Stabilization of a piezoelectric system

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Abstract. We consider a stabilization problem for a piezoelectric system. We prove an exponential stability result under some Lions geometric condition. Our method is based on an identity with multipliers that allows to show an appropriate observability estimate.

Key words. elasticity system, Maxwell’s system, piezoelectric system, stabilization
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1 Introduction

Let Ω be a bounded domain of \( \mathbb{R}^3 \) with a Lipschitz boundary \( \Gamma \). In that domain we consider the non-stationary piezoelectric system that consists in a coupling between the elasticity system with the Maxwell equation. More precisely we analyze the following partial differential equations:

\begin{align*}
&\sigma_{ij}(u,E) = a_{ijkl}\gamma_{kl}(u) - e_{kij}E_k \quad \forall \ i, j = 1, 2, 3, \\
&D_i = \varepsilon_{ij}E_j + e_{ikl}\gamma_{kl}(u) \quad \forall \ i = 1, 2, 3, \\
&B = \mu H.
\end{align*}

The equations of equilibrium are

\begin{align*}
\partial_t^2 u_i &= \partial_j \sigma_{ji} \quad \forall \ i = 1, 2, 3, \\
\partial_t D &= \text{curl} H, \quad \partial_t B = -\text{curl} E
\end{align*}

for the elastic displacement and electric/magnetic fields.

This system models the coupling between Maxwell’s system and the elastic one, in which \( E(x,t), H(x,t) \) are the electric and magnetic fields at the point \( x \in \Omega \) at time \( t \), \( u(x,t) \) is the displacement field at the point \( x \in \Omega \) at time \( t \), and \( \gamma_{ij}(u)_{i,j=1} \) is the strain tensor given by

\[ \gamma_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \]

Here \( \sigma = (\sigma_{ij})_{i,j=1}^3, D = (D_1, D_2, D_3), \) and \( B = (B_1, B_2, B_3) \) are the stress tensor, electric displacement, and magnetic induction, respectively. \( \varepsilon, \mu \) are the electric permittivity and

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magnetic permeability, respectively, and are supposed to be positive real numbers. The
elasticity tensor \((a_{ijkl})_{i,j,k,l=1,2,3}\) is made of constant entries such that
\[
a_{ijkl} = a_{jikl} = a_{klij}
\]
and satisfies the ellipticity condition
\[
a_{ijkl}\gamma_{ij}\gamma_{kl} \geq \alpha_0 \gamma_{ij}\gamma_{ij},
\]
for every symmetric tensor \((\gamma_{ij})\) and some \(\alpha_0 > 0\). The piezoelectric tensor \(e_{kij}\) is also made of
constant entries such that
\[
e_{kij} = e_{kji}.
\]
For shortness in the remainder of the paper introduce the tensor \(\sigma(u) = (a_{ijkl}\gamma_{kl}(u))_{i,j=1}^{3}\)
and let \(\nabla \sigma\) be the vector field defined by
\[
\nabla \sigma = (\partial_j \sigma_{ij})_{i=1}^{3},
\]
while for a tensor \(\gamma = (\gamma_{ij})_{i,j=1}^{3}\), and a vector \(F = (F_1, F_2, F_3)\), we set
\[
e\gamma = (e_{ijkl}\gamma_{kl})_{i,j=1}^{3} \quad e^\top F = (e_{ij}F_i)_{k,j=1}^{3}.
\]
These last notations mean that \(e\) corresponds to a linear mapping from \(\mathbb{R}^{3 \times 3}\) into \(\mathbb{R}^3\) and
that \(e^\top\) is its adjoint. With these notations, we see that (1.1) is equivalent to
\[
\sigma(u, E) = \sigma(u) - e^\top E,
\]
while (1.2) is equivalent to
\[
D = \varepsilon E + e\gamma(u).
\]

The system (1.1)-(1.3) is completed with the boundary and Cauchy conditions. This
means that we are considering the following system
\[
\begin{align*}
\partial_t^2 u - \nabla \sigma(u, E) &= 0 \text{ in } Q := \Omega \times [0, +\infty[,
\partial_t D - \text{curl} H &= 0 \text{ in } Q,
\mu \partial_t H + \text{curl} E &= 0 \text{ in } Q,
\text{div}(D) = \text{div}(\mu H) &= 0 \text{ in } Q,
H \times \nu - (Q^* \partial_t u) \times \nu + (E \times \nu) \times \nu &= 0 \text{ in } \Sigma = \Gamma \times (0, +\infty),
\sigma(u, E) \cdot \nu + Q(E \times \nu) + Au + \partial_t u &= 0 \text{ in } \Sigma,
u(u(0)) - u_0, \partial_t u(0) &= u_1 \text{ in } \Omega,
E(0) &= E_0, H(0) &= H_0 \text{ in } \Omega,
\end{align*}
\]
where \(\nu\) is the unit normal vector of \(\partial \Omega\) pointing towards the exterior of \(\Omega\), \(A\) is a positive
constant and \(Q\) is a function from \(\Gamma\) into the set of \(3 \times 3\) matrices with the regularity
\(Q \in L^\infty(\Gamma, \mathcal{C}^{3 \times 3})\).

**Remark 1.1** Note that the image of \(Q\) of normal vector fields plays no rule in the boundary
conditions appearing in (1.7). Indeed for \(X \in \mathbb{C}^3\), let \(X_N = (X \cdot \nu)\nu\) and \(X_T = X - X_N\) be
the normal and tangential components of \(X\) respectively, by writing
\[
Q_N X = QX_N, \quad Q_T X = QX_T,
\]
we get the splitting
\[
Q = Q_N + Q_T, \quad Q^* = Q_N^* + Q_T^*.
\]
But by definition \(Q_N X\) is orthogonal to the tangent plane. Therefore \(Q(E \times \nu) = Q_T(E \times \nu)\)
and \((Q^* \partial_t u) \times \nu = (Q_T^* \partial_t u) \times \nu\), which means that the normal part \(Q_N\) of \(Q\) does not
contribute to the boundary condition.
In [18] the authors treat the case $Q = 0$ with eventually discontinuous coefficients and an additional memory term and prove the exponential decay rate of the energy if $A$ is small enough and if $\Omega$ satisfies some geometrical conditions (star like shape). On the contrary in [32], the author treats the case $Q = I$ and some nonlinear feedback terms, but with the choice of $\epsilon$ such that $\nabla (\epsilon^T E) = \xi \text{curl} E$ for some real number $\xi$ (case excluding the natural condition $c_{kij} = c_{kj}$) and proves the exponential decay rate of the energy in the case of linear feedbacks if $\Omega$ is strictly star shaped with respect to a point. In that last paper the author combines the multiplier technique with the one from [3], where the authors uses some tangential integration by parts and a technique from [5]. Our goal is here to perform the same analysis for the general system (1.7). For $Q \in L^\infty(\Gamma, \mathbb{C}^{3x3})$ we prove that the system is well-posed using semigroup theory. On the other hand using the multiplier method (see [18]) and a technique inspired from [5, 13, 32] to absorb a zero order boundary term, we show that the system is exponential stable is $Q = \alpha I$ for some scalar continuously differentiable function $\alpha$ such that $\nabla \alpha$ is small enough.

The paper is organized as follows. The second section deals with the well-posedness of the problem. In the last section we give the main result of this paper which is the exponential stability of the piezoelectric system and its proof.

## 2 Well-posedness of the problem

We start this section with the well-posedness of problem (1.7). At the end we will check the dissipativeness of (1.7).

Let us introduce the Hilbert spaces (see e.g. [25, 30, 1])

\begin{equation}
J(\Omega, \varepsilon) = \{ E \in L^2(\Omega)^3 \text{div}(\varepsilon E) = 0 \text{ in } \Omega \},
\end{equation}

\begin{equation}
\mathcal{H} = H^1(\Omega)^3 \times L^2(\Omega)^3 \times L^2(\Omega)^3 \times J(\Omega, \mu),
\end{equation}

equipped with the norm induced by the inner product

\[
\langle E, E' \rangle_\varepsilon = \int_\Omega E(x) \cdot E'(x) \, dx, \forall E, E' \in J(\Omega, \varepsilon),
\]

\[
\langle (u, v, E, H), (u', v', E', H') \rangle_\mathcal{H} = \langle u, u' \rangle_1 + \langle v, v' \rangle_0 + \langle E, E' \rangle_\varepsilon + \langle H, H' \rangle_\mu, \forall (u, v, E, H), (u', v', E', H') \in \mathcal{H},
\]

where we have set

\[
\langle u, u' \rangle_0 = \int_\Omega u(x) \cdot u'(x) \, dx,
\]

\[
\langle u, u' \rangle_1 = \int_\Omega \sigma(u)(x) : \gamma(u')(x) \, dx + A \int_\Gamma u(x) \cdot u'(x) \, d\sigma,
\]

with the notation

\[
\sigma(v) : \gamma(v') := \sigma_{ij}(v)\gamma_{ij}(v').
\]

Now define the linear operator $A$ from $\mathcal{H}$ into itself as follows:

\begin{equation}
D(A) = \{ (u, v, E, H) \in \mathcal{H} | \nabla \sigma(u, E), \text{curl} E, \text{curl} H \in L^2(\Omega)^3; v \in H^1(\Omega)^3; E \times \nu, H \times \nu \in L^2(\Gamma)^3 \text{ satisfying }
\]

\begin{equation}
E \times \nu - (Q^* v) \times \nu + E \times \nu \times \nu = 0 \text{ on } \Gamma,
\end{equation}

\begin{equation}
\sigma(u, E) \cdot \nu + A u + v + Q(E \times \nu) = 0 \text{ on } \Gamma \}.
\end{equation}

For all $(u, v, E, H) \in D(A)$ we take

\[
A(u, v, E, H) = (v, \nabla \sigma(u, E), \epsilon^{-1}(\text{curl} H - e\gamma(v)), -\mu^{-1} \text{curl} E).
\]
The boundary conditions (2.11) and (2.12) are meaningful since for \((u, v, E, H) \in D(A)\), from section 2 of [2] the property \(\nabla \sigma(u, E) \in L^2(\Omega)^3\) implies that \(\sigma(u, E) \cdot \nu\) belongs to \(H^{-1/2}(\Gamma)^3\). Since the properties \(u, v \in H^1(\Omega)^3\) imply that \(Au + v\) belongs to \(H^{3/2}(\Gamma)^3\), the boundary condition (2.12) has a meaning (in \(H^{-1/2}(\Gamma)^3\)) and furthermore yields \(\sigma(u, E) \cdot \nu \in L^2(\Gamma)^3\) (because \(Q(E \times \nu) \in L^2(\Gamma)^3\)). Similarly the properties of \(H\) and \(v\) give meaning to the boundary condition (2.11) (as an equality in \(L^2(\Gamma)^3\)). In summary both boundary conditions (2.11) and (2.12) have to be understood as an equality in \(L^2(\Gamma)^3\).

We now see that formally problem (1.7) is equivalent to

\[
\begin{align*}
\frac{\partial U}{\partial t} &= AU, \\
U(0) &= U_0,
\end{align*}
\]

when \(U = (u, \partial_t u, E, H)\) and \(U_0 = (u_0, u_1, E_0, H_0)\).

We shall prove that this problem (2.13) has a unique solution using semigroup theory by showing that \(A\) is a maximal dissipative operator.

**Lemma 2.1** \(A\) is a maximal dissipative operator.

**Proof:** We start with the dissipativeness:

\[
(AU, U)_H \leq 0, \forall U \in D(A).
\]

From the definition of \(A\) and the inner product in \(H\), we have

\[
(\begin{align*}
(AU, U)_H &= (v, u)_1 + (\nabla \sigma(u, E), v)_0 \\
&+ \int_\Omega \{E \cdot (\text{curl}H - e\gamma(v)) - \text{curl}E \cdot H\} \, dx,
\end{align*}
\]

for any \((u, v, E, H) \in D(A)\). Lemma 2.2 of [31] and Green’s formula yield equivalently

\[
(\begin{align*}
(AU, U)_H &= (v, u)_1 - \int_\Omega \sigma(u, E) : \gamma(v) \, dx \\
&- \int_\Omega e\gamma(v) \cdot E \, dx \\
&+ \int_\Gamma \{[(\sigma(u, E) \cdot \nu) \cdot v + (E \times \nu) \cdot H] \, d\sigma,
\end{align*}
\]

for any \((u, v, E, H) \in D(A)\). Using the definition of the inner product \((\cdot, \cdot)_1\) and the boundary conditions (2.11) and (2.12), we arrive at

\[
(\begin{align*}
(AU, U)_H &= -\int_\Gamma \{|v|^2 + |E \times \nu|^2\} \, d\sigma \leq 0,
\end{align*}
\]

for any \((u, v, E, H) \in D(A)\).

Let us now pass to the maximality. This means that for at least one non negative real number \(\lambda\), \(\lambda I - A\) has to be surjective. Let us show that indeed \(I - A\) is surjective. This means that for all \((f, g, F, G)\) in \(H\), we are looking \((u, v, E, H) \in D(A)\) such that

\[
(2.14) \quad (I - A) (u, v, E, H) = (f, g, F, G).
\]

From the definition of \(A\), this equivalently means

\[
(2.15) \quad \begin{cases}
\begin{align*}
u &= f, \\
v - \nabla \sigma(u, E) &= g, \\
E - \varepsilon^{-1}(\text{curl}H - e\gamma(v)) &= F, \\
H + \mu^{-1}\text{curl}E &= G.
\end{align*}
\end{cases}
\]
The first and fourth equations allow to eliminate $H$ and $v$, since they are respectively equivalent to
\begin{align}
(2.16) & \quad v = u - f, \\
(2.17) & \quad H = G - \mu^{-1} \text{curl} E.
\end{align}
Substituting these expressions in the second and third equations yields formally
\begin{align}
(2.18) & \quad u - \nabla \sigma(u, E) = f + g, \\
(2.19) & \quad \varepsilon E + \text{curl}(\mu^{-1} \text{curl} E) + e\gamma(u) = \varepsilon F + \text{curl} G + e\gamma(f).
\end{align}
This system in $(u, E)$ will be uniquely defined by adding boundary conditions on $u$ and $E$. Indeed using the identities $(2.16)$ and $(2.17)$, we see that $(2.11)$ and $(2.12)$ are formally equivalent to
\begin{align}
(2.20) & \quad -\mu^{-1} \text{curl} E \times \nu + Q^* u \times \nu + (E \times \nu) \times \nu = -G \times \nu + Q^* f \times \nu \text{ on } \Gamma, \\
(2.21) & \quad \sigma(u, E) \cdot \nu + Au + u + Q(E \times \nu) = f \text{ on } \Gamma.
\end{align}
By formal integration by parts we remark that the variational formulation of the system $(2.18)$-$(2.19)$ with the boundary conditions $(2.20)$-$(2.21)$ is the following one: Find $(u, E) \in V$ such that
\begin{align}
(2.22) & \quad a((u, E), (u', E')) = F(u', E'), \quad \forall (u', E') \in V,
\end{align}
where the Hilbert space $V$ is given by $V = H^1(\Omega)^3 \times W$ when $W$ is defined by
\begin{align*}
W = \{ E \in L^2(\Omega)^3 | \text{curl} E \in L^2(\Omega)^3 \text{ and } E \times \nu \in L^2(\Gamma)^3 \},
\end{align*}
with the norm
\begin{align*}
||E||_W^2 &= \int_\Omega (|E|^2 + |\text{curl} E|^2) dx + \int_\Gamma |E \times \nu|^2 d\sigma,
\end{align*}
the form $a$ is defined by
\begin{align*}
a((u, E), (u', E')) &= \int_\Omega \{ \sigma(u, E) : \gamma(u') + u \cdot u' \} dx \\
&\quad + \int_\Omega \{ \mu^{-1} \text{curl} E \cdot \text{curl} E' + \varepsilon E \cdot E' + e\gamma(u) \cdot E' \} dx \\
&\quad + \int_\Gamma \{ (E \times \nu) \cdot (E' \times \nu) + (A + 1) u \cdot u' + Q(E \times \nu) \cdot u' - Q^* u \cdot (E' \times \nu) \} d\sigma,
\end{align*}
and finally the form $F$ is defined by
\begin{align*}
F(u', E') &= \int_\Omega \{ (f + g) \cdot u' + (\varepsilon F + e\gamma(f)) \cdot E' + G \cdot \text{curl} E' \} dx + \int_\Gamma (f \cdot u' - (Q^* f \times \nu) \cdot E') d\sigma.
\end{align*}
We easily see that the bilinear form $a$ is coercive on $V$ since
\begin{align*}
a((u, E), (u, E)) &= \int_\Omega \{ \sigma(u) : \gamma(u) + |u|^2 \} dx \\
&\quad + \int_\Omega \{ \mu^{-1} |\text{curl} E|^2 + \varepsilon |E|^2 \} dx \\
&\quad + \int_\Gamma \{ |E \times \nu|^2 + (A + 1) |u|^2 \} d\sigma,
\end{align*}
which is clearly greater than $||u||_{H^1(\Omega)^3}^2 + ||E||_W^2$ by the ellipticity assumption on the elasticity tensor. Hence by the Lax-Milgram lemma, problem $(2.22)$ has a unique solution $(u, E) \in V$.  
\[\text{5}\]
To end our proof we need to show that the solution \((u,E)\) in \(V\) of (2.22) and \(v, H\) given respectively by (2.16), (2.17) are such that \((u,v,E,H)\) belongs to \(D(A)\) and satisfies (2.14) (or equivalently (2.15)). First taking test functions \(u'\) in \(D(\Omega)^3\) and \(E' = 0\), we get

\[
\nabla \sigma (u,E) + v = g \text{ in } D'(\Omega).
\]

This implies the second identity in (2.15) as well as the regularity \(\nabla \sigma (u,E) \in L^2(\Omega)^3\) (from the fact that \(v, \text{curl} E\) as well as \(g\) belongs to that space).

Second we take test functions \(u' = 0\) and \(E' = \chi\) with \(\chi \in D(\Omega)^3\) by Lemma 2.3 of [31] we get

\[
\varepsilon E - \text{curl} H + e\gamma(u) = \varepsilon F \text{ in } D'(\Omega).
\]

This means that the third identity in (2.15) holds as well as the regularity \(\text{curl} H \in L^2(\Omega)^3\).

Thirdly taking test functions \(v' \in H^1(\Omega)^3\) and \(E' = \chi\) with \(\chi \in C^\infty(\Omega)^3\) and applying Green’s formula (see section 2 of [2] and Lemma 2.2 of [31]), we get

\[
\langle \sigma (u,E) \cdot \nu, v' \rangle - \int_\Gamma (H \times \nu) \cdot E' \, d\sigma + \int_\Gamma (Q (E \times \nu) \cdot u' - (Q^* u \times \nu) \cdot E') \, d\sigma + \int_\Gamma \{ (E \times \nu) \cdot (E' \times \nu) + (A + 1) u \cdot u' \} \, d\sigma = 0.
\]

This leads to the boundary conditions (2.11) and (2.12) since \(u'\) (resp. \(\chi\)) was arbitrary in \(H^1(\Omega)^3\) (resp. in \(C^\infty(\Omega)^3\)) whose trace belongs to a dense subspace of \(L^2(\Gamma)^3\).

Finally from (2.17) and the fact that \(\mu G\) is divergence free, \(\mu H\) is also divergence free.

Semigroup theory [33, 36] allows to conclude the following existence results:

**Corollary 2.2** For all \((u_0, u_1, E_0, H_0) \in H\), the problem (1.7) admits a unique (weak) solution \((u,E,H)\) satisfying \((u, \partial_t u, E, H) \in C(\mathbb{R}_+; H)\), or equivalently \(u \in C^1(\mathbb{R}_+, L^2(\Omega)^3) \cap C(\mathbb{R}_+, H^1(\Omega)^3)\), \(E \in C(\mathbb{R}_+, L^2(\Omega)^3)\) and \(H \in C(\mathbb{R}_+, J(\Omega, \mu))\). If moreover \((u_0, u_1, E_0, H_0)\) belongs to \(D(A)\) and satisfies

\[
div (e\gamma(u_0) + \varepsilon E_0) = 0 \text{ in } \Omega,
\]

then the problem (1.7) admits a unique (strong) solution \((u,E,H)\) satisfying \((u, \partial_t u, E, H) \in C^1(\mathbb{R}_+, H) \cap C(\mathbb{R}_+, D(A))\), or equivalently satisfying \(u \in C^2(\mathbb{R}_+, L^2(\Omega)^3) \cap C^1(\mathbb{R}_+, H^1(\Omega)^3)\), \(E \in C^1(\mathbb{R}_+, J(\Omega, \varepsilon)) \cap C(\mathbb{R}_+, W)\), \(H \in C^1(\mathbb{R}_+, J(\Omega, \mu)) \cap C(\mathbb{R}_+, W)\), satisfying (2.11)-(2.12) for a.e. \(t\) (with \(v = u_1\), as well as

\[
\nabla \sigma (u,E) \in C(\mathbb{R}_+, L^2(\Omega)^3).
\]

Note that, in that last case, \(D = e\gamma(u) + \varepsilon E\) satisfies in particular

\[
div D = 0 \text{ in } \Omega \times \mathbb{R}_+.
\]

We finish this section by showing the dissipativeness of our system.

**Lemma 2.3** The energy

\[
E(t) = \frac{1}{2} \int_\Omega |(\partial_t u(x,t)|^2 + \sigma(u(x,t) : \gamma(u)(x,t)) \, dx + \frac{A}{2} \int_\Gamma |u(x,t)|^2 \, d\sigma(x)
+ \frac{1}{2} \int_\Omega (\varepsilon(x)|E(x,t)|^2 + \mu(x)|H(x,t)|^2) \, dx
\]
is non-increasing. Moreover for \((u_0, u_1, E_0, H_0) \in D(A)\), we have for all \(0 \leq S < T < \infty\)

\begin{equation}
E(S) - E(T) = \int_S^T \int_{\Gamma} \{|E(x, t) \times \nu|^2 + |u'(x, t)|^2\} \, d\sigma dt,
\end{equation}

and for all \(t \geq 0\)

\begin{equation}
E'(t) = -\int_{\Gamma} \{|E(x, t) \times \nu|^2 + |u'(x, t)|^2\} \, d\sigma.
\end{equation}

**Proof:** Since \(D(A)\) is dense in \(H\) it suffices to show (2.25). For \((u_0, u_1, E_0, H_0) \in D(A)\), from the regularity of \(u, E, H\), we have

\[
E'(t) = \int_{\Omega} \left\{ \partial_t^2 u \cdot \partial_t u + \sigma(u) : \gamma(\partial_t u) \right\} dx + A \int_{\Gamma} \partial_t u \cdot u d\sigma \\
+ \int_{\Omega} \left\{ \varepsilon E \cdot \partial_t E + \mu H \cdot \partial_t H \right\} dx.
\]

By (1.7), we get

\[
E'(t) = \int_{\Omega} \left\{ \partial_t u \cdot \nabla \sigma(u, E) + \sigma(u) : \gamma(\partial_t u) \right\} dx + A \int_{\Gamma} \partial_t u \cdot u d\sigma \\
+ \int_{\Omega} \left\{ E \cdot (\text{curl}H - E \gamma(\partial_t u)) \right\} dx \\
= (A(u(t), \partial_t u(t), E(t), H(t)), (u(t), \partial_t u(t), E(t), H(t)))_{\mathcal{H}}.
\]

We conclude by Lemma 2.1.

**3 Exponential stability**

In this section we prove the main result of this paper, namely the exponential stability of our system (1.7) when \(\Omega\) is strictly star-shaped with respect to a point \(x_0\). This result is based on an identity with multipliers proved in [18] that allows to show the next observability estimate.

**Theorem 3.1** Assume that there exists \(x_0 \in \mathbb{R}^n\) and \(\delta > 0\) such that

\begin{equation}
m(x) \cdot \nu(x) \geq \delta \quad \forall x \in \partial \Omega,
\end{equation}

where \(m(x) = x - x_0\). Assume also that \(Q = \alpha I\) with a continuously differentiable function \(\alpha\) from \(\Gamma\) to \(\mathbb{C}\). Set \(c_{\alpha} = \max_{\Omega} |\nabla \alpha|\). Let \((u, E, H)\) be the strong solution of problem (1.7).

Then there exist two positive constants \(C, C'\) (independent of \(\alpha\)) such that for all \(T > 0\), and all \(\theta\), there exists a constant \(C(\theta)\) (independent of \(T\)) such that the next observability estimate holds:

\begin{equation}
TE(T) \leq C(\theta)(1 + c_{\alpha}T + \theta T)E(0) + C \int_{\Sigma_T} (|u|^2 + |E|^2).
\end{equation}

**Proof:** First the identity (3.9) of [18] with \(t_0 = 0\) and \(\varphi(x) = |x - x_0|^2/2\) yields

\begin{equation}
TE(T) = r + \int_{\Sigma_T} V(x, t) \, d\sigma(x) dt,
\end{equation}

where
where \( \Sigma_T = \Gamma \times (0, T) \),

\[
\begin{align*}
    r &= -2 \int_\Omega u_t \cdot \{ u + (m \cdot \nabla)u \} + \mu(m \times H) \cdot \{ \varepsilon E + e\gamma(u) \}^T, \\
    V &= 2\{ u_t + (m \cdot \nabla)u + u \} \cdot \sigma(u, E)\nu + m \cdot \nu \{ |u_t|^2 - \sigma(u) : \gamma(u) + \varepsilon|E|^2 + \mu|H|^2 \} \\
    &+ 2t(H \times E) \cdot \nu - 2\varepsilon E \cdot \nu E \cdot m - 2\mu H \cdot \nu H \cdot m \\
    &- 2(m \times e\gamma(u)) \cdot (E \times \nu).
\end{align*}
\]

Using the boundary conditions from (1.7), we see that

\[
\begin{align*}
    \Delta &= 2\{ (m \cdot \nabla)u + u \} \cdot \sigma(u, E)\nu + m \cdot \nu \{ |u_t|^2 - \sigma(u) : \gamma(u) \}. \\
\end{align*}
\]

By Young’s inequality, there exists \( C > 0 \) such that for all \( \beta_1, \beta_2 > 0 \)

\[
\begin{align*}
    V &\leq -2At u_t \\
    &- (m \cdot \nu - \beta_2)(\varepsilon|E_v|^2 + \mu|H_v|^2) \\
    &+ \Delta + C(1 + \frac{1}{\beta_2} + \frac{1}{\beta_1})|E \times \nu|^2 + C(1 + \frac{1}{\beta_2})|H \times \nu|^2 + \beta_1 \gamma(u) : \gamma(u).
\end{align*}
\]

By using again the first boundary condition from (1.7), we get for all \( \beta_1, \beta_2 > 0 \)

\[
\begin{align*}
    (3.29) \quad V &\leq -2At u_t + C(1 + \frac{1}{\beta_2})|u_t|^2 + C(1 + \frac{1}{\beta_2} + \frac{1}{\beta_1})|E \times \nu|^2 \\
    &- (m \cdot \nu - \beta_2)(\varepsilon|E_v|^2 + \mu|H_v|^2) + \Delta + \beta_1 \gamma(u) : \gamma(u).
\end{align*}
\]

Let us transform the first term of this right-hand side:

\[
-2A \int_{\Sigma_T} t u u_t = -A \int_{\Sigma_T} t \frac{d}{dt} u^2,
\]

and by an integration by parts in time, we get

\[
-2A \int_{\Sigma_T} t u u_t = A \int_{\Sigma_T} u^2 - A \int_\Omega u^2 |t|^T.
\]

This proves that

\[
(3.30) \quad -2A \int_{\Sigma_T} t u u_t \leq A \int_{\Sigma_T} u^2.
\]

Let us now estimate the term \( \Delta \). First using the second boundary condition from (1.7), we see that

\[
\Delta = -2\{ (m \cdot \nabla)u + u \} \cdot (Q(E \times \nu) + Au + u_t) + m \cdot \nu \{ |u_t|^2 - \sigma(u) : \gamma(u) \}.
\]

Using the ellipticity assumption (1.6) and the condition (3.26) we obtain

\[
(3.31) \Delta \leq -2\{ (m \cdot \nabla)u + u \} \cdot (Q(E \times \nu) + Au + u_t) + m \cdot \nu |u_t|^2 - \alpha_0 \delta \gamma(u) : \gamma(u) \\
\leq -2u \cdot Q(E \times \nu) - 2A|u|^2 - 2u \cdot u_t - 2(m \cdot \nabla)u \cdot Q(E \times \nu) \\
- 2A(m \cdot \nabla)u \cdot u - 2(m \cdot \nabla)u \cdot u_t + m \cdot \nu |u_t|^2 - \alpha_0 \delta \gamma(u) : \gamma(u).
\]
We need to estimate some terms of this right-hand side. First as before an integration by parts in time yields

\[-2 \int_{\Sigma_T} uu_t \leq A \int_\Gamma u^2(\cdot, t = 0) \leq 2E(0).\]

As in [3, 13], one can show that

\[(3.32) \quad -2A \int_{\Sigma_T} (m \cdot \nabla)u \cdot u d\sigma dt \leq \frac{C}{\theta_1} \int_{\Sigma_T} |u|^2 + \theta_1 \int_{\Sigma_T} \gamma(u) : \gamma(u) d\sigma dt,
\]

as well as

\[(3.33) \quad \int_{\Sigma_T} (m \cdot \nabla)u \cdot u_d\sigma dt \leq CE(0) + \frac{C}{\theta_2} \int_{\Sigma_T} (|u|^2 + |u|_1^2) + \theta_2 \int_{\Sigma_T} \gamma(u) : \gamma(u) d\sigma dt, \forall \theta_1, \theta_2 > 0.
\]

By Young’s inequality we clearly have

\[(3.34) \quad \int_{\Sigma_T} u \cdot Q(E \times \nu) \leq C \int_{\Sigma_T} (|u|^2 + |E|^2).
\]

Now we notice that

\[(m \cdot \nabla)u \cdot Q(E \times \nu) = (Q^*(m \cdot \nabla)u) \cdot (E \times \nu),\]

and for any $k = 1, 2, 3$, we may write

\[(Q^*(m \cdot \nabla)u)_k = Q^*_k m \cdot \partial_i u_j = 2Q^*_k m \gamma_{ij}(u) - Q^*_k m_i \partial_j u_i = 2Q^*_k m_i \gamma_{ij}(u) + Q^*_k u_i \partial_j m_i - Q^*_k \partial_j (m_i u_i).
\]

The two first terms of this right-hand side will be estimated by Young’s inequality and it therefore remains to estimate the last term, namely by the previous identities we have

\[(3.35) \quad \int_{\Sigma_T} (m \cdot \nabla)u \cdot Q(E \times \nu) \leq \theta_3 \int_{\Sigma_T} \gamma(u) : \gamma(u) d\sigma dt
\]

\[+ C \int_{\Sigma_T} (|u|^2 + (1 + \frac{1}{\theta_3}))|E|^2 - \int_{\Sigma_T} (Q^* \nabla (m \cdot u)) \cdot (E \times \nu), \forall \theta_3 > 0.
\]

Now using Green’s formula, we see that

\[\int_{\Sigma_T} (Q^* \nabla (m \cdot u)) \cdot (E \times \nu) = \int_{Q_T} \{\mathrm{curl}(Q^* \nabla (m \cdot u)) \cdot E - Q^* \nabla (m \cdot u) \cdot \mathrm{curl} E\}.
\]

Now using the fact that $Q(x) = \alpha(x) I$, and that $\mathrm{curl} E = \mu H_t$, we obtain

\[\int_{\Sigma_T} (Q^* \nabla (m \cdot u)) \cdot (E \times \nu) = \int_{Q_T} \{(\nabla \alpha \times \nabla (m \cdot u)) \cdot E - Q^* \nabla (m \cdot u) \cdot \mu H_t\}.
\]

For this last term, we first integrate by parts in time and get

\[\int_{Q_T} Q^* \nabla (m \cdot u) \cdot \mu H_t = - \int_{Q_T} Q^* \nabla (m \cdot u_t) \cdot \mu H + \int_{\Omega} Q^* \nabla (m \cdot u) \cdot \mu H|_0^T.
\]

An integration by parts in space leads to

\[\int_{Q_T} Q^* \nabla (m \cdot u) \cdot \mu H_t = \int_{Q_T} (Q^* m \cdot u_t \varphi(m H) + m \cdot u_t \nabla \alpha \cdot (\mu H))
\]

\[\quad - \int_{\Sigma_T} Q^* m \cdot u_t (\mu H) \cdot \nu + \int_{\Omega} Q^* \nabla (m \cdot u) \cdot \mu H|_0^T.
\]
These two identities and reminding that \( \text{div}(\mu H) = 0 \) lead to
\[
\int_{\Sigma_T} (Q^* \nabla (m \cdot u)) \cdot (E \times \nu) = \int_{\Sigma_T} \left\{ (\nabla \alpha \times \nabla (m \cdot u)) \cdot E - m \cdot u_t \nabla \alpha \cdot (\mu H) \right\} + \int_{\Sigma_T} Q^* m \cdot u_t (\mu H) \cdot \nu - \int_{\Omega} Q^* \nabla (m \cdot u) \cdot \mu H \bigg|_0^T.
\]
By Young’s inequality we find that
\[
\int_{\Sigma_T} (Q^* \nabla (m \cdot u)) \cdot (E \times \nu) \leq C(1 + c_0 T) E(0) + \int_{\Sigma_T} \left\{ \frac{C}{\theta_4} |u_t|^2 + \theta_4 |H_\nu|^2 \right\}, \forall \theta_4 > 0.
\]
This last estimate in (3.35) leads to
\[
(3.36) \quad \int_{\Sigma_T} (m \cdot \nabla) u \cdot Q(E \times \nu) \leq C(1 + c_0 T) E(0) + \theta_3 \int_{\Sigma_T} \gamma(u) : \gamma(u) d\sigma dt + C \int_{\Sigma_T} \left\{ (1 + \frac{1}{\theta_2}) |u_t|^2 + (1 + \frac{1}{\theta_3}) |E_\tau|^2 \right\} + \theta_4 |H_\nu|^2 \}
+ C \left\{ (1 + \frac{1}{\theta_1}) + \frac{1}{\theta_2} \right\} \int_{\Sigma_T} |u|^2, \forall \theta_1, \theta_2, \theta_3, \theta_4 > 0,
\]
This estimate in (3.29) and using (3.30), we get finally
\[
\int_{\Sigma_T} V \leq C(1 + c_0 T) E(0) + (-\alpha_0 \delta + \beta_1 + \theta_1 + \theta_2 + \theta_3) \int_{\Sigma_T} \gamma(u) : \gamma(u) d\sigma dt + C \int_{\Sigma_T} \left\{ (1 + \frac{1}{\theta_2}) + \frac{1}{\theta_3} \right\} |u_t|^2
+ C \int_{\Sigma_T} \left\{ (1 + \frac{1}{\beta_1}) + \frac{1}{\beta_2} + \frac{1}{\beta_3} \right\} |E_\tau|^2
+ C \int_{\Sigma_T} \left\{ (-m \cdot \nu + \beta_2) \epsilon |E_\nu|^2 + ((-m \cdot \nu + \beta_2) \mu + \theta_4) |H_\nu|^2 \right\}
+ C \left\{ (1 + \frac{1}{\theta_1}) + \frac{1}{\theta_2} \right\} \int_{\Sigma_T} |u|^2, \forall \beta_1, \beta_2, \theta_1, \theta_2, \theta_3, \theta_4 > 0.
\]
By choosing \( \beta_1, \beta_2, \theta_1, \theta_2, \theta_3 \) and \( \theta_4 \) small enough, we have found that
\[
(3.37) \quad \int_{\Sigma_T} V \leq C(1 + c_0 T) E(0) + C \int_{\Sigma_T} (|u|^2 + |u_t|^2 + |E_\tau|^2).
\]
Coming back to (3.28) and using again Young’s and Korn’s inequalities to estimate \( \tau \), we obtain
\[
TE(T) \leq C(1 + c_0 T) E(0) + C \int_{\Sigma_T} (|u|^2 + |u_t|^2 + |E_\tau|^2).
\]
Now invoking Lemma 3.5 below, we arrive at
\[
TE(T) \leq C(\theta)(1 + c_0 T) E(0) + C \int_{\Sigma_T} (|u|^2 + |E_\tau|^2) + \theta \int_0^T E(t) dt \leq (C(\theta)(1 + c_0 T) + \theta T) E(0) + C \int_{\Sigma_T} (|u|^2 + |E_\tau|^2), \forall \theta > 0,
\]
reminding that the energy is non increasing. This is the requested estimate (3.27).
Remark 3.2 Note that the last term of the estimate (3.35) is zero if $Q^* = Q^*_N$, but according to Remark 1.1, this assumption is meaningless.

Theorem 3.3 Under the assumptions of the previous theorem and if $c_\alpha$ is small enough, there exist two positive constants $M$ and $\omega$ such that

\begin{equation}
E(t) \leq Me^{-\omega t}E(0),
\end{equation}

for all strong solution $(u, E, H)$ of (1.7).

Remark 3.4 By same method, we can obtain the same exponential stability result in the case where $\varepsilon, \mu$ are positive functions satisfy some regularity and technical conditions.

Proof: The estimate (3.27) and Lemma 2.3 yield

\[ TE(T) \leq (C(\theta)(1 + c_\alpha T) + \theta T)E(0) + C(E(0) - E(T)), \forall \theta > 0, \]

which may be equivalently written

\[ E(T) \leq \frac{C(\theta)(1 + c_\alpha T) + \theta T}{C + T} E(0), \forall \theta > 0. \]

Now we choose $\theta = \frac{1}{2}$ and $c_\alpha \leq \frac{1}{12\sqrt{3}}$, with this choice $\frac{C(\theta)(1 + c_\alpha T) + \theta T}{C + T}$ tends to $C(\frac{1}{2})c_\alpha + \frac{1}{2} \leq \frac{3}{4}$ as $T$ goes to infinity. Therefore for $T$ large enough, we have found $r \in (0, 1)$ such that

\[ E(T) \leq rE(0). \]

Since our system is invariant by translation, standard arguments about uniform stabilization of hyperbolic system (see for instance [35, 31]) yield the conclusion.

The key point in the above proof is to estimate appropriately the term $\int_{\Sigma_T} |u|^2 d\sigma dt$. This estimate is obtained as in [5, 13, 32] but the technique is strongly linked to the piezoelectric system we are analyzing.

Lemma 3.5 Let $(u, E, H)$ be a strong solution of (1.7). Then for all $\theta > 0$ there exists a constant $C(\theta) > 0$ (which does not depend on $T$ but depends on $\theta$, the domain and the coefficients $a_{ijkl}, \varepsilon, \mu, \varepsilon_{ijk}, A$) such that

\begin{equation}
\int_{\Sigma_T} |u|^2 d\sigma dt \leq C(\theta)E(0) + \theta \int_0^T E(t)dt.
\end{equation}

Proof: Inspired from [5, 13, 32] for each $t \geq 0$ we consider the solution $(z, \chi)$ (depending on $t$) of

\begin{equation}
\begin{cases}
\nabla(\sigma(z) - \varepsilon^T \nabla \chi) = 0 \text{ in } \Omega, \\
\text{div}(\varepsilon \nabla \chi + \varepsilon \gamma(z)) = 0 \text{ in } \Omega, \\
z = u, \chi = 0 \text{ on } \Gamma.
\end{cases}
\end{equation}

This solution is characterized by $z = w + u$ where $(w, \chi) \in \tilde{V} := H^1_0(\Omega)^3 \times H^1_0(\Omega)$ is the unique solution of

\begin{equation}
\tilde{a}((w, \chi), (w', \chi')) = a((u, 0), (w', \chi'), \forall (w', \chi') \in \tilde{V},
\end{equation}

where

\[ \tilde{a}((w, \chi), (w', \chi')) = \int_\Omega \left( (\sigma(w) - \varepsilon^T \nabla \chi) : \gamma(w') + (\varepsilon \nabla \chi + \varepsilon \gamma(w)) \cdot \nabla \chi' dx, \forall (w', \chi') \in V. \]
The above problem has a unique solution since the bilinear form $a$ is coercive on $V$.

A direct consequence of (3.41) is that
\[
\tilde{a}(z, \chi, (w', \chi')) = 0, \forall (w', \chi') \in \tilde{V}.
\]
By taking as test function $w' = w = z - u$ and $\chi' = \chi$, we find that
\[
\tilde{a}(z, \chi, (z, \chi)) = \tilde{a}(z, \chi, (u, 0)),
\]
which implies
\[
(3.42) \quad \int_{\Omega} \sigma(z) : \varepsilon(u) - e^T \nabla \chi : \gamma(u) \, dx = \tilde{a}(z, \chi, (z, \chi)) \geq 0.
\]

Note further that the coerciveness of $\tilde{a}$ leads to
\[
\|w\|_{1, \Omega} + \|\chi\|_{1, \Omega} \leq C \|u\|_{1, \Omega},
\]
and then to
\[
(3.43) \quad \|z\|_{1, \Omega} + \|\chi\|_{1, \Omega} \leq C \|u\|_{1, \Omega} \leq CE(t),
\]
where $\|u\|_{s, \Omega} = \|u\|_{H^s(\Omega)}$.

Now we consider the adjoint problem: Find $(w^*, \chi^*) \in \bar{V}$ solution of
\[
(3.44) \quad \begin{cases}
\nabla(\sigma(w^*) + e^T \nabla \chi^*) = z \text{ in } \Omega, \\
\text{div}(\varepsilon \nabla \chi^* - e\gamma(w^*)) = 0 \text{ in } \Omega, \\
z^* = 0, \chi^* = 0 \text{ on } \Gamma,
\end{cases}
\]
which is the unique solution of
\[
(3.45) \quad \tilde{a}^*((w^*, \chi^*), (w', \chi')) = \int_{\Omega} z \cdot w' \, dx, \forall (w', \chi') \in V,
\]
where
\[
\tilde{a}^*((w, \chi), (w', \chi')) = \int_{\Omega} \{(\sigma(w) + e^T \nabla \chi) : \gamma(w') + (\varepsilon \nabla \chi - e\gamma(w)) \cdot \nabla \chi' \} \, dx, \forall (w', \chi') \in V.
\]

Again this problem has a unique solution since the bilinear form $\tilde{a}^*$ is also coercive on $\bar{V}$. Since the system (3.44) is strongly elliptic (see Theorem 4.8 of [7]), we deduce that $(w^*, \chi^*)$ belongs to $H^2(\Omega)^3 \times H^2(\Omega)$ with the estimate
\[
(3.46) \quad \|w^*\|_{2, \Omega} + \|\chi^*\|_{2, \Omega} \leq C \|z\|_{0, \Omega},
\]
where here and below $C$ is a positive constant that depends only on $a_{ijkl}, \varepsilon, \mu, e_{ijkl}$ and on $\Omega$.

By using the differential equations from (3.44), we may write
\[
\int_{\Omega} |z|^2 \, dx = \int_{\Omega} \nabla(\sigma(w^*) + e^T \nabla \chi^*) \cdot z = \int_{\Omega} \{\nabla(\sigma(w^*) + e^T \nabla \chi^*) \cdot z + \text{div}(\varepsilon \nabla \chi^* - e\gamma(w^*))\} \, dx.
\]
Applying Green’s formula we get

\[
\int_{\Omega} |z|^2 \, dx = - \int_{\Omega} \{ (\sigma(w^*) + e^T \nabla \chi^*) : \gamma(z) + (\varepsilon \nabla \chi^* - e^T \gamma(w^*)) \cdot \nabla \chi \} \, dx
+ \int_{\Gamma} (\sigma(w^*) + e^T \nabla \chi^*) \nu \cdot z \, d\sigma
= - \int_{\Omega} \{ (\sigma(z) - e^T \nabla \chi) : \gamma(w^*) + (\varepsilon \nabla \chi + e^T \gamma(z)) \cdot \nabla \chi^* \} \, dx
+ \int_{\Gamma} (\sigma(w^*) + e^T \nabla \chi^*) \nu \cdot z \, d\sigma.
\]

Applying again Green’s formula and reminding problem (3.40), we have found that

\[
\int_{\Omega} |z|^2 \, dx = \int_{\Gamma} (\sigma(w^*) + e^T \nabla \chi^*) \nu \cdot u \, d\sigma.
\]

By Cauchy-Schwarz’s inequality and the estimate (3.46) (with the help of a trace theorem), we obtain finally

\[
(3.47) \quad \int_{\Omega} |z|^2 \, dx \leq C \int_{\Gamma} |u|^2 \, d\sigma \leq \frac{2C}{A} E(t).
\]

By deriving the system (3.40) in time, this estimate also shows that

\[
(3.48) \quad \int_{\Omega} |\partial_t z|^2 \, dx \leq C \int_{\Gamma} |\partial_t u|^2 \, d\sigma \leq -CE'(t),
\]

this last estimate coming from the identity (2.25).

Now we further remark that (see (3.40)) \( \varepsilon \nabla \chi + e^T \gamma(z) \) is divergence free in \( \Omega \), hence as \( \Omega \) is simply connected, we deduce that there exists \( \psi \in X_T(\Omega) \) such that

\[
(3.49) \quad \varepsilon \nabla \chi + e^T \gamma(z) = \text{curl}\psi,
\]

where (see Theorem I. 3.5 in [11])

\[
X_T(\Omega) = \{ \phi \in H^1(\Omega)^3 : \text{div} \phi = 0 \text{ in } \Omega, \text{ and } \phi \cdot \nu = 0 \text{ on } \Gamma \},
\]

with the estimate

\[
\|\psi\|_{1,\Omega} \leq C \|\varepsilon \nabla \chi + e^T \gamma(z)\|_{1,\Omega}.
\]

Thanks to (3.43), we get

\[
(3.50) \quad \|\psi\|_{1,\Omega} \leq C \|u\|_{1,\Omega} \leq CE(t)
\]

Let us finally consider the problem: find \( \tilde{\chi} \) solution of

\[
\begin{align*}
\text{curlcurl} \tilde{\chi} &= \psi \text{ in } \Omega, \\
\text{div} \tilde{\chi} &= 0 \text{ in } \Omega, \\
\tilde{\chi} \cdot \nu &= 0, \text{curl} \tilde{\chi} \times \nu &= 0 \text{ on } \Gamma.
\end{align*}
\]

The variational formulation of this problem is: find \( \tilde{\chi} \in X_T(\Omega) \) solution of

\[
(3.52) \quad b(\tilde{\chi}, \theta) = \int_{\Omega} \psi \cdot \theta \, dx, \forall \theta \in H_T(\Omega),
\]

\]
where
\[
b(\tilde{\chi}, \theta) = \int_\Omega \{ \operatorname{curl}\tilde{\chi} \operatorname{curl}\theta + \operatorname{div}\tilde{\chi} \operatorname{div}\theta \} \, dx, \quad \forall \tilde{\chi}, \theta \in H_T(\Omega),
\]
and
\[
H_T(\Omega) = \{ \phi \in H^1(\Omega)^3 : \phi \cdot \nu = 0 \text{ on } \Gamma \}.
\]
It is well known (see for instance [6]) that \( b \) is coercive on \( H_T(\Omega) \) and therefore problem (3.52) is well posed, its solution \( \tilde{\chi} \) furthermore satisfies (3.51) because \( \psi \) is divergence free.

Moreover as the system \( \operatorname{curl}\operatorname{curl} - \nabla \operatorname{div} = -\Delta \) is strongly elliptic and the boundary conditions in (3.51) cover this system, we get that \( \tilde{\chi} \) belongs to \( H^2(\Omega)^3 \) with (see for instance Theorem 4.8 of [7])

\[
\|\tilde{\chi}\|_{2,\Omega} \leq C\|\psi\|_{0,\Omega}.
\]

Now as before we can write by using Green’s formula and the identity (3.49)

\[
\|\psi\|^2_{0,\Omega} = \int_\Omega \psi \cdot \operatorname{curl}\operatorname{curl}\tilde{\chi} \, dx
\]
\[
= \int_\Omega \operatorname{curl}\psi \cdot \operatorname{curl}\tilde{\chi} \, dx
\]
\[
= \int_\Omega (\varepsilon \nabla \chi + e\gamma(z)) \cdot \operatorname{curl}\tilde{\chi} \, dx
\]
\[
= -\int_\Omega \nabla(e^T \operatorname{curl}\tilde{\chi}) \cdot z \, dx + \int_\Gamma (e^T \operatorname{curl}\tilde{\chi}) \nu \cdot z \, d\sigma.
\]

By the estimate (3.53) and reminding that \( z = u \) on \( \Gamma \), we obtain

\[
\|\psi\|_{0,\Omega} \leq C(\|z\|_{0,\Omega} + \|u\|_{0,\Gamma}).
\]

By the estimate (3.47), we arrive at

\[
\|\psi\|^2_{0,\Omega} \leq C\|u\|^2_{0,\Gamma} \leq \frac{2C}{A} E(t),
\]

as well as

\[
\|\psi_t\|^2_{0,\Omega} \leq C\|u_t\|^2_{0,\Gamma} \leq -CE'(t).
\]

Now multiplying the first identity of (1.7) by \( z \) and integrating on \( Q_T \) we get

\[
\int_{Q_T} z \cdot (\partial_t^2 u - \nabla \sigma(u, E)) \, dx \, dt = 0.
\]

By Green’s formula we obtain

\[
\int_{Q_T} (z \cdot \partial_t^2 u + \sigma(u, E) : \gamma(z)) \, dx \, dt - \int_{\Sigma_T} z \cdot \langle \sigma(u, E) \cdot \nu \rangle \, d\sigma \, dt = 0.
\]

Using the second boundary condition in (1.7) and the boundary condition in (3.40), we obtain

\[
A \int_{\Sigma_T} |u|^2 \, d\sigma \, dt = -\int_{\Sigma_T} u \cdot \partial_t u \, d\sigma \, dt - \int_{Q_T} (z \cdot \partial_t^2 u + \sigma(u, E) : \gamma(z)) \, dx \, dt.
\]

Owing to (3.42) we arrive at

\[
A \int_{\Sigma_T} |u|^2 \, d\sigma \, dt \leq 14.
\]
\[-\int_{\Sigma_{T}} u \cdot (u_{t} + Q(E \times \nu))\, d\sigma dt - \int_{Q_{T}} \left( z \cdot \partial_{t}^{2} u + e^{T} \nabla \chi : \gamma(u) - e^{T} E : \gamma(z) \right) \, dx dt.\]

By using the identity \(e_{\gamma}(u) = D - \varepsilon E\), we get

\[(3.56) \quad A \int_{\Sigma_{T}} |u|^{2}\, d\sigma dt \leq - \int_{\Sigma_{T}} u \cdot (u_{t} + Q(E \times \nu))\, d\sigma dt - \int_{Q_{T}} \left( z \cdot \partial_{t}^{2} u + \nabla \chi \cdot D - E \cdot (e_{\gamma}(z) + \varepsilon \nabla \chi) \right) \, dx dt.\]

We now transform the two last terms of this identity, first by Green’s formula in space, we see that

\[\int_{Q_{T}} \nabla \chi \cdot D \, dx dt = - \int_{Q_{T}} \chi \text{div} D - d(x dt) + \int_{\Sigma_{T}} \chi D \cdot \nu \, d\sigma dt = 0,\]

since \(D\) is divergence free and \(\chi = 0\) on \(\Gamma\). On the other hand, by the identity (3.49) we have

\[\int_{Q_{T}} E \cdot (e_{\gamma}(z) + \varepsilon \nabla \chi) \, dx dt = \int_{Q_{T}} E \cdot \text{curl} \psi \, dx dt,\]

and by Green’s formula in space

\[\int_{Q_{T}} E \cdot (e_{\gamma}(z) + \varepsilon \nabla \chi) \, dx dt = \int_{Q_{T}} \text{curl} E \cdot \psi \, dx dt + \int_{\Sigma_{T}} (E \times \nu) \cdot \psi \, d\sigma dt.\]

Now reminding that \(\mu H_{t} = \text{curl} E\) and using an integration by parts in time, we arrive at

\[\int_{Q_{T}} E \cdot (e_{\gamma}(z) + \varepsilon \nabla \chi) \, dx dt = \int_{Q_{T}} \mu H \cdot \psi_{t} \, dx dt + \int_{\Omega} \mu H \cdot \psi \, dx^{T}_{0} + \int_{\Sigma_{T}} (E \times \nu) \cdot \psi \, d\sigma dt.\]

In the same manner an integration by parts in time yields

\[\int_{Q_{T}} z \cdot \partial_{t}^{2} u \, dx dt = - \int_{Q_{T}} z_{t} \cdot u_{t} \, dx dt + \int_{\Omega} z \cdot u_{t} \, dx^{T}_{0}\]

These identities in (3.56) lead to

\[(3.57) \quad A \int_{\Sigma_{T}} |u|^{2}\, d\sigma dt \leq - \int_{\Sigma_{T}} u \cdot (u_{t} + Q(E \times \nu)) + (E \times \nu) \cdot \psi \, d\sigma dt + \int_{Q_{T}} (z_{t} u_{t} + \mu H \cdot \psi_{t}) \, dx dt - \int_{\Omega} z \cdot u_{t} \, dx^{T}_{0} + \int_{\Omega} \mu H \cdot \psi \, dx^{T}_{0}.\]

It remains to estimate each term of this right-hand side. For the first term applying successively Cauchy-Schwarz’s inequality, Young’s inequality and the identity (2.25) we may write

\[
\left| \int_{\Sigma_{T}} u \cdot (u_{t} + Q(E \times \nu))\, d\sigma dt \right| \leq \frac{A}{2} \int_{\Sigma_{T}} |u|^{2}\, d\sigma dt + \frac{C}{2A} \int_{\Sigma_{T}} (|u|^{2} + |E \times \nu|^{2})\, d\sigma dt
\]

\[
\leq \frac{A}{2} \int_{\Sigma_{T}} |u|^{2}\, d\sigma dt + \frac{C}{2A} \int^{T}_{0} E'(t)\, dt.
\]

Since the energy is non-negative, we arrive at

\[(3.58) \quad \left| \int_{\Sigma_{T}} u \cdot \partial_{t} u \, d\sigma dt \right| \leq \frac{A}{2} \int_{\Sigma_{T}} |u|^{2}\, d\sigma dt + \frac{C}{2A} E(0).\]
For the second term by using Cauchy-Schwarz’s inequality, Young’s inequality, a trace theorem, the estimate (3.50) and again the identity (2.25)

$$
\left| \int_{\Sigma_T} (E \times \nu) \cdot \psi \, d\sigma \, dt \right| \leq \theta \int_0^T \| \psi \|_{L^2(\Omega)}^2 \, dt + \frac{C}{\theta} \int_{\Sigma_T} |E \times \nu|^2 \, d\sigma \, dt
\leq \theta \int_0^T E(t) \, dt + \frac{C}{\theta} \int_{\Sigma_T} |E \times \nu|^2 \, d\sigma \, dt
\leq \theta \int_0^T E(t) \, dt - \frac{C}{\theta} \int_0^T E'(t) \, dt.
$$

As before the energy being non-negative, we arrive at

(3.59) \quad \left| \int_{\Sigma_T} (E \times \nu) \cdot \psi \, d\sigma \, dt \right| \leq \theta \int_0^T E(t) \, dt + \frac{C}{\theta} E(0).

For the third term we use successively Cauchy-Schwarz’s inequality, Young’s inequality, the estimate (3.48) and the definition of the energy to get for all $\theta > 0$

$$
\left| \int_{Q_T} z_t \cdot u_t \, dx \, dt \right| \leq \frac{1}{2\theta} \int_{Q_T} |z_t|^2 \, dx \, dt + \frac{\theta}{2} \int_{Q_T} |u_t|^2 \, dx \, dt
\leq -\frac{C}{2\theta} \int_0^T E'(t) \, dt + \theta \int_0^T E(t) \, dt.
$$

Again we get

(3.60) \quad \left| \int_{Q_T} z_t \cdot u_t \, dx \, dt \right| \leq \frac{C}{\theta} E(0) + \theta \int_0^T E(t) \, dt.

As for the third term replacing the estimate (3.48) by (3.55) we get for the fourth term

(3.61) \quad \left| \int_{Q_T} \mu H \cdot \psi_t \, dx \, dt \right| \leq \frac{C}{\theta} E(0) + \theta \int_0^T E(t) \, dt.

For the fifth term the application of Cauchy-Schwarz’s inequality, the estimate (3.47) and the definition of the energy directly

(3.62) \quad \left| \int_{\Omega} z \cdot u_t \, dx \right|_0^T \leq C(E(0) + E(T)) \leq 2CE(0)

since the energy is non-decreasing.

Similarly using (3.54) instead of (3.47), we have

(3.63) \quad \left| \int_{\Omega} \mu H \cdot \psi \, dx \right|_0^T \leq CE(0).

The estimates (3.58) to (3.63) into the estimate (3.57) yields the conclusion. \hfill \blacksquare
References


