 0)^n$, the cycle $\sum_i m(\pi|_{N_i}) a_i \overline{\pi(N_i)}$ is called *the projection* $\pi_* N$ of the cycle N.

Let $f_1, \ldots, f_m$ be Laurent polynomials on $(\mathbb{C} \setminus 0)^n$, such that $\operatorname{codim}\{f_1 = \ldots = f_m = 0\} = m$. Denote the intersection cycle of the divisors of the polynomials $f_1, \ldots, f_m$ by $[f_1 = \ldots = f_m = 0]$.

Definition **1.3.** Let $f_0, \ldots, f_k$ be Laurent polynomials on $(\mathbb{C} \setminus 0)^n$, such that $\operatorname{codim}\{f_0 = \ldots = f_k = 0\} = k + 1$. The Laurent polynomial $\pi_{f_0,\ldots,f_k}$ on $(\mathbb{C} \setminus 0)^{n-k}$, such that $[\pi_{f_0,\ldots,f_k} = 0] = \pi_*[f_0 = \ldots = f_k = 0]$, is called *the composite polynomial* of polynomials $f_0, \ldots, f_k$ with respect to the projection π.

The composite polynomial $\pi_{f_0,\ldots,f_k}$ is defined up to a monomial factor. To describe its Newton polytope, we need Kouchnirenko-Bernstein's formula for the number of roots of a system of polynomial equations.

Kouchnirenko-Bernstein's formula. The set of all convex bodies in $\mathbb{R}^m$ is a semigroup with respect to the operation of *Minkowskii summation* $A+B = \{a + b \,|\, a \in A,\, b \in B\}$.

Definition **1.4.** *The mixed volume* MV_μ, induced by a volume form μ on $\mathbb{R}^m$, is the symmetric Minkowski-multilinear function of m convex bodies in $\mathbb{R}^m$, such that $\mathrm{MV}_\mu(\Delta, \ldots, \Delta) = \int_\Delta \mu$ for every convex body $\Delta \subset \mathbb{R}^m$. The mixed volume, induced by the standard volume form, is denoted by MV.

The *restriction* $f|_B$ of a Laurent polynomial $f(x) = \sum_{a\in\mathbb{Z}^n} c_a x^a$ onto a set $B \subset \mathbb{Z}^n$ is the polynomial $\sum_{a\in B} c_a x^a$. *The Newton polytope* Δ_f of a Laurent polynomial f is the convex hull of the set A such that $f(x) = \sum_{a\in A} c_a x^a$ and $c_a \neq 0$.

Definition **1.5.** Laurent polynomials $f_0, \ldots, f_k$ on $(\mathbb{C}\setminus 0)^n$ are said to be *Newton-nondegenerate* if, for any collection of faces $A_0 \subset \Delta_{f_0}, \ldots, A_k \subset \Delta_{f_k}$, such that the sum $A_0 + \ldots + A_k$ is at most a k-dimensional face of the sum $\Delta_{f_0} + \ldots + \Delta_{f_k}$, the restrictions $f_0|_{A_0}, \ldots, f_k|_{A_k}$ have no common zeros in $(\mathbb{C} \setminus 0)^n$.

Newton-nondegenerate collections of polynomials form a dense subset in the space of all collections of polynomials with given Newton polytopes.

Theorem **1.6** (Kouchnirenko-Bernstein, [B]). *1) The number of common roots of Newton-nondegenerate Laurent polynomials $f_1, \ldots, f_n$ in $(\mathbb{C} \setminus 0)^n$, taking multiplicities into account, is equal to $n!\,\mathrm{MV}(\Delta_{f_1}, \ldots, \Delta_{f_n})$.*
2) Without the assumption of Newton-nondegeneracy, the number of isolated common roots of $f_1, \ldots, f_n$ in $(\mathbb{C} \setminus 0)^n$, taking multiplicities into account, is not greater than $n!\,\mathrm{MV}(\Delta_{f_1}, \ldots, \Delta_{f_n})$.

The Newton polytope of the composite polynomial. The Newton polytope of the polynomial $\pi_{f_0,\ldots,f_k}$ is uniquely determined up to a shift by the condition $(*)$ below. This condition is a corollary of Kouchnirenko-Bernstein's formula, and can be seen as its generalization (see Lemma 1.1.1).

THEOREM **1.7.** *1) Let $\pi^\times : \mathbb{Z}^{n-k} \hookrightarrow \mathbb{Z}^n$ be the inclusion of character lattices, defined by the epimorphism $\pi : (\mathbb{C} \setminus 0)^n \mapsto (\mathbb{C} \setminus 0)^{n-k}$. Let $A_0, \ldots, A_k \subset \mathbb{Z}^n$ and $A \subset \mathbb{Z}^{n-k}$ be the Newton polytopes of polynomials $f_0, \ldots, f_k$ and $\pi_{f_0,\ldots,f_k}$. Then, for any convex bodies $B_1, \ldots, B_{n-k-1} \subset \mathbb{Z}^{n-k}$,*

$$(n-k)!\,\mathrm{MV}(A, B_1, \ldots, B_{n-k-1}) =$$

$$= n!\,\mathrm{MV}(A_0, \ldots, A_k, \pi^\times B_1, \ldots, \pi^\times B_{n-k-1}), \qquad (*)$$

provided that the polynomials $f_0, \ldots, f_k$ are Newton-nondegenerate.
2) Without the assumption of Newton-nondegeneracy,

$$(n-k)!\,\mathrm{MV}(A, B_1, \ldots, B_{n-k-1}) \leqslant$$

$$\leqslant n!\,\mathrm{MV}(A_0, \ldots, A_k, \pi^\times B_1, \ldots, \pi^\times B_{n-k-1}). \qquad (**)$$

This theorem gives rise to *"elimination theory for convex bodies"*, which describes the polytope A in terms of $A_0, \ldots, A_k$, proceeding from the equality $(*)$, and estimates it, proceeding from the inequality $(**)$. See Section 2 for details.

PROOF. By continuity and linearity of the mixed volume, it is enough to prove this theorem under the assumption that $B_1, \ldots, B_{n-k-1}$ are polytopes with integer vertices. Under this assumption, consider generic Laurent polynomials $g_1, \ldots, g_{n-k-1}$ on $(\mathbb{C} \setminus 0)^{n-k}$ with the Newton polytopes $B_1, \ldots, B_{n-k-1}$. Since $\pi_{f_0,\ldots,f_k}$ is not identically zero, the collection $\pi_{f_0,\ldots,f_k}$, $g_1, \ldots, g_{n-k-1}$ is Newton-nondegenerate. If the collection $f_0, \ldots, f_k$ is Newton-nondegenerate, then the collection $f_0, \ldots, f_k, g_1 \circ \pi, \ldots, g_{n-k-1} \circ \pi$ is also Newton-nondegenerate.

By Kouchnirenko-Bernstein's formula, the number of solutions of the systems $f_0 = \ldots = f_k = g_1 \circ \pi = \ldots = g_{n-k-1} \circ \pi = 0$ and $\pi_{f_0,\ldots,f_k} = g_1 = \ldots = g_{n-k-1} = 0$ equal $n!V(A_0, \ldots, A_k, B_1, \ldots, B_{n-k-1})$ and $(n-k)!V(A, B_1, \ldots, B_{n-k-1})$ respectively. On the other hand, the solutions of the second system are the projections of the solutions of the first one. □

2 Elimination theory for convex bodies.

Theorem 1.7 motivates the following definition, which gives rise to "elimination theory for convex bodies".

A convex body B in an $(n-k)$-dimensional subspace $L \subset \mathbb{R}^n$ is called *a composite body* of convex bodies $\Delta_0, \ldots, \Delta_k \subset \mathbb{R}^n$, if the mixed volume $n!\,\mathrm{MV}(\Delta_0, \ldots, \Delta_k, B_1, \ldots, B_{n-k-1})$ in $\mathbb{R}^n$ equals the mixed volume $(n-k)!\,\mathrm{MV}(B, B_1, \ldots, B_{n-k-1})$ in L for every collection of convex bodies $B_1, \ldots, B_{n-k-1} \subset L$. See Definition 2.1 for details.

For every collection of convex bodies $\Delta_0, \ldots, \Delta_k$, there exists a unique up to a shift composite body (Theorem 2.2). The existence of composite bodies follows from the fact that the mixed fiber body of bodies $\Delta_0, \ldots, \Delta_k$ satisfies the definition of a composite body up to a shift and dilatation (Definition 2.11 and Theorem 2.12).

Thus, the theory of composite bodies is a version of the theory of mixed fiber polytopes, conjectured in [McD] and constructed in [McM]. Since [McM] is based on the paper [McM2], which have not been published yet, we prefer to present another approach to mixed fiber polytopes in Section 3 to make our paper self-contained. At the same time, we prove some basic facts about composite bodies:
- A composite body of polytopes is a polytope (Theorem 2.10.2).
- A composite body of integer polytopes (i. e. polytopes such that all their vertices are integer lattice points) is a shifted integer polytope (Theorem 3.20).
- Composite bodies are monotonous (Theorem 2.3).
- The linear span of a composite body depends on the linear spans of its arguments (Theorem 2.6).
- Codimension m faces of a composite polytope depend on codimension m faces of its arguments (Theorem 3.13).
- In particular, vertices of the composite polytope of polytopes $\Delta_0, \ldots, \Delta_k$ can be expressed in terms of moments of their k-dimensional faces (Theorem 3.19).

Composite bodies.

Definition **2.1.** Let $L \subset \mathbb{R}^n$ be a vector subspace of codimension k, let μ be a volume form on $\mathbb{R}^n/L$, and let $\Delta_0, \ldots, \Delta_k$ be convex bodies in $\mathbb{R}^n$. A convex body $B \subset L$ is called a *composite body* of $\Delta_0, \ldots, \Delta_k$ in L and is denoted by $\mathrm{CB}_\mu(\Delta_0, \ldots, \Delta_k)$ if, for every collection of convex bodies $B_1, \ldots, B_{n-k-1} \subset L$,

$$n!\,\mathrm{MV}_{\mu'\wedge\mu}(\Delta_0, \ldots, \Delta_k, B_1, \ldots, B_{n-k-1}) = (n-k)!\,\mathrm{MV}_{\mu'}(B, B_1, \ldots, B_{n-k-1}),$$

where μ' is a volume form on L.

Theorem **2.2.**
1) For any collection of convex bodies $\Delta_0, \dots, \Delta_k \subset \mathbb{R}^n$*, there exists a composite body* $\mathrm{CB}_\mu(\Delta_0, \dots, \Delta_k)$.
2) A composite body $\mathrm{CB}_\mu(\Delta_0, \dots, \Delta_k)$ *is unique up to a shift.*

Proof. Part 1 follows from an explicit formula for composite bodies, see Theorem 2.12. Part 2 follows from monotonicity, see Theorem 2.3. □

The proof of uniqueness implies that, in Definition 2.1, it is enough to consider collections $B_1, \dots, B_{n-k-1}$ such that $B_1, \dots, B_{n-k-1}$ are simplices. Since composite bodies are unique up to a shift, all the statements about composite bodies are implied to be valid up to a shift of a composite body.

Monotonicity of a composite body.

Theorem **2.3.** *If* $\Delta_i \subset \Delta_i'$ *for* $i = 0, \dots, k$*, then* $\mathrm{CB}_\mu(\Delta_0, \dots, \Delta_k) \subset \mathrm{CB}_\mu(\Delta_0', \dots, \Delta_k')$.

This is a corollary of monotonicity of mixed volume and the following fact.

Lemma **2.4.** *Let* Δ *and* Δ' *be convex bodies in* $\mathbb{R}^m$*. Suppose that*

$$\mathrm{MV}(\Delta, B, \dots, B) \leqslant \mathrm{MV}(\Delta', B, \dots, B)$$

for every simplex B*. Then, for some shift* $a \in \mathbb{R}^n$*, the shifted body* $\Delta + a$ *is contained in* Δ'.

Proof. Choose a such that the minimax distance

$$\mathrm{dist}(\Delta + a, \Delta') = \max_{x \in \Delta + a} \min_{y \in \Delta'} |x - y|$$

is minimal. Suppose that $\mathrm{dist}(\Delta + a, \Delta') > 0$. Then the set of all covectors $\gamma \in (\mathbb{R}^m)^*$, such that

$$\max_{x \in \Delta + a} \langle \gamma, x \rangle > \max_{y \in \Delta'} \langle \gamma, y \rangle,$$

is not contained in a half-space. In particular, it contains covectors $\gamma_0, \dots, \gamma_m$ such that none of them is a linear combination of the others with non-negative coefficients. Denote by B an m-dimensional simplex with external normal covectors $\gamma_0, \dots, \gamma_m$. Then $\mathrm{MV}(\Delta, B, \dots, B) > \mathrm{MV}(\Delta', B, \dots, B)$ because of the following formula for mixed volumes. □

LEMMA **2.5.** *Let Δ be a convex body, let $B_1, \ldots, B_{m-1}$ be polytopes, and let μ be a volume form in $\mathbb{R}^m$. Let $\Gamma \subset (\mathbb{R}^n)^*$ be a set that contains one external normal covector for each $(m-1)$-dimensional face of the sum $B_1 + \ldots + B_{m-1}$. Then*

$$\mathrm{MV}_\mu(\Delta, B_1, \ldots, B_{m-1}) = \frac{1}{m} \sum_{\gamma \in \Gamma} \max_{x \in \Delta} \langle \gamma, x \rangle \, \mathrm{MV}_{\mu/\gamma}(B_1^\gamma, \ldots, B_{m-1}^\gamma),$$

where B_i^γ is the maximal face of B_i on which γ attains its maximum as a function on B_i.

The mixed volume in the right hand side makes sense, since its arguments are all parallel to the same $(m-1)$-dimensional subspace $\ker \gamma$.

PROOF. If $\Delta = B_1 = \ldots = B_{m-1}$ contains the origin, then this formula states that the volume of Δ equals the sum of volumes of the convex hulls $\mathrm{conv}(\{0\} \cup F)$, where F runs over all $(m-1)$-dimensional faces of Δ. In general, the formula follows from this special case by additivity and continuity of the mixed volume. □

Linear span of a composite body. We need one more fact about composite bodies, which, in the context of Newton polytopes, reflects the fact that elimination of variables preserves homogeneity of equations. Namely, the following theorem expresses the linear span of a composite body in terms of linear spans of its arguments.

For a set $\Delta \subset \mathbb{R}^n$, denote the linear span of all vectors of the form $a - b$, where $a \in \Delta$ and $b \in \Delta$, by $\langle \Delta \rangle$. For a subspace $L \subset \mathbb{R}^n$, denote the projection $\mathbb{R}^n \mapsto \mathbb{R}^n / L$ by p. Recall that μ is a volume form on $\mathbb{R}^n / L$.

THEOREM **2.6.** *1) If $\dim p(\Delta_{i_1} + \ldots + \Delta_{i_q}) < q - 1$ for some numbers $0 \leqslant i_1 < \ldots < i_q \leqslant k$, then $\mathrm{CB}_\mu(\Delta_0, \ldots, \Delta_k)$ consists of one point.*
2) Otherwise, there exists a unique minimal non-empty set $\{i_1, \ldots, i_q\} \subset \{1, \ldots, n\}$ such that $\dim p(\Delta_{i_1} + \ldots + \Delta_{i_q}) = q - 1$. In this case

$$\langle \mathrm{CB}_\mu(\Delta_0, \ldots, \Delta_k) \rangle = \langle \Delta_{i_1} + \ldots + \Delta_{i_q} \rangle \cap L.$$

PROOF. By definition of a composite body, this theorem follows from a similar fact about mixed volumes, namely, from D. Bernstein's criterion for vanishing of the mixed volume (see below). The uniqueness of a minimal non-empty set $\{i_1, \ldots, i_q\} \subset \{1, \ldots, n\}$, such that $\dim p(\Delta_{i_1} + \ldots + \Delta_{i_q}) = q - 1$, follows from the fact that the family of all such sets is closed under the operation of intersection (see [S], Theorem 1.1 for details). □

LEMMA **2.7** (D. Bernstein's criterion, [Kh1]). *The mixed volume of convex bodies* $B_1, \ldots, B_n$ *in* $\mathbb{R}^n$ *is equal to 0 iff* $\dim\langle B_{i_1} + \ldots + B_{i_q}\rangle < q$ *for some numbers* $1 \leqslant i_1 < \ldots < i_q \leqslant n$.

Mixed fiber bodies and existence of composite bodies. The notion of a composite body turns out to be a version of the notion of a mixed fiber body. We use this relation to prove the existence and some basic properties of composite bodies. Recall the definition of a mixed fiber body.

Let $L \subset \mathbb{R}^n$ be a vector subspace of codimension k, let μ be a volume form on $\mathbb{R}^n/L$, denote by p the projection $\mathbb{R}^n \to \mathbb{R}^n/L$.

DEFINITION **2.8** ([BS]). For a convex body $\Delta \subset \mathbb{R}^n$, the set of all points of the form $\int_{p(\Delta)} s\mu \in \mathbb{R}^n$, where $s : p(\Delta) \to \Delta$ is a continuous section of the projection p, is called the *Minkowski integral* of Δ and is denoted by $\int p|_\Delta \mu$.

The following fact explains the relation between composite bodies and Minkowski integrals.

LEMMA **2.9.** *The convex body*

$$(k+1)! \int p|_\Delta \mu$$

is contained in a fiber of the projection p *and, up to a shift, satisfies the definition of the composite body* $\mathrm{CB}_\mu(\Delta, \ldots, \Delta)$.

PROOF. 1) If $\Delta = A + B$, where $B \subset L$ and the restriction $p|_A$ is injective, then the statement follows from the additivity of the mixed volume. Indeed, for arbitrary convex bodies $B_1, \ldots, B_{n-k-1} \subset L$ and a volume form μ' on L,

$$n!\,\mathrm{MV}_{\mu\wedge\mu'}(A+B, \ldots, A+B, B_1, \ldots, B_{n-k-1}) =$$

$$= (k+1)\cdot n!\,\mathrm{MV}_{\mu\wedge\mu'}(A, \ldots, A, B, B_1, \ldots, B_{n-k-1}) =$$
$$= (k+1)\cdot k!\big(\textstyle\int_{p(A)}\mu\big)\cdot (n-k)!\,\mathrm{MV}_{\mu'}(B, B_1, \ldots, B_{n-k-1}) =$$
$$= (n-k)!\,\mathrm{MV}_{\mu'}\Big((k+1)!\big(\textstyle\int_{p(A)}\mu\big)\cdot B, B_1, \ldots, B_{n-k-1}\Big) =$$
$$= (n-k)!\,\mathrm{MV}_{\mu'}\Big((k+1)!\cdot \textstyle\int p|_\Delta \mu, B_1, \ldots, B_{n-k-1}\Big).$$

2) In general, one can subdivide the projection $p(\Delta)$ into small pieces, and subdivide Δ into the inverse images Δ_i of these pieces. Representing

the mixed volume $\mathrm{MV}_{\mu\wedge\mu'}(\Delta,\ldots,\Delta,B_1,\ldots,B_{n-k-1})$ as the sum of mixed volumes $\sum_i \mathrm{MV}_{\mu\wedge\mu'}(\Delta_i,\ldots,\Delta_i,B_1,\ldots,B_{n-k-1})$ for arbitrary convex bodies $B_1,\ldots,B_{n-k-1}$ in L, and approximating each Δ_i by a sum A_i+B_i, such that $B_i\subset L$ and the restriction $p|_{A_i}$ is injective, one can reduce the general case to the special case (1). □

The following theorem provides a way to generalize Lemma 2.9 to composite bodies of arbitrary collections of convex bodies.

THEOREM **2.10.** *Choose a linear projection $u:\mathbb{R}^n\to L$. 1) There exists a unique symmetric multilinear mapping $\mathrm{MF}_{\mu,u}$ from collections of $k+1$ convex bodies in $\mathbb{R}^n$ to convex bodies in L, such that $\mathrm{MF}_{\mu,u}(\Delta,\ldots,\Delta)=u\int p|_\Delta\mu$ for each convex body $\Delta\subset\mathbb{R}^n$.*
2) This mapping assigns polytopes to polytopes.

Proof of this theorem is given below.

DEFINITION **2.11.** The convex body $\mathrm{MF}_{\mu,u}(\Delta_0,\ldots,\Delta_k)$ is called the *mixed fiber body* of bodies $\Delta_0,\ldots,\Delta_k$.

THEOREM **2.12.** *The convex body*

$$(k+1)!\,\mathrm{MF}_{\mu,u}(\Delta_0,\ldots,\Delta_k)$$

is contained in a fiber of the projection p and, up to a shift, satisfies the definition of the composite body $\mathrm{CB}_\mu(\Delta_0,\ldots,\Delta_k)$.

PROOF. By additivity of mixed fiber bodies and mixed volumes, one can reduce the statement to the special case $\Delta_0=\ldots=\Delta_k$ considered in Lemma 2.9. □

Virtual bodies. It is more convenient to prove Theorem 2.10 in the context of virtual bodies instead of convex bodies, because an explicit formula for mixed fiber bodies (see Lemma 2.16) involves subtraction of convex bodies.

Recall that *the Grothendieck group K_G* of a commutative semigroup K is the group of formal differences of elements from K. In more details, it is the quotient of the set $K\times K$ by the equivalence relation $(a,b)\sim(c,d)\Leftrightarrow\exists k:\ a+d+k=b+c+k$ with operations $(a-b)+(c-d)=(a+c)-(b+d)$ and $-(a,b)=(b,a)$. For each semigroup K with cancellation law $a+c=b+c\Rightarrow a=b$, the mapping $a\to(a+a,a)$ induces the inclusion $K\hookrightarrow K_G$. An element of the form $(a+a,a)\in K_G$ is said to be *proper* and is usually identified with $a\in K$. Under this convention, one can write $(a,b)=a-b$.

DEFINITION **2.13.** *The group of virtual bodies* in $\mathbb{R}^n$ is the Grothendieck group of the semigroup of convex bodies in $\mathbb{R}^n$ with the operation of Minkowski summation. It contains *the group of virtual polytopes* in $\mathbb{R}^n$, i. e. the Grothendieck group of the semigroup of convex polytopes in $\mathbb{R}^n$.

These commutative groups are real vector spaces with the operation of scalar multiplication defined as dilatation.

DEFINITION **2.14.** For a virtual body Δ in $\mathbb{R}^n$, its *support function* $\Delta(\cdot) : (\mathbb{R}^n)^* \to \mathbb{R}$ is defined as

$$\Delta(\gamma) = \max_{x \in \Delta_1} \langle \gamma, x \rangle - \max_{x \in \Delta_2} \langle \gamma, x \rangle,$$

where Δ_1 and Δ_2 are convex bodies such that $\Delta = \Delta_1 - \Delta_2$.

The following statement describes the group of virtual bodies more explicitely. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *positively homogeneous* if $f(tx) = tf(x)$ for each $t \geqslant 0$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *a d. c. function* if it can be represented as the difference of two convex functions.

LEMMA **2.15.** *1) The mapping $\Delta \to \Delta(\cdot)$ induces the isomorphism between the group of virtual bodies in $\mathbb{R}^n$ and the group of positively homogenious d. c. functions on $(\mathbb{R}^n)^*$.*
2) This isomorphism induces an isomorphism between the group of virtual polytopes and the group of continuous piecewise linear positively homogenious functions.

PROOF. The mapping $\Delta \to \Delta(\cdot)$ is surjective by definition of a d. c. function. It is injective, since a convex body is uniquely determined by its support function. Part 2 follows from the fact that each continuous piecewise linear function can be represented as a difference of two convex piecewise linear functions. □

The operations of taking the mixed volume, the composite body and the mixed fiber body can be extended to virtual bodies by linearity. This extension is unique, but its properties are quite different. For example, The mixed volume of virtual polytopes is not monotonous (for example, $\mathrm{MV}(-A, A) > \mathrm{MV}(-A, 2A)$ for a convex polygon A) and is not non-degenerate in the sense of Lemma 2.7 (for example, $\mathrm{MV}(B - C, 2B + 2C) = 0$ for non-parallel segments B and C in the plane). As a result, virtual composite bodies do not satisfy Theorems 2.3 and 2.6.

Proof of Theorem 2.10. The uniqueness and Part 2 are corollaries of the following formula for mixed fiber bodies:

Lemma **2.16.** *For any convex bodies* $\Delta_0, \ldots, \Delta_k \subset \mathbb{R}^n$*,*

$$\mathrm{MF}_{\mu,u}(\Delta_0, \ldots, \Delta_k) = \frac{1}{(k+1)!} \sum_{0 \leqslant i_1 < \ldots < i_q \leqslant k} (-1)^{k+1-q} u \textstyle\int p|_{(\Delta_{i_1} + \ldots + \Delta_{i_q})}\, \mu.$$

Proof. Let $m : A \times \ldots \times A \to B$ be a symmetric multilinear mapping, where A and B are semigroups. Then

$$m(a_1, \ldots, a_k) = \sum_{0 \leqslant i_1 < \ldots < i_q \leqslant k} (-1)^{k+1-q} m(a_{i_1} + \ldots + a_{i_q}, \ldots, a_{i_1} + \ldots + a_{i_q}).$$

To prove this formula, open the brackets in the right hand side by linearity of m and cancel similar terms. $\square$

Lemma **2.17.** *(see Section 3 or [McM]) Let* $u : \mathbb{R}^n \to L$ *be a linear projection and let* μ *be a volume form on* $\mathbb{R}^n / L$*.*
1) There exists a symmetric multilinear mapping $\mathrm{MF}_{\mu,u}$ *from collections of* $k+1$ *virtual polytopes in* $\mathbb{R}^n$ *to virtual polytopes in* L *such that* $\mathrm{MF}_{\mu,u}(\Delta, \ldots, \Delta) = u \int p|_\Delta \mu$ *for each convex polytope* $\Delta \subset \mathbb{R}^n$*.*
2) $\mathrm{MF}_{\mu,u}$ *maps convex polytopes to convex polytopes.*

The existence of mixed fiber bodies can be reduced to this special case as follows. For arbitrary convex bodies $\Delta_0, \ldots, \Delta_k$ in $\mathbb{R}^n$, define the virtual body $\mathrm{MF}_{\mu,u}(\Delta_0, \ldots, \Delta_k)$ as in Lemma 2.16 above. It follows from the definition that
1) $\mathrm{MF}_{\mu,u}$ is symmetric,
2) $\mathrm{MF}_{\mu,u}(\Delta, \ldots, \Delta) = u \int p|_\Delta \mu$ for each convex body $\Delta \subset \mathbb{R}^n$,
3) $\mathrm{MF}_{\mu,u}$ is continuous in the sense of the norm $|\Delta| = \max_{\gamma \in B} |\Delta(\gamma)|$, where $B \in (\mathbb{R}^n)^*$ is a compact neighborhood of the origin, since the Minkowski integral is continuous in this sense.

Lemma 2.17 implies that $\mathrm{MF}_{\mu,u}$ is multilinear and preserves convexisty under the assumption that the arguments are polytopes. Namely, for any virtual polytopes $\Delta_0, \Delta_0', \Delta_1, \ldots, \Delta_k$,
4) $\mathrm{MF}_{\mu,u}(\Delta_0 + \Delta_0', \Delta_1, \ldots, \Delta_k) = \mathrm{MF}_{\mu,u}(\Delta_0, \ldots, \Delta_k) + \mathrm{MF}_{\mu,u}(\Delta_0', \ldots, \Delta_k)$,
5) $\mathrm{MF}_{\mu,u}(t \cdot \Delta_0, \ldots, \Delta_k) = t \cdot \mathrm{MF}_{\mu,u}(\Delta_0, \ldots, \Delta_k)$,
6) $\mathrm{MF}_{\mu,u}(\Delta_0, \ldots, \Delta_k)$ is covex if $\Delta_0, \ldots, \Delta_k$ are convex.

Approximating arbitrary convex bodies with convex polytopes and using the continuity of $\mathrm{MF}_{\mu,u}$ (property 3), one can extend properties 4, 5 and 6 to arbitrary convex bodies. $\square$

3 Mixed fiber polytopes.

In this section, we prove the existence of mixed fiber polytopes (Lemma 2.17). Namely, let $L \subset \mathbb{R}^n$ be a vector subspace of codimension k, let $u : \mathbb{R}^n \to L$ be a linear projection, and let μ be a volume form on $\mathbb{R}^n/L$. Denote the projection $\mathbb{R}^n \to \mathbb{R}^n/L$ by p. Then
1) there exists a symmetric multilinear mapping $\mathrm{MF}_{\mu,u}$ from collections of $k+1$ virtual polytopes in $\mathbb{R}^n$ to virtual polytopes in L such that $\mathrm{MF}_{\mu,u}(\Delta,\ldots,\Delta) = u\int p|_\Delta \mu$ for each convex polytope $\Delta \subset \mathbb{R}^n$.
2) $\mathrm{MF}_{\mu,u}$ maps convex polytopes to convex polytopes.
PROOF follows from the fact that the Minkowski integral is a polynomial mapping from the space of virtual polytopes in $\mathbb{R}^n$ to the space of virtual polytopes in L. Every polynomial mapping of vector spaces gives rise to a certain symmetric multilinear function, which is called the polarization of the polynomial. In more details, Part 1 follows from Theorems 3.3, 3.5 and 3.7; Part 2 follows from Corollary 3.15. □

Polarizations of polynomials on Zariski dense sets. The existence of mixed fiber polytopes is a corollary of the following general construction.

DEFINITION **3.1.** A set A in a vector space W is said to be *Zariski dense* if each finite-dimensional subspace $U \subset W$ is contained in a finite-dimensional subspace $V \subset W$ such that $A \cap V$ is Zariski closed in V (i. e. $A \cap V$ is not contained in a proper algebraic subset of V).

DEFINITION **3.2.** A map $f : A \to V$ from a subset A of a vector space W to a vector space V is said to be (homogeneous) polynomial of degree k if, for each finite-dimensional subspace $U \subset W$ and for each linear function $l : V \to \mathbb{R}$, the composition $l \circ f|_U : A \cap U \to \mathbb{R}$ is a restriction of a (homogeneous) polynomial of degree at most k on U.

THEOREM **3.3.** *1) A (homogeneous) polynomial map of degree k on a Zariski dense subset of a vector space W has a unique extension to a (homogeneous) polynomial map of degree k on W.*
2) For a homogeneous polynomial map $f : W \to V$ of degree k, there exists a unique symmetric multilinear function $Mf : \underbrace{W \oplus \ldots \oplus W}_{k} \to V$ such that $Mf(w,\ldots,w) = f(w)$ for every $w \in W$.

DEFINITION **3.4.** The function Mf is called *the polarization* of a polynomial f.

Proof. 1) Let A be a Zariski dense subset in W and let $f : A \to V$ be a (homogeneous) polynomial map of degree k. For a subspace U such that $A \cap U$ is Zariski dense in U, there exists a unique (homogeneous) polynomial map $f_U : U \to V$ of degree k such that $f_U = f$ on $U \cap A$. For any two such finite-dimensional subspaces U and U', the sum $U + U'$ is contained in a finite-dimensional subspace subspace U'' such that $U'' \cap A$ is Zariski dense. Thus $f_{U''} = f_{U'}$ on U' and $f_{U''} = f_U$ on U. In particular, $f_U = f_{U'}$ on the intersection $U \cap U'$. This implies that polynomials f_U glue up into a mapping $\tilde{f} : W \to V$ such that $\tilde{f} = f$ on A.
2) For numbers $t_1, \dots, t_k$ and vectors $w_1, \dots, w_k \in W$, the expression $f(t_1 w_1 + \dots + t_k w_k)/k!$ is a homogeneous polynomial as a function of $t_1, \dots, f_k$. The coefficient of the monomial $t_1 \dots t_k$ in this polynomial satisfies the definition of the polarization Mf. □

We apply polarizations in the following context. Let $V(K)$ be the space of virtual polytopes in a k-dimensional vector space K. Let $A(K) \subset V(K)$ be the set of convex polytopes.

Theorem **3.5.** *$A(K)$ is a Zariski dense subset of $V(K)$.*

Definition **3.6.** A polytope $\Delta' \in V(K)$ is said to be compatible with a polytope $\Delta \in V(K)$ if the support function $\Delta'(\cdot)$ is linear on every domain of linearity of $\Delta(\cdot)$.

Let $V(\Delta) \subset V(K)$ be the space of all virtual polytopes compatible with $\Delta \in V(K)$. Theorem 3.5 is a corollary of the following facts.
1) For every polytope $\Delta \in V(K)$, the space $V(\Delta)$ is finite dimensional. Indeed, the space of piecewise-linear functions with the prescribed domains of linearity is finite-dimensional.
2) For every convex polytope $\Delta \in A(K)$ the intersection $V(\Delta) \cap A(K)$ is Zariski dense in $V(K)$.
3) Every finite dimensional vector subspace $U \subset V(K)$ is contained in the space $V(\Delta)$ for some convex polytope $\Delta \in A(K)$. Indeed, if U is generated by differences $A_i - B_i$ of convex polytopes A_i and B_i, then one can choose $\Delta = \sum_i A_i + B_i$. □

Minkowski integral is a polynomial. Let $u : \mathbb{R}^n \to L$ be a linear projection, let μ be a volume form on the k-dimensional vector space $\mathbb{R}^n/L$, and let p be the projection $\mathbb{R}^n \to \mathbb{R}^n/L$.

Theorem **3.7.** *The Minkowski integral $\mathcal{M}(\Delta) = u \int p|_\Delta \mu$ is a homogeneous polynomial mapping $A(\mathbb{R}^n) \to A(L)$ of degree $k+1$.*

Proof. For a convex polytope $\Delta \in \mathbb{R}^n$, define $A(\Delta)$ as the set of all convex polytopes, compatible with Δ. For a convex k-dimensional polytope Δ, the restriction of $\mathcal{M}$ to $A(\Delta)$ is a homogeneous polynomial mapping of degree $k+1$ because of the following two facts (the first one follows from the definition of the Mnikowski integral, and the second one is well-known).

Lemma **3.8.** *The Minkowski integral $\mathcal{M}(\Delta)$ of a convex k-dimensional polytope Δ consists of one point, and this point equals the projection u of the first moment $\int_\Delta xp^*(\mu)$ of Δ, where x runs over Δ and $p^*(\mu)$ is the volume form μ on $\mathbb{R}^n/L$ lifted to Δ.*

Lemma **3.9.** *The first moment is a homogeneous polynomial of degree $k+1$ on the space $A(\Delta)$, if Δ is a convex k-dimensional polytope.*

One can reduce Theorem 3.7 to k-dimensional polytopes as follows. For a covector $\gamma \in (\mathbb{R}^n)^*$ and a convex polytope $\Delta \subset \mathbb{R}^n$, let Δ^γ be the maximal face, where γ attains its maximum as a function on Δ.

Lemma **3.10.** *For every covector $\gamma \in L^*$,*

$$(u{\textstyle\int} p|_\Delta \mu)^\gamma = \sum_{\substack{\delta \in (\mathbb{R}^n)^* \\ \delta|_L = \gamma}} u{\textstyle\int} p|_{\Delta^\delta} \mu.$$

This equality easily follows from the definition of the Minkowski integral, and we omit the proof. The sum in the right hand side makes sense, since it contains finitely many non-zero summands. Note that Lemmas 3.8 and 3.10 are similar to Propositions 5.1 and 5.2 from [McM] respectively.

Lemma **3.11.** *The mapping $\mathcal{M}$ preserves compatibility of convex polytopes: $\mathcal{M}\big(A(\Delta)\big) \subset A\big(\mathcal{M}(\Delta)\big)$.*

Proof. The integral of a continuous family of convex functions is a linear function iff every function in the family is linear. Apply this fact to the following description of the support function of $\mathcal{M}(\Delta)$. □

For a convex body $\Delta \subset \mathbb{R}^n$ and a point $a \in \mathbb{R}^n/L$, denote the convex body $u\Big(\Delta \cap p^{(-1)}(a)\Big) \subset L$ by Δ_a; roughly speaking, this is a fiber of Δ over the point a.

Lemma **3.12.** *The support function of the body $u\big(\mathcal{M}(\Delta)\big)$ equals the integral of the support functions of bodies Δ_a over $a \in p(\Delta)$.*

This equality easily follows from the definition of the Minkowski integral, and we omit the proof.

Proof of Theorem 3.7. For a face B of a polytope Δ, let $\tilde{B} : A(\Delta) \to A(B)$ be the mapping which maps every $\Delta' \in A(\Delta)$ to its face $B' \in A(B)$, such that $B + B'$ is a face of $\Delta + \Delta'$.

For an n-dimensional convex polytope $\Delta \in A(\mathbb{R}^n)$, denote vertices of $\mathcal{M}(\Delta)$ by $a_1, \ldots, a_I$, and denote k-dimensional faces of Δ by $B_1, \ldots, B_J$. By Lemma 3.11, the points $\tilde{a}_1\big(\mathcal{M}(\Delta')\big), \ldots, \tilde{a}_I\big(\mathcal{M}(\Delta')\big)$ are the vertices of the polytope $\mathcal{M}(\Delta')$ for every convex polytope $\Delta' \in A(\Delta)$. By Lemma 3.10, each vertex $\tilde{a}_i\big(\mathcal{M}(\Delta')\big)$ equals a finite sum of the Minkowski integrals of k-dimensional faces $\tilde{B}^j(\Delta')$. By Lemmas 3.8 and 3.9, the Minkowski integral $\mathcal{M}$ is a homogeneous polynomial of degree $k+1$ on the image of each linear mapping $\tilde{B}_j$. □

Faces and convexity of mixed fiber polytopes. By Theorems 3.5 and 3.7, there exists a unique polarization of the Minkowski integral of a polytope in $\mathbb{R}^n$ with respect to a volume form μ on $\mathbb{R}^n/L$. It is denoted by $\mathrm{MF}_{\mu,u}(\Delta_0, \ldots, \Delta_k)$ and is called the mixed fiber polytope. To prove that it preserves convexity, we extend Lemma 3.10 to mixed fiber polytopes as follows.

For a virtual polytope Δ, which equals the difference of convex polytopes A and B in $\mathbb{R}^n$, and for a covector $\gamma \in (\mathbb{R}^n)^*$, the *support face* Δ^γ is defined as $A^\gamma - B^\gamma$.

Theorem **3.13.** *For virtual polytopes $\Delta_0, \ldots, \Delta_k \subset \mathbb{R}^n$ and a covector $\gamma \in L^*$, the face $\big(\mathrm{MF}_{\mu,u}(\Delta_0, \ldots, \Delta_k)\big)^\gamma$ coincides with the Minkowski sum*

$$\sum_{\substack{\delta \in (\mathbb{R}^n)^* \\ \delta|_L = \gamma}} \mathrm{MF}_{\mu,u}(\Delta_0^\delta, \ldots, \Delta_k^\delta).$$

This theorem follows from Lemma 3.10 by linearity of mixed fiber polytopes.

The length of a one-dimensional Minkowski integral of a convex polytope Δ is by definition equal to the volume of Δ. This fact extends by linearity as follows.

Lemma **3.14.** *Suppose that convex polytopes $\Delta_0, \ldots, \Delta_k$ are all parallel to a $(k+1)$-dimensional subspace $K \subset \mathbb{R}^n$. Choose a coordinate t on the line $K \cap L$. Then a mixed fiber body $\mathrm{MF}_{\mu,u}(\Delta_0, \ldots, \Delta_k)$ is a segment, parallel* *to the line $K \cap L$, and its length (in the sense of the coordinate t) equals* $\mathrm{MV}_{dt \wedge p^*\mu}(\Delta_0, \ldots, \Delta_k)$.

This mixed volume makes sense because its arguments are all parallel to the same $(k+1)$-dimensional subspace K. The volume form $dt \wedge p^*\mu$ makes sense on K, because $\ker(p|_K) = K \cap L$.

In particular, a one-dimensional mixed fiber polytope of convex polytopes is convex. Since, by Theorem 3.13, every edge of a mixed fiber polytope is a sum of one-dimensional mixed fiber polytopes, every edge of a mixed fiber polytope of convex polytopes is convex. A polytope with all convex edges is convex.

Corollary 3.15. *The mixed fiber polytope of convex polytopes is convex.*

Vertices and integrality of mixed fiber polytopes. The proof of Theorem 3.7 is based on the fact that vertices of the Minkowski integral of Δ can be expressed in terms of the first moments of faces of Δ. We extend this fact to mixed fiber polytopes in order to prove their integrality. To formulate this, we need the polarization of the first moment, which exists by Lemma 3.9. For virtual polytopes $\Delta_0, \ldots, \Delta_k$ in $\mathbb{R}^n$, the subspace $\langle \Delta_0, \ldots, \Delta_k \rangle \subset \mathbb{R}^n$ is defined as the minimal subspace containing convex polytopes B_i^j such that $\Delta_i = B_i^0 - B_i^1$ up to a shift for $i = 0 \ldots, k$.

Lemma **3.16.** *There exists a unique symmetric multilinear function* MM_μ *of $k+1$ convex bodies such that*
1) The domain of MM_μ *consists of all collections of virtual polytopes* $\Delta_0, \ldots, \Delta_k \subset \mathbb{R}^n$ *such that* $\dim\langle \Delta_0, \ldots, \Delta_k \rangle \leqslant k$*;*
2) For each k-dimensional convex polytope $\Delta \subset \mathbb{R}^n$,

$$\mathrm{MM}_\mu(\Delta, \ldots, \Delta) = \int_\Delta x p^*(\mu),$$

where x runs over Δ and $p^*(\mu)$ is the volume form μ on $\mathbb{R}^n/L$ lifted to Δ.

Definition **3.17.** The point $\mathrm{MM}_\mu(\Delta_0, \ldots, \Delta_k) \in \mathbb{R}^n$ is called the *mixed moment* of $\Delta_0, \ldots, \Delta_k$.

By linearity, Lemma 3.8 extends to mixed fiber polytopes as follows.

Lemma **3.18.** *If* $\dim\langle \Delta_0, \ldots, \Delta_k \rangle \leqslant k$*, then the Mixed fiber polytope* $\mathrm{MF}_{\mu,u}(\Delta_0, \ldots, \Delta_k) \in \mathbb{R}^n$ *consists of one point* $u\,\mathrm{MM}_\mu(\Delta_0, \ldots, \Delta_k) \in L$.

Lemma 3.18 and Theorem 3.13 give the following expression for vertices of a mixed fiber polytope.

Theorem **3.19.** *In the notation of Theorem 3.13,*
1) If $\dim\langle\Delta_0^\delta,\ldots,\Delta_k^\delta\rangle \leqslant k$ *for each covector* $\delta\in(\mathbb{R}^n)^*$ *such that* $\delta|_L=\gamma$*, then the face* $\bigl(\mathrm{MF}_{\mu,u}(\Delta_0,\ldots,\Delta_k)\bigr)^\gamma$ *is a vertex of* $\mathrm{MF}_{\mu,u}(\Delta_0,\ldots,\Delta_k)$*, and this vertex equals* $\sum_{\delta|_L=\gamma} u\,\mathrm{MM}_\mu(\Delta_0^\delta,\ldots,\Delta_k^\delta)$*.*
2) Almost all covectors $\gamma\in L^*$ *satisfy the condition of Part 1.*
3) The set of all points of the form $\sum_{\delta|_L=\gamma} u\,\mathrm{MM}_\mu(\Delta_0^\delta,\ldots,\Delta_k^\delta)$*, where* $\gamma\in L^*$ *satisfies the condition of Part 1, coincides with the set of all vertices of* $\mathrm{MF}_{\mu,u}(\Delta_0,\ldots,\Delta_k)$*.*

In Part 2, "almost all (co)vectors in a space V" means "all covectors from the complement of a finite union of proper vector subspaces of V".

In particular, since the mixed moment of integer polytopes is a rational number with the denominator $(k+1)!$, the same is true for mixed fiber polytopes.

Theorem **3.20.** *If* $\Delta_0,\ldots,\Delta_k$ *are integer polytopes (i. e. their vertices are integer lattice points),* $L\subset\mathbb{R}^n$ *is a* k*-dimensional rational subspace,* $u(\mathbb{Z}^n)=L\cap\mathbb{Z}^n$*, and* μ *is the integer volume form on* $\mathbb{R}^n/L$ *(i. e.* $\int_{\mathbb{R}^n/(L+\mathbb{Z}^n)}\mu=1$*), then* $(k+1)!\,\mathrm{MF}_{\mu,u}(\Delta_0,\ldots,\Delta_k)$ *is an integer polytope.*

4 Leading coefficients of a composite polynomial in terms of composite polynomials of fewer variables.

We present some technical facts in this section about how to compute leading coefficients of a composite polynomial in terms of composite polynomials of fewer variables. In the next section, we use these facts to compute leading coefficients of a composite polynomial $\pi_{f_0,\ldots,f_k}$ explicitly under the assumption that the Newton polytopes of polynomials $f_0,\ldots,f_k$ satisfy some condition of general position.

Recall that, for a covector $\gamma\in(\mathbb{R}^n)^*$ and a convex polytope $A\subset\mathbb{R}^n$, the polytope A^γ is defined as the maximal face of A, where γ attains its maximum as a function on A.

Definition **4.1.** For a covector $\gamma \in (\mathbb{R}^n)^*$ and a Laurent polynomial $f(x) = \sum_{a \in A} c_a x^a$ on $(\mathbb{C} \setminus 0)^n$, the polynomial $\sum_{a \in A^\gamma} c_a x^a$ is called the *truncation* of f in the direction γ and is denoted f^γ.

Theorem 4.3 expresses a truncation of a composite polynomial in terms of composite polynomials of truncations. Theorem 4.4 represents a homogeneous composite polynomial as a composite polynomial of fewer variables. Since truncations of polynomials are homogeneous, one can use Theorem 4.4 to simplify the answer in the formulation of Theorem 4.3. As a result, one can express a truncation of a composite polynomial in terms of composite polynomials of fewer variables.

Definition **4.2.** *The vertex coefficients* of a polynomial f are the coefficients of its monomials which correspond to the vertices of the Newton polytope Δ_f.

Since a composite polynomial is unique up to a monomial factor, we are interested in ratios of its vertex coefficients rather than in individual vertex coefficients. Theorem 4.6 expresses the ratio of two vertex coefficients of a composite polynomial as the product of values of some monomial over the roots of some system of polynomial equations. By Lemma 1.1.2, this product over roots can be seen as a vertex coefficient of a corresponding composite polynomial of one variable.

Truncation and dehomogenization. The operations of truncating and taking the composite polynomial commute in the following sense.

Theorem **4.3.** *Let $\pi : (\mathbb{C} \setminus 0)^n \to (\mathbb{C} \setminus 0)^{n-k}$ and $\pi^\times : \mathbb{Z}^{n-k} \hookrightarrow \mathbb{Z}^n$ be an epimorphism of complex tori and the corresponding embedding of their character lattices, and let $f_0, \ldots, f_k$ be Newton-nondegenerate Laurent polynomials on $(\mathbb{C} \setminus 0)^n$. Then, for every $\gamma \in (\mathbb{Z}^{n-k})^*$, the truncation $\pi^\gamma_{f_0,\ldots,f_k}$ equals the product $\prod_\delta \pi_{f_0^\delta,\ldots,f_k^\delta}$ over all $\delta \in (\mathbb{Z}^n)^*$ such that $\delta|_{\pi^\times \mathbb{Z}^{n-k}} = \gamma$.*

Since composite polynomials are defined up to a monomial multiplier, we can assume that whenever $\pi_{f_0^\delta,\ldots,f_k^\delta}$ is a monomial, it is equal to 1. Under this assumption, the product $\prod\limits_{\delta \in \mathbb{Z}^n,\, \delta|_{\pi^\times \mathbb{Z}^{n-k}} = \gamma} \pi_{f_0^\delta,\ldots,f_k^\delta}$ contains a finite number of factors different from 1. The proof of this theorem is given at the end of this section. Theorem 3.13 is the geometrical counterpart of this theorem.

A Laurent polynomial $f : (\mathbb{C} \setminus 0)^n \to \mathbb{C}$ is said to be *homogeneous*, if there exist an epimorphism of complex tori $(\mathbb{C} \setminus 0)^n \to (\mathbb{C} \setminus 0)^{n'}$ and a Laurent polynomial $g : (\mathbb{C} \setminus 0)^{n'} \to \mathbb{C}$, such that $n' < n$ and $f = g \circ h$ up to a monomial factor. The polynomial g is called a *dehomogenization* of f. Theorem 4.4 below implies that the operations of dehomogenization and taking the composite polynomial commute in the following sense: if polynomials $f_0, \ldots, f_k$ are "homogeneous enough", then their composite polynomial is also homogeneous, and its dehomogenization equals the composite polynomial of dehomogenizations of $f_0, \ldots, f_k$, raised to some power.

Every pair of tori epimorphisms $(\mathbb{C} \setminus 0)^{n-k} \stackrel{\pi}{\leftarrow} (\mathbb{C} \setminus 0)^n \stackrel{h}{\to} (\mathbb{C} \setminus 0)^{n'}$ and corresponding character lattice embeddings $\mathbb{Z}^{n-k} \stackrel{\pi^\times}{\hookrightarrow} \mathbb{Z}^n \stackrel{h^\times}{\hookleftarrow} \mathbb{Z}^{n'}$ can be included into the commutative squares

$$\begin{array}{ccc} (\mathbb{C} \setminus 0)^n & \stackrel{h}{\mapsto} & (\mathbb{C} \setminus 0)^{n'} \\ \downarrow \pi & & \downarrow \pi' \\ (\mathbb{C} \setminus 0)^{n-k} & \stackrel{h'}{\mapsto} & (\mathbb{C} \setminus 0)^{n'-k} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{Z}^n & \stackrel{h^\times}{\hookleftarrow} & \mathbb{Z}^{n'} \\ \uparrow \pi^\times & & \uparrow \pi'^\times \\ \mathbb{Z}^{n-k} & \stackrel{h'^\times}{\hookleftarrow} & \mathbb{Z}^{n'-k}, \end{array}$$

such that the image of $\mathbb{Z}^{n'-k}$ in $\mathbb{Z}^n$ equals the intersection $\pi^\times \mathbb{Z}^{n-k} \cap h^\times \mathbb{Z}^{n'}$.

THEOREM **4.4.** *In this notation, if Laurent polynomials $f_0, \ldots, f_k$ on $(\mathbb{C} \setminus 0)^n$ are homogeneous in the sense that $f_i = g_i \circ h$ up to a monomial factor for some Laurent polynomials $g_0, \ldots, g_k$ on $(\mathbb{C} \setminus 0)^{n'}$, then their composite polynomial $\pi_{f_0,\ldots,f_k}$ is homogeneous in the sense that it equals $g \circ h'$, where $g = (\pi'_{g_0,\ldots,g_k})^{\left|\mathbb{Z}^n / (\pi^\times \mathbb{Z}^{n-k} + h^\times \mathbb{Z}^{n'})\right|}$ is a Laurent polynomial on $(\mathbb{C} \setminus 0)^{n'-k}$.*

The proof is given at the end of this section.

Vertex coeficients.

DEFINITION **4.5.** *The product over roots* $R_{A_1,\ldots,A_m}(g_0; g_1, \ldots, g_m)$ is a raitonal function on the space of collections of Laurent polynomials $(g_1, \ldots, g_m)$ such that the Newton polytope of g_i is $A_i \subset \mathbb{Z}^m$. By definition, this function equals the product of values of a polynomial g_0 over the roots of the system $g_1 = \ldots = g_m = 0$ for Newton-nondegenerate polynomials $g_1, \ldots, g_m$.

Part 2 of Lemma 1.1 is a formula for the vertex coefficient of a composite polynomial of one variable in terms of products over roots. The following theorem extends this formula to composite polynomials of several variables. Let $\pi : \mathbb{C}^k \times \mathbb{C}^{n-k} \to \mathbb{C}^{n-k}$ be the standard projection, and let $u_1, \ldots, u_k$ be the standard coordinates on $\mathbb{C}^k$. Suppose that $f_0, \ldots, f_k$ are polynomials on $\mathbb{C}^k \times \mathbb{C}^{n-k}$, and their Newton polytopes $A_0, \ldots, A_k \subset \mathbb{Z}^k \times \mathbb{Z}^{n-k}$ intersect all coordinate hyperplanes. Denote the Newton polytope of the composite polynomial $\pi_{f_0,\dots,f_k}$ by $A \subset \mathbb{Z}^{n-k}$, and consider covectors γ_1 and γ_2 in $(\mathbb{Z}^{n-k})^*$ with positive integer coordinates. Let $\tilde{f}_i(u,t)$ be a Laurent polynomial $f_i(u,\, t^{\gamma_2}+t^{-\gamma_1})$ of $k+1$ variables $u_1,\dots,u_k,t$, and let $\tilde{A}_i$ be its Newton polytope.

THEOREM **4.6.** *If the polynomials $f_0,\dots,f_k$ are Newton-nondegenerate, and covectors γ_1 and γ_2 are generic in the sense that, for every $a \in \mathbb{Z}^k$, the face $\Big((\{a\}\times\mathbb{Z}^{n-k})\cap\sum_i A_i\Big)^{\gamma_j}$ is a vertex, then*
1) the face A^{γ_j} of the polytope A is a vertex (denote it by B_j), and the difference $B_1 - B_2$ equals

$$(k+1)!\sum_{\delta\in(\mathbb{Z}^k)^*}\mathrm{MM}_\mu(A_0^{\gamma_1+\delta},\dots,A_k^{\gamma_1+\delta}) - \mathrm{MM}_\mu(A_0^{\gamma_2+\delta},\dots,A_k^{\gamma_2+\delta}),$$

where μ is the unit volume form on $\mathbb{Z}^k$, and the mixed moment MM *is defined in Lemma 3.16;*
2) the ratio of the coefficients of the composite polynomial $\pi_{f_0,\dots,f_k}$ at the vertices B_1 and B_2 equals

$$(-1)^{\gamma_1\cdot B_1+\gamma_2\cdot B_2}R_{\tilde{A}_0,\dots,\tilde{A}_k}(t;\tilde{f}_0,\dots,\tilde{f}_k).$$

After an appropriate monomial change of coordinates and multiplication polynomials f_i by appropriate monomials, one can use this theorem to find the ratio of coefficients of the composite polynomial $\pi_{f_0,\dots,f_k}$ at two arbitrary vertices B_1 and B_2 of its Newton polytope. If $\pi_{f_0,\dots,f_k}$ is homogeneous, then a monomial change of coordinates is not necessary. If the Newton polytopes of the polynomials $f_0,\dots,f_k$ satisfy some condition of general position (see Definition 5.2), then one can use Theorem 5.13 to compute $R(t;\tilde{f}_0,\dots,\tilde{f}_k)$ explicitly.

PROOF. Part 1 follows from Theorem 3.19. To prove Part 2, apply the following lemma to the composite polynomial $\pi_{f_0,\dots,f_k}$, multiplied by a monomial in such a way that its Newton polytope belongs to the positive octant and intersects all coordinate hyperplanes.

LEMMA **4.7.** *Suppose that the Newton polytope A of a polynomial g intersects all coordinate hyperplanes, and γ_1 and γ_2 are covectors with positive integer components. Then the ratio of the coefficients of g at the vertices A^{γ_1} and A^{γ_2} equals $(-1)^{\gamma_1\cdot B_1+\gamma_2\cdot B_2}$ times the product of roots of the Laurent poynomial in one variable $g(t^{\gamma_2}+t^{-\gamma_1})$.*

This lemma is a corollary of the Vieta theorem.

Proof of Theorem 4.4. Extend the commutative square

$$\begin{array}{ccc} (\mathbb{C}\setminus 0)^n & \overset{h}{\mapsto} & (\mathbb{C}\setminus 0)^{n'} \\ \downarrow \pi & & \downarrow \pi' \\ (\mathbb{C}\setminus 0)^{n-k} & \overset{h'}{\mapsto} & (\mathbb{C}\setminus 0)^{n'-k} \end{array} \quad \text{to} \quad \begin{array}{ccccc} (\mathbb{C}\setminus 0)^n & \overset{p}{\mapsto} & T & \overset{p_2}{\mapsto} & (\mathbb{C}\setminus 0)^{n'} \\ & & \downarrow p_1 & & \downarrow \pi' \\ & & (\mathbb{C}\setminus 0)^{n-k} & \overset{h'}{\mapsto} & (\mathbb{C}\setminus 0)^{n'-k}, \end{array}$$

where p_1 and p_2 are the projections of $(\mathbb{C}\setminus 0)^{n-k}\times(\mathbb{C}\setminus 0)^{n'}$ to the multipliers, T is the kernel of the epimorphism $h'\circ p_1 - \pi'\circ p_2 : (\mathbb{C}\setminus 0)^{n-k}\times(\mathbb{C}\setminus 0)^{n'} \to (\mathbb{C}\setminus 0)^{n'-k}$, and $p=(\pi,h):(\mathbb{C}\setminus 0)^n \to (\mathbb{C}\setminus 0)^{n-k}\times(\mathbb{C}\setminus 0)^{n'}$. The corresponding commutative diagram of embeddings of character lattices implies that the image of $p^\times$ is a sublattice of index $q=\left|\mathbb{Z}^n \,/\, (\pi^\times\mathbb{Z}^{n-k}+h^\times\mathbb{Z}^{n'})\right|$ in $\mathbb{Z}^n$. Thus, a fiber of the epimorphism p consists of q points.

For a cycle $N=\sum_i a_iN_i$ in a complex torus $(\mathbb{C}\setminus 0)^m$ and an epimorphism $p:(\mathbb{C}\setminus 0)^n \to (\mathbb{C}\setminus 0)^m$, denote the cycle $\sum_i a_i p^{(-1)}(N_i)$ by $p^{(-1)}(N)$. If $m=n$ and a fiber of p consists of q points, then $p_*\circ p^{(-1)}(N)=q\cdot N$. Thus, $\pi_*\circ h^{(-1)}=(p_1)_*\circ p_*\circ p^{(-1)}\circ p_2^{(-1)} = q\cdot(p_1)_*\circ p_2^{(-1)} = q\cdot h^{(-1)}\circ\pi'_*$. To prove the statement of the theorem, apply both sides of this equality to the cycle $[g_0=\ldots=g_k=0]$. $\square$

Truncations of varieties. The proof of theorem 4.3 is based on the following definition of a truncation of a variety (just a more geometric reformulation of the usual one, see [K]). By varieties we mean formal sums of irreducible algebraic varieties of the same dimension with positive coefficients. By the intersection of varieties we mean the intersection counting multiplicities, which makes sense for proper intersections only (the intersection of varieties V_i is said to be *proper*, if its codimension equals the sum of codimensions of V_i). For an algebraic curve $C\subset(\mathbb{C}\setminus 0)^n$, there exists a unique compactification $\tilde{C}=C\sqcup\{p_1,\ldots,p_I\}$ which is smooth near all infinite points p_i. A variety $N\subset(\mathbb{C}\setminus 0)^n$ is said to be *γ-homogeneous* for a linear function γ on the character lattice of the torus $(\mathbb{C}\setminus 0)^n$, if N is invariant under the action of the corresponding one-parameter subgroup $\{t^\gamma \,|\, t\in(\mathbb{C}\setminus 0)\}\subset(\mathbb{C}\setminus 0)^n$.

Definition 4.8. 1) The truncation of an irreducible curve $C\subset(\mathbb{C}\setminus 0)^n$ in the direction $\gamma\in\mathbb{Z}^n$ is a curve $C^\gamma=\sum A_i$, where the summation is over all infinite points p_i of its compactification $\tilde{C}$, and a curve A_i is given by a parameterization c_it^γ, if C is given by a parameterization $c_it^\gamma+\ldots$ near p_i.
2) The truncation of an arbitrary curve $C=\sum m_iC_i$ in the direction $\gamma\in\mathbb{Z}^n$ is a curve $C^\gamma=\sum m_iC_i^\gamma$.

3) The truncation of an m-dimensional variety $M \subset (\mathbb{C} \setminus 0)^n$ in the direction $\gamma \in \mathbb{Z}^n$ is an m-dimensional γ-homogeneous variety M^γ, such that for any γ-homogeneous variety N of dimension $\operatorname{codim} M + 1$
a) if $M^\gamma \cap N$ is a curve, then $M \cap N$ is a curve, and
b) under this assumption, $M^\gamma \cap N = (M \cap N)^\gamma$.

Lemma **4.9.** *1) There exists a unique truncation of a given variety in a given direction.*
2) Let $f_1 = \ldots = f_k = 0$ be a Newton-nondegenerate complete intersection. Then its truncation in a direction γ is the complete intersection $f_1^\gamma = \ldots = f_k^\gamma = 0$.
3) There is a finite number of different truncations of a given variety.

Proof. Uniqueness follows from the definition. Existence is a corollary of the following explicit construction for the truncation of $M \subset (\mathbb{C} \setminus 0)^n$ in the direction $\gamma \in \mathbb{Z}^n$. Without loss of generality we can assume that $\gamma = (k, 0, \ldots, 0)$ and define M^γ as $p_1^{-1}(\overline{M} \cap \{x_1 = 0\})$, wher $x_1, \ldots, x_n$ are the standard coordinates in $\mathbb{C}^n$, $p_1 : (\mathbb{C} \setminus 0)^n \to \{x_1 = 0\}$ is the standard projection, and $\overline{M} \subset \mathbb{C} \times (\mathbb{C} \setminus 0)^{n-1}$ is the closure of the variety $M \subset (\mathbb{C} \setminus 0)^n \subset \mathbb{C} \times (\mathbb{C} \setminus 0)^{n-1}$ counting multiplicities.

Part 2 also follows from this construction. Indeed, the variety $p_1^{-1}(\overline{\{f_1^\gamma = \ldots = f_k^\gamma = 0\}} \cap \{x_1 = 0\})$ is given by the ideal I, generated by γ-truncations of all elements of the ideal $\langle f_1, \ldots, f_k \rangle$. The ideal I equals $\langle f_1^\gamma, \ldots, f_k^\gamma \rangle$, because, for any relation $\sum g_i f_i^\gamma = 0$, polynomials g_i are contained in the ideal $\langle f_1^\gamma, \ldots, f_k^\gamma \rangle$ (the last fact is equivalent to vanishing of the first homology group of the Koszul complex for a regular sequence $f_1^\gamma, \ldots, f_k^\gamma$).

In general, Part 3 follows from the existance of the c-fan, or the Grobner fan of the ideal of a variety M (see [K]). If M is a Newton-nondegenerate complete intersection, which is the only important case for the proof of theorem 4.3, then Part 3 follows from Part 2. Indeed, γ_1 and γ_2-truncations of a Newton-nondegenerate complete intersection $f_1 = \ldots = f_k = 0$ coincide, if $A^{\gamma_1} = A^{\gamma_2}$, where A is the sum of the Newton polytopes Δ_{f_i}. □

Proof of Theorem 4.3. Theorem 4.3 is a special case of the following fact:

Theorem **4.10.** *Let $\pi : (\mathbb{C} \setminus 0)^n \to (\mathbb{C} \setminus 0)^{n-k}$ and $\pi^\times : \mathbb{Z}^{n-k} \hookrightarrow \mathbb{Z}^n$ be an epimorphism of complex tori and the corresponding embedding of their character lattices, and let $M \subset (\mathbb{C} \setminus 0)^n$ be a variety. Then the truncation $\big(\pi_*(M)\big)^\gamma$ equals the sum $\sum_\delta \pi_*(M^\delta)$ over all $\delta \in (\mathbb{Z}^n)^*$ such that $\delta|_{\pi^\times \mathbb{Z}^{n-k}} = \gamma$. (in particular, there is a finite number of non-empty summands).*

If M is 1-dimensional, then this theorem follows from the definition of the truncation of a curve. If the dimension is arbitrary, then the number of non-empty summands is finite by Lemma 4.9.3, since $M^{\delta_1} = M^{\delta_2}$, $\delta_1 \neq \delta_2$, $\delta_1|_{\pi\times\mathbb{Z}^{n-k}} = \delta_2|_{\pi\times\mathbb{Z}^{n-k}}$ implies $\pi_* M^{\delta_1} = \pi_* M^{\delta_2} = \varnothing$. If a (codim $M-k+1$)-dimensional γ-homogeneous variety $N \subset (\mathbb{C}\setminus 0)^{n-k}$ intersects all summands $\pi_*(M^\delta)$ properly, then

$$N \cap \sum_{\substack{\delta\in(\mathbb{Z}^n)^*,\\ \delta|_{\pi\times\mathbb{Z}^{n-k}}=\gamma}} \pi(M^\delta) = \bigl(N\cap\pi(M)\bigr)^\gamma \quad \overset{(1)}{\Leftrightarrow}$$

$$\pi_*\Bigl(\sum_{\substack{\delta\in(\mathbb{Z}^n)^*,\\ \delta|_{\pi\times\mathbb{Z}^{n-k}}=\gamma}} \pi^{(-1)}(N)\cap M^\delta\Bigr) = \Bigl(\pi_*\bigl(\pi^{(-1)}(N)\cap M\bigr)\Bigr)^\gamma \quad \overset{(2)}{\Leftrightarrow}$$

$$\pi_*\Bigl(\sum_{\substack{\delta\in(\mathbb{Z}^n)^*,\\ \delta|_{\pi\times\mathbb{Z}^{n-k}}=\gamma}} \pi^{(-1)}(N)\cap M^\delta\Bigr) = \pi_*\Bigl(\sum_{\substack{\delta\in(\mathbb{Z}^n)^*,\\ \delta|_{\pi\times\mathbb{Z}^{n-k}}=\gamma}} \bigl(\pi^{(-1)}(N)\cap M\bigr)^\delta\Bigr).$$

Here the last equation follows from the definition of a truncation of the variety M. Equivalence (2) is the statement of the theorem for a curve $\pi^{(-1)}(N)\cap M$. Equivalence (1) is a corollary of the following fact: $\pi_*(A\cap \pi^{(-1)}B) = (\pi_* A)\cap B$ for any varieties $A\subset(\mathbb{C}\setminus 0)^n$ and $B\subset(\mathbb{C}\setminus 0)^{n-k}$.

5 Leading coefficients of a composite polynomial: explicit answers for generic Newton polytopes.

DEFINITION **5.1.** *The edge coefficients* of a polynomial f are the coefficients of its monomials which correspond to the integer lattice points on the edges of the Newton polytope Δ_f.

We can compute explicitly the Newton polytope and the vertex and edge coefficients of a composite polynomial $\pi_{f_0,\ldots,f_k}$, provided that the Newton polytopes of the polynomials $f_0,\ldots,f_k$ satisfy the following condition of general position.

DEFINITION **5.2.** Polytopes $A_0,\ldots,A_k$ in $\mathbb{R}^n$ are said to be *developed*, if the following condition is satisfied:

> Faces $B_0, \ldots, B_k$ of polytopes $A_0, \ldots, A_k$ sum up to a k-dimensional face of the Minkowski sum $A_0 + \ldots + A_k \Rightarrow B_i$ is a vertex of A_i for some i.

Elimination theory for polynomials with developed Newton polytopes. If the Newton polytopes of polynomials $f_0, \ldots, f_k$ are developed, then the explicit computation of the Newton polytope and the vertex and edge coefficients of the composite polynomial $\pi_{f_0,\ldots,f_k}$ is based on the following facts:

- polynomials $f_0, \ldots, f_k$ are Newton-nondegenerate, and the assumption of Newton nondegeneracy in Theorems 1.7.1, 4.3, 4.4 and 4.6 is redundant.

- Theorems 4.3, 4.4 and 4.6 express the vertex and edge coefficients of a composite polynomial of several variables in terms of composite polynomials of one variable.

- Passing to the right hand side in the formulation of Theorems 4.3, 4.4 and 4.6 preserves the property of Newton polytopes to be developed (see Lemmas 5.3 and 5.5 below).

- If $\pi_{f_0,\ldots,f_k}$ is a composite polynomial of one variable, then Lemma 1.1 implies that Khovanskii's product formula (Theorems 5.11 and 5.13) and Gelfand-Khovanskii formula (Theorem 5.8) can be seen as explicit formulas for the vertex coefficient and the edge coefficients of $\pi_{f_0,\ldots,f_k}$ respectively.

LEMMA **5.3.** *1) In the notation of Theorem 4.3, if the Newton polytopes of polynomials $f_0, \ldots, f_k$ are developed, then the Newton polytopes of the polynomials $f_0^\delta, \ldots, f_k^\delta$ are also developed for every covector δ.*
2) In the notation of Theorem 4.4, if the Newton polytopes of polynomials $f_0, \ldots, f_k$ are developed, then the Newton polytopes of the polynomials $g_0, \ldots, g_k$ are developed.

These facts follow from definitions, and we omit the proof.

However, in the notation of Theorem 4.6, the Newton polytopes of polynomials $\tilde{f}_0, \ldots, \tilde{f}_k$ are not usually developed (regardless of the Newton polytopes of polynomials $f_0, \ldots, f_k$), and we have to consider the following (weaker) condition.

DEFINITION **5.4.** Polytopes $A_1, \ldots, A_n$ in $\mathbb{R}^n$ are said to be *developed with respect to a point* $b \in \mathbb{R}^n$, if the following condition is satisfied:

> Faces $B_1, \ldots, B_n$ of polytopes $A_1, \ldots, A_n$ sum up to a face of the Minkowski sum $A_1 + \ldots + A_n \Rightarrow B_i$ is a vertex of A_i for some

i, unless $B_1 + \ldots + B_n$ contains a segment parallel to the vector $b \in \mathbb{R}^n$.

LEMMA **5.5.** *In the notation of Theorem 4.6, if the Newton polytopes of polynomials $f_0, \ldots, f_k$ are developed, then the Newton polytopes of the polynomials $\tilde{f}_0, \ldots, \tilde{f}_k$ are developed with respect to the degree of the monomial t.*

Gelfond-Khovanskii's formula and Khovanskii's product formula.

DEFINITION **5.6.** For a collection of polytopes $A_1, \ldots, A_n$ in $\mathbb{R}^n$, let ϕ_i be a non-negative real-valued function on the boundary $\partial(A_1 + \ldots + A_n)$, such that its zero set is the union of all faces of the form $B_1 + \ldots + B_n$, where $B_1, \ldots, B_n$ are faces of $A_1, \ldots, A_n$ respectively, and B_i is a vertex. *The combinatorial coefficient* C_a of a vertex $a \in (A_1 + \ldots + A_n)$ is the local degree of the map $(\phi_1, \ldots, \phi_n) : \partial(A_1 + \ldots + A_n) \to \partial\mathbb{R}^n_+$ near a, provided that $\phi_1 \cdot \ldots \cdot \phi_n = 0$ near a.

In particular, the definition of the combinatorial coefficient makes sense for all vertices of the sum of developed polytopes.

DEFINITION **5.7.** Let $f_1, \ldots, f_n, g$ be Laurent polynomials of variables $x_1, \ldots, x_n$, and suppose that their Newton polytopes $A_1, \ldots, A_n$ are developed. *The residue* $\mathrm{res}_a \, \omega_{f.,g}$ of a form $\omega_{f.,g} = \frac{g dx_1 \wedge \ldots \wedge dx_n}{f_1 \ldots f_n x_1 \ldots x_n}$ at a vertex a of the polytope $\sum A_i$ is defined as the constant term of the series $g \frac{1}{p(a)} \frac{1}{p/p(a)}$, where p is the product $f_1 \cdot \ldots \cdot f_n$, $p(a)$ is its term of degree a of the polynomial p, and $\frac{1}{p/p(a)}$ is the inverse of the polynomial $p/p(a)$ near the origin.

THEOREM **5.8** ([GKh])**.** *Let $f_1, \ldots, f_n, h$ be Laurent polynomials on $(\mathbb{C} \setminus 0)^n$, and suppose that their Newton polytopes $A_1, \ldots, A_n$ are developed. Then the sum of values of h over the roots of the system $f_1 = \ldots = f_n = 0$ (multiplicities of the roots being taken into account) equals $(-1)^n \sum_a C_a \, \mathrm{res}_a \, \omega_{f.,h \det \frac{\partial f.}{\partial x.}}$, where a runs over all vertices of the polytope $\sum A_i$.*

Let $\mathbb{Z}_2^{n \times m}$ be the space of $\mathbb{Z}_2$-matrices with n rows and m columns.

DEFINITION **5.9.** There exists a unique non-zero function $\det_2 : \mathbb{Z}_2^{n \times (n+1)} \to \mathbb{Z}_2$ which is linear and symmetric as a function of columns and vanishes at degenerate matrices. It is called the *2-determinant.*

Definition **5.10.** Let $f_1, \ldots, f_n$ be Laurent polynomials on $(\mathbb{C} \setminus 0)^n$, and suppose that their Newton polytopes $A_1, \ldots, A_n$ are developed. *The Parshin symbol* $[f_1, \ldots, f_n, x^b]_a$ of the monomial x^b at a vertex a of the polytope $\sum A_i$ is the product

$$(-1)^{\det_2(a_1,\ldots,a_n,b)} f_1(a_1)^{-\det(b,a_2,\ldots,a_n)} \ldots f_n(a_n)^{-\det(b,a_1,\ldots,a_{n-1})},$$

where $f_i(a)$ is the term of degree a of the polynomial f_i.

Theorem **5.11** ([Kh2]). *Let $f_1, \ldots, f_n$ be Laurent polynomials on $(\mathbb{C} \setminus 0)^n$, and suppose that their Newton polytopes $A_1, \ldots, A_n$ are developed. Then the product of values of the monomial x^{A_0} over the roots of the system $f_1 = \ldots = f_n = 0$ (taking multiplicities of the roots into account) equals $\prod_a [f_1, \ldots, f_n, x^{A_0}]_a^{(-1)^n C_a}$, where a runs over all vertices of the polytope $\sum A_i$.*

In particular, this product is a monomial as a function of the vertex coefficients of polynomials $f_1, \ldots, f_n$, and this theorem can be seen as a multidimensional generalization of the fact that the constant term of a polynomial in one variable equals the product of the negatives of its roots.

Khovanskii's product formula for Newton polytopes, developed with respect to a point. Lemma 5.5 implies, that we have to generalize Theorem 5.11 to polytopes, developed with respect to a point, to make it applicable in the context of Theorem 4.6. For a polytope $A \subset \mathbb{R}^n$ and a concave piecewise-linear function $v : A \to \mathbb{R}$, denote the polyhedron $\{(a,t) | a \in A, t \leq v(a)\} \subset \mathbb{R}^n \oplus \mathbb{R}^1$ by $N(v)$. Let $v_1, \ldots, v_n$ be piecewise-linear functions on polytopes $A_1, \ldots, A_n \subset \mathbb{R}^n$. Denote the union of all bounded faces of the polytope $\sum_i N(v_i) \subset \mathbb{R}^n \oplus \mathbb{R}^1$ by Γ. Γ is a topological disc. Let $\Gamma_j \subset \partial\Gamma$ be the union of all faces that can be represented as $\sum_i B_i$ where B_i are faces of $N(v_i), i = 1, \ldots, n$, and B_j is a point. Consider a continuous mapping $(\phi_1, \ldots, \phi_n) : \partial\Gamma \to \mathbb{R}^n_+$, such that the zero set of a function ϕ_j is Γ_j.

Definition **5.12.** Functions $v_1, \ldots, v_n$ are called *developed*, if the image of the mapping $(\phi_1, \ldots, \phi_n)$ is contained in the boundary of the positive octant $\mathbb{R}^n_+$. A point $a \in \partial(A_1 + \ldots + A_n)$ is called *a vertex of the sum $A_1 + \ldots + A_n$ with respect to the functions v_j*, if it equals the projection of some vertex $b \subset \partial\Gamma$ of the sum $N(v_1) + \ldots + N(v_n)$. In this case $b = b_1 + \ldots + b_n$, where b_j is a vertex of the polyhedron $N(v_j)$, and we denote the projection of b_j by a_j.

The combinatorial coefficient C_a of a vertex a is the local topological degree of the mapping $(\phi_1, \ldots, \phi_n) : \Gamma \to \partial\mathbb{R}^n_+$ at the point b. *The Parshin symbol* $[f_1, \ldots, f_n, x^k]_a$ of the monomial x^k at this vertex is the product

$$(-1)^{\det_2(a_1,\ldots,a_n,k)} f_1(a_1)^{-\det(k,a_2,\ldots,a_n)} \ldots f_n(a_n)^{-\det(k,a_1,\ldots,a_{n-1})},$$

where $f_i(a)$ is the term of degree a of the polynomial f_i.

THEOREM **5.13.** *If polytopes* $A_1, \ldots, A_n \subset \mathbb{Z}^n$ *are developed with respect to* A_0*, then the function* $R_{A_1,\ldots,A_n}(x^{A_0}; f_1, \ldots, f_n)$ *(see Definition 4.5) equals the following monomial in vertex coefficients of polynomials* $f_1, \ldots, f_n$*:*

$$\prod_{a \text{ is a vertex of } A_1+\ldots+A_n \text{ with respect to } v_1,\ldots,v_n} [f_1, \ldots, f_n, x^{A_0}]_a^{(-1)^n C_a},$$

where $v_1, \ldots, v_n$ *are arbitrary developed functions on the polytopes* $A_1, \ldots, A_n$ *such that all vertices of* $A_1 + \ldots + A_n$ *with respect to* $v_1, \ldots, v_n$ *are integer.*

Theorem 5.11 is a special case of this theorem for developed polytopes $A_1, \ldots, A_n$ and developed functions $v_i = 0$ on them. The statement of Theorem 5.13 is true for Newton-degenerate polynomials $f_1, \ldots, f_n$, but $R_{A_1,\ldots,A_n}(x^{A_0}; f_1, \ldots, f_n)$ is not equal to the product of values of the monomial x^{A_0} over the roots of the system $f_1 = \ldots = f_n = 0$ in this case.

PROOF. The main point in the proof of Theorem 5.11 (see [Kh2]) is the following fact: if polynomials $f_1, \ldots, f_n$ depend on a parameter $s \in (\mathbb{C} \setminus 0)$, and their Newton polytopes are developed and do not depend on s, then the product of values of the monomial x^{A_0} over the roots of the system $f_1 = \ldots = f_n = 0$ as a function of s is a monomial, because it is a rational function of s which has no zeroes and no poles. One can easily verify that the same is true under the assumption that the Newton polytopes of $f_1, \ldots, f_n$ are developed with respect to A_0, if we consider the function $R_{A_1,\ldots,A_n}(x^{A_0}; f_1, \ldots, f_n)$ instead of the product of x^{A_0} over the roots of the system $f_1 = \ldots = f_n = 0$. □

6 Other versions of elimination theory in the context of Newton polytopes.

In this paper, we discuss common zeros of Laurent polynomials, the multiplicities of zeros being taken into account. Of course, one can develop the same theory in many other contexts. Let us give some examples.

Square free composite polynomials. This point of view is usual when discussing Newton polytopes of multidimensional resultants. For a finite set $A \in \mathbb{Z}^n$, denote the set of all Laurent polynomials $\sum\limits_{a\in A} c_a x^a$ by $\mathbb{C}[A]$. Consider an epimorphism $\pi : (\mathbb{C} \setminus 0)^n \to (\mathbb{C} \setminus 0)^{n-k}$ and finite sets $A_0, \ldots, A_k$ in the character lattice $\mathbb{Z}^n$ of the complex torus $(\mathbb{C} \setminus 0)^n$.

The composite polynomial $\pi_{f_0,\ldots,f_k}$ is not square free for a collection of polynomials $(f_0, \ldots, f_k) \in \mathbb{C}[A_0] \oplus \ldots \oplus \mathbb{C}[A_k]$, if the sets $A_0, \ldots, A_k \subset \mathbb{Z}^n$ are degenerate in some sense. Let $\pi^0_{f_0,\ldots,f_k}$ be the square free polynomial that has the same zeros as $\pi_{f_0,\ldots,f_k}$. The theorem stated below expresses *the square free composite polynomial* $\pi^0_{f_0,\ldots,f_k}$ in terms of $\pi_{f_0,\ldots,f_k}$.

Definition **6.1.** Let $L \subset \mathbb{Z}^n$ be an $(n-k)$-dimensional lattice, and let $p : \mathbb{Z}^n \to \mathbb{Z}^k$ be the projection along L. *The multiplicity* $d(A_0, \ldots, A_k, L)$ of the collection of finite sets $A_0, \ldots, A_k \subset \mathbb{Z}^n$ with respect to L is defined as follows:

1) if $\dim p(A_{i_1} + \ldots + A_{i_q}) < q-1$ for some numbers $0 \leqslant i_1 < \ldots < i_q \leqslant k$, then $d(A_0, \ldots, A_k, L) = 0$;

2) Otherwise, choose the minimal non-empty set $\{i_1, \ldots, i_q\} \subset \{0, \ldots, k\}$ such that $\dim p(A_{i_1} + \ldots + A_{i_q}) = q - 1$, choose the minimal sublattice $M \subset \mathbb{Z}^n$ that contains the sum $A_{i_1} + \ldots + A_{i_q} + L$ up to a shift, and note that $\operatorname{codim} M = k + 1 - q$. Denote the projection $\mathbb{Z}^n \to \mathbb{Z}^{k+1-q}$ along M by r, and denote the set $\{0, \ldots, k\} \setminus \{i_1, \ldots, i_q\}$ by $\{j_1, \ldots, j_{k+1-q}\}$. In this notation,

$$d(A_0, \ldots, A_k, L) = (k + 1 - q)!\,\mathrm{MV}(rA_{j_1}, \ldots, rA_{j_{k+1-q}}) \cdot |\ker r/M|.$$

For example, suppose that $L \subset \mathbb{Z}^2$ is the horizontal coordinate axis. In this case $d(A_1, A_2, L) = 0$ iff both A_1 and A_2 are contained in horizontal segments. If one of them is contained in a horizontal segment, then $d(A_1, A_2, L)$ equals the height of the other one. If neither A_1 nor A_2 is contained in a horizontal segment, then $d(A_1, A_2, L)$ equals the GCD of lengths of vertical segments, connecting points of the set $A_1 + A_2 + L$.

Theorem **6.2.** *Consider an epimorphism of complex tori* $\pi : (\mathbb{C}\setminus 0)^n \to (\mathbb{C} \setminus 0)^{n-k}$*, the corresponding embedding of their character lattices* $L \subset \mathbb{Z}^n$*, and finite sets* $A_0, \ldots, A_k \subset \mathbb{Z}^n$*.*

1) If $d(A_0, \ldots, A_k, L) = 0$*, then* $\pi^0_{f_0,\ldots,f_k} = \pi_{f_0,\ldots,f_k} = 1$ *for all collections of polynomials* $(f_0, \ldots, f_k) \in \mathbb{C}[A_0] \oplus \ldots \oplus \mathbb{C}[A_k]$*.*

2) Otherwise, $(\pi^0_{f_0,\dots,f_k})^{d(A_0,\dots,A_k,L)} = \pi_{f_0,\dots,f_k}$ *for all collections of polynomials* $f_0,\dots,f_k$ *from some Zariski open subset of the space* $\mathbb{C}[A_0]\oplus\ldots\oplus\mathbb{C}[A_k]$.

Note that this Zariski open subset neither contains nor is contained in the set of all Newton-nondegenerate collections of polynomials. In particular, this theorem implies that the Newton polytope of $\pi^0_{f_0,\dots,f_k}$ is $d(A_0,\dots,A_k,L)$ times smaller, than the Newton polytope of $\pi_{f_0,\dots,f_k}$ for a generic collection of polynomials $(f_0,\dots,f_k) \in \mathbb{C}[A_0]\oplus\ldots\oplus\mathbb{C}[A_k]$.

Proof. Part 1 is a corollary of Theorem 2.6.1. Applying Theorem 2.6.2 and Theorem 4.4, one can reduce Part 2 to the following special case. □

Definition **6.3.** Let $L \subset \mathbb{Z}^n$ be an $(n-k)$-dimensional lattice $L \subset \mathbb{Z}^n$, and let $p : \mathbb{Z}^n \to \mathbb{Z}^k$ be the projection along L. A collection of finite sets $A_0,\dots,A_k \subset \mathbb{Z}^n$ is said to be *essential* with respect to $L \subset \mathbb{Z}^n$ if $\dim p(A_{i_1} + \ldots + A_{i_q}) > q-1$ for every collection of numbers $0 \leqslant i_1 < \ldots < i_q \leqslant k$, $q \leqslant k$, and the sum $A_0 + \ldots + A_k + L$ is not contained in a shifted proper sublattice of $\mathbb{Z}^n$.

Note that $d(A_0,\dots,A_k,L) = 1$ if the collection $A_0,\dots,A_k \subset \mathbb{Z}^n$ is essential with respect to L, but the converse is not true. For example, if $L \subset \mathbb{Z}^2$ is the horizontal coordinate axis, then A_1 and A_2 form an essential collection iff neither of them is contained in a horizontal segment and $d(A_1,A_2,L) = 1$.

Lemma **6.4.** *If* $A_0,\dots,A_k$ *are essential with respect to* L*, then* $\pi^0_{f_0,\dots,f_k} = \pi_{f_0,\dots,f_k}$ *for all collections of polynomials* $f_0,\dots,f_k$ *from some Zariski open subset of the space* $\mathbb{C}[A_0]\oplus\ldots\oplus\mathbb{C}[A_k]$.

The proof is the straightforward generalization of a similar argument for multidimensional resultants (see [S], Theorem 1.1, where the notion of essential sets was introduced for $k = n$).

Composite functions of rational functions.

Definition **6.5.** *A vertex* of a virtual polytope $A - B$ is a pair of vertices (a,b) of polytopes A and B, such that $a + b$ is a vertex of the sum $A + B$. *The Newton polytope of a rational function* $\frac{f}{g}$ is the difference of the Newton polytopes of f and g. *The vertex coefficient of a rational function* $\frac{f}{g}$ at the vertex (a,b) of its Newton polytope is the ratio of the vertex coefficients of polynomials f and g at the vertices a and b respectively.

One can readily generalize elimination theory from Laurent polynomials and convex polytopes to rational functions and virtual polytopes.

Composite functions of germs of analytic functions. A *convex polyhedron* in $\mathbb{R}^n$ is an intersection of a finite number of half-spaces (which may be unbounded). Two convex polyhedra in $\mathbb{R}^n$ are said to be *parallel* if their support functions have the same domain. For a germ of an analytic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ of variables $x_1, \ldots, x_n$, the Newton polyhedron is defined as the minimal polyhedron parallel to the positive octant in the lattice of monomials in $x_1, \ldots, x_n$ and containing all monomials of the Taylor expansion of f. One can readily generalize elimination theory from Laurent polynomials and bounded polyhedra to germs of analytic functions and polyhedra parallel to the positive octant. It requires the following version of Bernstein's theorem.

Definition **6.6** ([E1], [E2])**.** Let P_C be the set of all pairs of polyhedra (A, B), such that A and B are both parallel to a cone C and the difference $A \triangle B$ is bounded. The notions of Minkowski sum $(A, B) + (C, D) = (A + C, B + D)$ and volume $\mathrm{Vol}\big((A, B)\big) = \mathrm{Vol}(A \setminus B) - \mathrm{Vol}(B \setminus A)$ for such pairs give rise to the mixed volume $V_C : \underbrace{P_C \times \ldots \times P_C}_{n} \to \mathbb{R}$, which is the polarization of the volume with respect to Minkowski summation.

If the cone C consists of one point $*$, then P_C is the set of pairs of bounded polyhedra, $\mathrm{Vol}\big((A, B)\big) = \mathrm{Vol}(A) - \mathrm{Vol}(B)$, and thus

$$V_*((A_1, B_1), \ldots, (A_n, B_n)) = \mathrm{MV}(A_1, \ldots, A_n) - \mathrm{MV}(B_1, \ldots, B_n).$$

If $C \neq \{*\}$, then the mixed volumes on the right-hand side are infinite, but "their difference is well defined".

Lemma **6.7** ([E2])**.** $V_C((A_1, B_1), \ldots, (A_n, B_n)) +$

$$+ V_C((B_1, C_1), \ldots, (B_n, C_n)) = V_C((A_1, C_1), \ldots, (A_n, C_n)).$$

Let μ be the unit volume form in $\mathbb{R}^n$, let S be the positive octant in $(\mathbb{R}^n)^*$, and let $S_0 \subset S$ be a set of covectors that contains a unique multiple of every covector from S.

Lemma **6.8** ([E2])**.** $V_C((A_1, B_1), \ldots, (A_n, B_n)) =$

$$= \frac{1}{n} \sum_{\gamma \in S_0} \sum_{i=1}^{n} (\max \gamma(A_i) - \max \gamma(B_i))\, \mathrm{MV}_{\mu/\gamma}(A_1^\gamma, \ldots, A_{i-1}^\gamma, B_{i+1}^\gamma, \ldots, B_n^\gamma).$$

Note that the right hand side of this formula is not symmetric with respect to permutations of pairs. The sum in the right hand side makes sense, since it containts finitely many non-zero summands (which correspond to normal covectors of bounded $(n-1)$-dimensional faces of the sum $A_1 + B_1 + \ldots + A_n + B_n$). The $(n-1)$-dimensional mixed volume in the right hand side makes sense, since all arguments are contained in the $(n-1)$-dimensional space $\ker \gamma$.

DEFINITION **6.9.** A polyhedron is called an *M-far stabilization* of a polyhedron $\Delta \subset \mathbb{R}^n_+$ parallel to the positive octant $\mathbb{R}^n_+$, if it can be represented as the convex hull of a union $\Delta \cup \Gamma$ for some polyhedron $\Gamma \subset \mathbb{R}^n_+$, such that the distance between Γ and the origin is greater than M, and the difference $\mathbb{R}^n_+ \setminus \Gamma$ is bounded. *The mixed volume of (unbounded) polyhedra $\Delta_1, \ldots, \Delta_n \subset \mathbb{R}^n_+$* parallel to $\mathbb{R}^n_+$ is defined as the mixed volume of pairs $(\mathbb{R}^n_+, \tilde{\Delta}_1), \ldots, (\mathbb{R}^n_+, \tilde{\Delta}_n)$, where $\tilde{\Delta}_i$ is an M-far stabilization of Δ_i, provided that the mixed volume of these pairs is independent of the choice of M-far stabilizations for some M (we say that the mixed volume of $\Delta_1, \ldots, \Delta_n$ is *well-defined* in this case).

THEOREM **6.10.** *1) The mixed volume of polyhedra $\Delta_1, \ldots, \Delta_n \subset \mathbb{R}^n_+$ parallel to the positive octant $\mathbb{R}^n_+$ is well-defined if and only if each k-dimensional coordinate plane intersects at least k of these polyhedra.*
2) If the germs of functions $f_1, \ldots, f_n : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ have an isolated common root of multiplicity μ, then the mixed volume V of their Newton polyhedra is well-defined, and $\mu \geqslant n!V$.
3) If the mixed volume V of integer polyhedra $\Delta_1, \ldots, \Delta_n \subset \mathbb{R}^n_+$ parallel to the positive octant $\mathbb{R}^n_+$ is well defined, then the germs of analytic functions $f_1, \ldots, f_n : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ have an isolated common root of multiplicity $n!V$, provided that their Newton polyhedra are equal to $\Delta_1, \ldots, \Delta_n$ and their leading coefficients are in general position in the following sense:

> *For any collection of bounded faces $A_1 \subset \Delta_1, \ldots, A_n \subset \Delta_n$, such that the sum $A_1 + \ldots + A_n$ is a face of the sum $\Delta_1 + \ldots + \Delta_n$, the Laurent polynomials $f_1|_{A_1}, \ldots, f_n|_{A_n}$ have no common zeros in $(\mathbb{C} \setminus 0)^n$.*

Part 1 follows from Lemma 6.7, Parts 2 and 3 follow from Part 1 and a local version of Bernstein's formula (see [E2], Theorem 3).

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