Representations of bicircular matroids

Collette R. Coullard*

Department of Industrial Engineering and Management Sciences, Northwestern University, Evanston, IL 60208, USA

John G. del Greco

Department of Mathematical Sciences, Loyola University, Chicago, IL 60626, USA

Donald K. Wagner**

Mathematical Sciences Division, Office of Naval Research, Arlington, VA 22217, USA; and Department of Operations Research and Applied Statistics, George Mason University, Fairfax, VA 22030, USA

Received 3 June 1988
Revised 20 October 1989

Abstract


A bicircular matroid is a matroid defined on the edge set of a graph. Two different graphs can have the same bicircular matroid. The first result of this paper is a characterization of the collection of graphs having the same bicircular matroid as a given arbitrary graph. A bicircular matroid can be represented by a matrix over the real numbers that has at most two nonzeros per column. Such a matrix can be viewed as an incidence matrix of a graph. The second result of this paper is that given almost any (in a sense to be made precise) collection of graphs $G_1, \ldots, G_t$ having the same bicircular matroid $M$, there exist row-equivalent matrices $N_1, \ldots, N_t$ each representing $M$, such that $N_i$ is an incidence matrix of $G_i$ for $1 \leq i \leq t$. These results form the basis for an algorithm (presented in a subsequent paper) that under certain conditions converts a given linear-programming problem to a generalized-network flow problem.

* Research partially supported by ONR Grant N00014-86-K-0689 at Purdue University and by the Alexander von Humboldt Foundation while the author was on leave at the Institut für Operations Research, Universität Bonn, Bonn, West Germany.

** Research partially supported by NSF Grant ECS-8307796 and ONR Grant N00014-86-K-0689 at Purdue University and by a Nato Postdoctoral Fellowship from NSF while the author was on leave at the Institut für Operations Research, Universität Bonn, Bonn, West Germany.
1. Introduction

Let \( G = (V, E) \) be a graph and for \( I \subseteq E \) define \( G[I] \) to be the edge-induced subgraph of \( G \). The collection

\[
\{ I \subseteq E \mid \text{each component of } G[I] \text{ has at most one cycle} \}
\]

is the collection of independent sets of a matroid on \( E \). This matroid is called the bicircular matroid of \( G \) and is denoted by \( B(G) \). A matroid \( M \) is bicircular if there exists a graph \( G \) such that \( M = B(G) \). The graph \( G \) is a representation of \( M \).

Let \( G \) and \( G' \) be graphs on the same edge set. If \( B(G) = B(G') \), then what is the relationship between \( G \) and \( G' \)? They need not be equal as shown by the graphs in Fig. 1. Section 4 of this paper answers this question by showing there exists a small set of operations that when applied to \( G \) produce \( G' \). For example, in Fig. 1, \( G' \) is obtained from \( G \) by the operation “replace edge 1 by a loop at one of its ends”. This graph representation result is the first of the two main results of the paper.

Let \( A \) be a matrix having columns indexed on a set \( E \). Then \( M(A) \) denotes the matroid on \( E \) such that a subset of \( E \) is independent in \( M(A) \) if and only if the corresponding columns are linearly independent. A matroid \( M \) is matric if there exists a matrix \( A \) such that \( M = M(A) \). The matrix \( A \) is a representation of \( M(A) \).

Matrices \( A \) and \( A' \) are row-equivalent if there exists a nonsingular matrix \( T \) such that \( A' = TA \). Clearly, if \( A \) is a row-equivalent to \( A' \), then \( M(A) = M(A') \). The converse is in general not true.

As shown in the next section, every bicircular matroid has a matrix representation \( N \) such that \( N \) has either one or two nonzeros per column. Such a matrix is called a generalized-incidence matrix, for associated with such a matrix is a graph that is constructed as follows. Each column corresponds to an edge and each row to a vertex. The set of ends of an edge is given by the nonzero entries of the corresponding column. In Section 5, the above representation result is strengthened by showing that for almost any (in a sense to be made precise) collection of graphs \( G_1, \ldots, G_t \) having the same bicircular matroid \( M \), there exist row-equivalent matrices \( N_1, \ldots, N_t \) each representing \( M \) such that \( N_i \) is a generalized-incidence matrix of \( G_i \) for \( 1 \leq i \leq t \). This matrix representation result is the second main result of the paper.
The graph representation result and the matrix representation result form the theoretical basis of an algorithm that under certain conditions converts a linear-programming problem to a generalized-network flow problem. The algorithm will be the topic of a subsequent paper. This subsequent paper as well as the present paper are based on the Ph.D. dissertation of del Greco [3].

The outline of the paper is as follows. Section 2 describes a relationship between bicircular matroids and the class of linear-programming problems known as generalized-network flow problems. Section 3 contains the necessary background material. Section 4 contains the results on graph representations of bicircular matroids and Section 5 contains the results on matrix representations. Some related problems are remarked upon in Section 6.

2. A connection with linear programming

There exists a connection between bicircular matroids and linear programming. In particular, bicircular matroids are related to the constraint matrices of those linear-programming problems called generalized-network flow problems (Kennington and Helgason [6]) or flows-with-gain problems (Gondran and Minoux [5]). A generalized-network flow problem is a linear-programming problem the constraint matrix of which has either one or two nonzeros per column.

To make the above relationship clear, a more general class of matroids is introduced. Let D = (V, A) be a directed graph and let g(e) be a nonzero real number assigned to arc e, called the gain of e. A unicycle of D is a non-loop cycle such that the product of the gains of the forward arcs divided by that of the reverse arcs is 1. Zaslavsky [15] showed that the set \( \{I \subseteq A \mid D[I] \text{ has no unicycle and each component of } D[I] \text{ has at most one cycle} \} \) is the collection of independent sets of a matroid on A, called the gain matroid of \((D,g)\). A matroid M is a gain matroid if there exists a directed graph D and a function g such that M is the gain matroid of \((D,g)\). Zaslavsky [14] also observed the following.

**Proposition 2.1.** Bicircular matroids are gain matroids.

**Proof.** Let M be a bicircular matroid and let G be a graph such that \( M = B(G) \). Let D be the directed graph obtained by arbitrarily assigning directions to the edges of G. Where \( e_1, \ldots, e_m \) is an enumeration of the arcs of D, define \( g(e_i) = i \text{th prime number, for } 1 \leq i \leq m \). Then \((D,g)\) has no unicycles. \( \square \)

**Proposition 2.2.** Let \( D = (V, A) \) be a directed graph and let \( g : A \to \mathbb{R} \setminus \{0\} \). Then the gain matroid of \((D,g)\) has a matrix representation N over the reals such that N has either one or two nonzeros per column.

**Proof.** Define N to be a generalized-incidence matrix associated with D as follows:
Using straightforward linear algebra it can now be verified that the set of bases of $N$ coincides with the set of bases of the gain matroid of $(D,g)$. (This has been done in the context of generalized-network linear programming—see, e.g., Kennington and Helgason [6, Chapter 5].) □

The above two propositions imply that bicircular matroids are the matroids of the constraint matrices of generalized network flow problems having no unicycles.

Generalized-network flow models are important for at least two reasons. First, they have several applications, and second, they can be solved efficiently in practice; see Kennington and Helgason [6]. In addition, Goldberg, Plotkin and Tardos [4] have developed a polynomial-time combinatorial algorithm for a subclass of generalized-network flow problems.

Because of their efficient solvability, a natural problem to consider is whether an arbitrary linear-programming problem can be somehow converted to a generalized-network flow problem. For example, if $A$ is the constraint matrix of a linear programming problem $P$, and if there exists a nonsingular matrix $T$ such that $TA$ has at most two nonzeros per column, then a solution to $P$ is readily obtained by solving a certain generalized-network flow problem having constraint matrix $TA$.

Because matrices $A$ and $TA$ have the same matric matroid, the study of the matroids of generalized-network-constraint matrices may prove useful in the development of such a conversion procedure. Indeed, this was the motivation for many of the results of this paper. A polynomial-time algorithm that, under certain conditions, produces a matrix $T$ like the one above is described in del Greco [3], and will appear in a subsequent paper. Using a different approach Shull, Orlin, Shuchat and Gardner [8,9] and Shull, Shuchat, Orlin and Gardner [10] have independently developed such an algorithm.

3. Background material

Many facts about bicircular matroids can be found in [7,11–15]. Given a graph $G=(V,E)$ the circuits of $B(G)$, which are called the bicycles of $G$, are the edge sets of subgraphs of $G$ that are a subdivision of one of the graphs of Fig. 2. The rank

![Fig. 2.](image-url)
of a set $A \subseteq E$ is given by $r(A) = |V(G[A])| - |T(G[A])|$, where $T(G[A])$ is the set of acyclic components of $G[A]$. A subset of $E$ is a cocircuit of $B(G)$ if it is a minimal set of edges the deletion of which increases the number of acyclic components, where an isolated vertex counts as an acyclic component.

In [7, 13] a study of matroid connectivity in bicircular matroids was undertaken. Let $M$ be a matroid on $E$ having rank function $r$. A partition $\{E_1, E_2\}$ of $E$ is a $k$-separation of $M$, for a positive integer $k$, if $|E_1| \geq k \leq |E_2|$ and $r(E_1) + r(E_2) - r(E) = k - 1$. The matroid $M$ is $n$-connected, for a positive integer $n$, if it has no $k$-separation for any $k < n$. Every matroid is 1-connected. Thus, a connected matroid is one that is 2-connected. Alternatively, connected matroids are characterized as those matroids in which every pair of elements is contained in a circuit.

The next result is from Wagner [13]; the connected case was also proved by Matthews [7]. A cut vertex of a connected graph is one the deletion of which produces a disconnected graph. A polygon is a connected graph every vertex of which has degree 2.

**Proposition 3.1.** Let $G$ be a connected graph with at least three vertices. Then $B(G)$ is connected if and only if $G$ is not a polygon and has no degree-1 vertices, and $B(G)$ is 3-connected if and only if $G$ has no vertex that is either a cut vertex, a vertex of degree less than 3 or a vertex incident to more than one loop.

Finally, some graph definitions are given. The symbol "\" denotes deletion. The star of a vertex $u$ in $G$ is the set of edges of $G$ incident to $u$, denoted by $St_G(u)$. For a proper subgraph $H$ of $G = (V, E)$ the set of vertices common to $H$ and $G[E - E(H)]$ are the vertices of attachment of $H$. By convention, if $H = G$, then every vertex is a vertex of attachment. Cycles and paths are regarded as edge sets, but, when convenient, are equated with the subgraphs they induce.

A block of $G$ is a maximal subgraph satisfying the property that every pair of edges is contained in a cycle. An end-block of $G$ is a block $H$ having exactly one vertex of attachment, called the tip of $H$. By convention, if $G$ is a block, then $G$ is an end-block of itself and every vertex of $G$ is a tip of $G$.

A balloon of $G$ is a maximal set of edges $B$ such that $G[B]$ is a subdivision of one of the graphs in Fig. 3 and the vertex of attachment of $G[B]$ is $u$, where $u$ is the indicated vertex. A line of $G$ is a set of edges not contained in a balloon, that forms a path, the internal vertices of which have degree 2 in $G$ and the end vertices of which have degree at least 3. Observe that if $B(G)$ is connected and $E$ is not a bicycle, then for each edge there exists a unique line or balloon containing that edge.

![Fig. 3.](image-url)
4. Graph representations

4.1. The operations

This section begins with a description of operations that, if applied to a given graph, produce a graph with the same bicircular matroid. Throughout the section \( G = (V; E) \) denotes a connected graph.

Let \( S \) be a line of \( G \) having end vertices \( u \) and \( v \). Let \( e \) be the unique edge of \( S \) incident to \( u \). Define \( G' \) to be the graph obtained from \( G \) by redefining the incidence relation of \( e \) so that \( e \) is incident to a vertex \( w \neq u \) of \( S \) instead of \( u \). Then \( G' \) is obtained from \( G \) by a rolling of \( S \) away from \( u \), and \( G \) is obtained from \( G' \) by an unrolling of \( S \) to \( u \). Observe that \( S \) is a balloon of \( G' \).

The next two theorems were first proved by Wagner [13] for the case when \( B(G) \) is 3-connected.

Theorem 4.1. Let \( G \) and \( G' \) be graphs such that \( B(G) \) is connected and \( G' \) is obtained from \( G \) by a rolling of a line \( S \) away from the vertex \( u \). Then \( B(G) = B(G') \) if and only if there exists an end-block \( H \) of \( G \) such that \( S \subseteq E(H) \), \( u \) is a tip of \( H \) and every cycle of \( H \) contains \( u \).

Proof. (\( \Rightarrow \)) Since \( B(G) \) is connected, \( S \) is properly contained in some cycle of \( G \), implying that \( S \subseteq E(H) \), for some block \( H \) of \( G \). Let \( \{u, v\} \) be the vertices of attachment of \( S \).

Suppose that either \( H \) is not an end-block or \( u \) is not a vertex of attachment of \( H \) or there is a cycle of \( H \) that does not contain \( u \). Each of these cases implies the existence of a cycle \( C \) that does not contain \( u \). (In the first two cases, \( C \) can be taken to be a cycle of \( G \setminus E(H) \).) Now, where \( P \) is a path from \( u \) to \( C \) that does not contain \( v \), \( C \cup P \cup S \) is a bicycle of \( G' \) and not of \( G \), a contradiction.

(\( \Leftarrow \)) Clearly it suffices to consider those bicycles of \( G \) and \( G' \) that contain \( S \). Moreover, bicycles of \( G \) (respectively \( G' \)) that contain \( S \) and some edge of \( G \setminus E(H) \) are readily seen to be bicycles of \( G' \) (respectively \( G \)), since such bicycles meet \( H \setminus S \) in precisely a path from \( u \) to \( v \).

Let \( K \) be a bicycle of \( G \) contained in \( H \) and containing \( S \). Since each cycle of \( H \) contains \( v \), \( K - S \) consists of a cycle \( C \) containing \( v \) together with a path from \( C \) to \( u \). Now clearly \( K \) is a bicycle of \( G' \). Similarly, each bicycle of \( G' \) contained in \( H \) and containing \( S \) is a bicycle of \( G \).

Let \( v \) be a vertex incident to exactly three lines in \( G \), and let \( L_1, L_2 \) and \( L_3 \) be the lines of \( G \) having end vertex \( v \). Suppose the other end vertex of \( L_1 \) is \( u \), and the other end of \( L_2 \) and \( L_3 \) is \( w \neq u \). Let \( e_1 \) be the edge of \( L_1 \) incident to \( u \), and let \( e_2 \) be the edge of \( L_2 \) incident to \( w \). Define \( G' \) to be the graph obtained from \( G \) redefining the incidence relations of \( e_1 \) and \( e_2 \) so that \( e_1 \) is incident to \( w \) instead of \( u \), and
Representations of bicircular matroids

Let $e_2$ be incident to $u$ instead of $w$. Then $G'$ is obtained from $G$ by a rotation of $L_1$ and $L_2$ at the vertex $v$.

The proof of the next result is similar to that of Theorem 4.1 and is left to the reader.

**Theorem 4.2.** Let $G$ and $G'$ be graphs such that $B(G)$ is connected and $G'$ is obtained from $G$ by a rotation of lines $L_1$ and $L_2$ at $v$. Then $B(G) = B(G')$ if and only if there exists an end-block $H$ of $G$ such that $L := \bigcup_{i=1}^{3} L_i$ is contained in $E(H)$, $u$ is a tip of $H$ and every cycle of $H \setminus L$ contains $u$, where $L_3$ is the line having the same ends as $L_2$ and $u$ is the end of $L_3$ not equal to $v$.

Let $S$ be a line (respectively balloon) of a graph $G$. Let $G'$ be a graph obtained from $G$ by replacing $S$ with another line (respectively balloon) on the same edge set and having the same vertices (respectively vertex) of attachment. Then $G'$ is obtained from $G$ by a replacement. In addition, if $E(G) = E(G')$ and $E(G)$ is a bicycle of $G$ and $G'$, then $G'$ is obtained from $G$ by a replacement. Since every bicycle of $G$ either contains $S$ or is disjoint from $S$, the next result follows.

**Theorem 4.3.** Let $G$ and $G'$ be graphs such that $G'$ is obtained from $G$ by a replacement. Then $B(G) = B(G')$.

Rollings, unrollings, rotations and replacements are operations. If $G'$ is obtained from $G$ by an operation and $B(G) = B(G')$, then the operation is legitimate. (All replacements are legitimate.) A graph $G''$ is r-equivalent to $G$ if there exist graphs $G_1, G_2, \ldots, G_t$ such that $G = G_1, G'' = G_t$ and $G_{i+1}$ is obtained from $G_i$ by a legitimate operation, for $1 \leq i \leq t-1$. Define two graphs to be b-equivalent if they have the same bicircular matroid. Thus, r-equivalence implies b-equivalence. The converse is almost true. However, there does exist a well-defined class of graphs for which the converse does not hold; this class is characterized in the next section.

4.2. Preliminary lemmas

This section is devoted to characterizing a class of pairs of graphs that are exceptions to the statement “b-equivalent graphs are r-equivalent.” The next two results are technical lemmas.

Let $S$ be a balloon of a graph $G$ and let $C$ be the unique cycle of $S$. Let $e = uv$ be an edge of $C$ such that the degree of $u$ in $G$ is greater than 2. Consider the graph $G'$ that is obtained by redefining the incidence relation of $e$ such that $e$ is a loop at $u$. Then $G'$ is obtained from $G$ by a replacement of $S$, called a contraction.

**Lemma 4.4.** Let $G$ be a graph such that $B(G)$ is connected. Then there exists a graph representation $H$ of $B(G)$ such that the star of every vertex of $H$ is a cocircuit
of $B(G)$. Moreover, $H$ is obtained from $G$ by a sequence of legitimate rollings and contractions.

**Proof.** Let $v$ be a vertex of $G$ such that its star is not a cocircuit. Then $G \setminus st_G(v)$ has more than one acyclic component. Let $F$ be such a component that is not equal to $v$. Then there exists an end-block $K$ of $G$ containing $F$. Moreover, $v$ is a tip of $K$ and every cycle of $K$ contains $v$. Let $e$ be any edge of $K$ incident to $v$. If $e$ is in a line $S$, then roll $S$ away from $v$. By Theorem 4.1, this rolling is legitimate. If $e$ is in a balloon $S$, then contract $S$. Continuing in this way produces $H$. 

An easy proof of the next lemma is in [13].

**Lemma 4.5.** Let $G = (V, E)$ be a graph with $|V| \geq 3$, and let $D$ be a cocircuit of $B(G)$. If $B(G[E - D])$ is connected, then $D$ is the star of a vertex of $G$.

Observe that a pair of b-equivalent graphs having at most two vertices are r-equivalent. The next two lemmas handle the cases when the graphs have three or four vertices and the bicircular matroid is 3-connected. For a vertex $v$ of a graph $H$, define $A_v = E(H) - st_H(v)$.

**Lemma 4.6.** Let $G$ and $G'$ be b-equivalent, but not r-equivalent, graphs such that $|V(G)| = 3$ and $B(G)$ is 3-connected. Then $G$ is the graph of Fig. 4(a) and $G'$ is the...
graph of Fig. 4(b) (with the indicated edge labels), or both $G$ and $G'$ are the graph of Fig. 4(c). (Note any labeling of the edges of the graph of Fig. 4(c) yields the same bicircular matroid.)

**Proof.** Let $H$ (respectively $H'$) be the graph obtained by applying Lemma 4.4 to $G$ (respectively $G'$). Then $H$ and $H'$ are b-equivalent, but not r-equivalent. It suffices to prove the lemma for $H$ and $H'$; since none of the graphs of Fig. 4 have loops it will follow that $G = H$ and $G' = H'$.

Let $V(H) = \{x, y, z\}$ and $V(H') = \{x', y', z'\}$. First suppose $|A_x|, |A_y| \geq 3$. Since $|A_x| \geq 3$, $B(H[A_x])$ is connected. Thus, by Lemma 4.5, $st_H(x)$ is the star of some vertex, say $x'$, of $H'$. Likewise, $st_H(y)$ is the star of $y'$. If $|A_z| \geq 3$, then $st_H(z) = st_{H'}(z')$, implying $H = H'$. Otherwise, $|A_z| \leq 2$, and since an edge must join $x$ to $y$, by 3-connectivity of $B(G)$, there is at most one loop, say $p$, at $x$ or at $y$. Now, $st_H(z') \subseteq st_H(z) \cup \{p\}$. Clearly if $st_H(z') \neq st_H(z)$, then $H$ is obtained from $H'$ by a sequence of rollings at $z'$.

Suppose $|A_x| \geq 3$ and $|A_y|, |A_z| \leq 2$. Then there are at most four edges not joining $y$ and $z$. Moreover, the 3-connectivity of $B(G)$ implies there exist edges joining $x$ to $y$ and $x$ to $z$. Thus, there is only a small number of possibilities for $H$, and likewise for $H'$. These are easily checked.

Similarly, if $|A_x|, |A_y|, |A_z| < 2$, then the result is easily checked. In particular, the 3-connectivity implies there are edges joining all three pairs of vertices of $H$ and $H'$, and there can be at most six edges. \[\square\]

---

![Fig. 5](image-url)
Lemma 4.1. Let $G$ and $G'$ be $b$-equivalent, but not $r$-equivalent, graphs such that $|V(G)| = 4$ and $B(G)$ is 3-connected. Then $G$ is the graph of Fig. 5(a) and $G'$ is the graph of Fig. 5(b) (with the indicated edge labels), or both $G$ and $G'$ are one of the graphs of Fig. 5(c). (Note any labeling of the edges of the graphs of Fig. 5(c) yields the same bicircular matroid.)

**Proof.** As in the proof of Lemma 4.6, apply Lemma 4.4 to $G$ and $G'$ to obtain $H$ and $H'$, respectively. Let $V(H) = \{w, x, y, z\}$ and $V(H') = \{w', x', y', z'\}$. If $st_H(w)$, $st_H(x)$ and $st_H(y)$ are stars of $H'$, then $H$ and $H'$ are $r$-equivalent, a contradiction. Thus, by Lemma 4.5 assume $B(H[A_,])$ and $B(H[A_\bar{z}])$ are not connected. By Proposition 3.1, $H[A,]$ either is a triangle or has a degree-1 vertex. Likewise for $H[A_\bar{z}]$.

Suppose $H[A,]$ is a triangle. By the 3-connectivity of $B(H)$ there is an edge from $y$ to every other vertex. Therefore $H[A,]$ is also a triangle. Now $H$ is easily seen to be the graph of Fig. 5(a) or the graph on the left of Fig. 5(c). So, by symmetry, it can be assumed that both $H[A,]$ and $H[A_\bar{z}]$ have a degree-1 vertex. Suppose first that $z$ has degree 1 in $H[A,]$. Then $H$ has at least two edges $yz$. Without loss of generality, assume $yz \in E(H)$. Then $wz \in E(H)$ and $wy \in E(H)$. Now $y$ must have degree 1 in $H[A_\bar{z}]$. It follows that either $H$ has at least one edge parallel to $wx$, or the only other edges are loops at $w$ and $x$. In the latter case, $H$ is the graph in Fig. 6 and one can check that $H$ is the unique representation for $B(H)$. In the former case, $H$ is the graph on the right in Fig. 5(c), unless there are more than two edges $yz$ or more than two edges $xz$ or a loop at $y$ or $z$. Each of these three alternatives implies $H$ is the unique representation for $B(H)$ or any two representations of $B(H)$ are $r$-equivalent (using only rotations).

Now assume that $z$ is not degree 1 in $H[A,]$. Assume $x$ is degree 1 in $H[A,]$. Then there are at least two edges $xy$. Suppose first that $xz \in E(H)$. Then both $wz \in E(H)$ and $wy \in E(H)$. Thus, $w$ is degree 1 in $H[A,]$, implying there are at least two edges $wz$. If there are more than two $xy$ edges, more than two $wz$ edges, or there is a loop at $y$ or $z$, then, as above, any two representations of $B(H)$ are $r$-equivalent. It follows that $H$ is the graph on the right in Fig. 5(c). Finally, assume $xz \notin E(H)$. Then $wz \in E(H)$, $y_z \in E(H)$, and $w$ is degree 1 in $H[A,]$. It follows that there are at least two edges $wz$. If there is exactly one edge $yz$, then as before, $H$ is the graph on the right in Fig. 5(c). Otherwise, there are at least two edges $yz$. If there are more than two edges $xy$ or $wz$, or if there is a loop at $y$ or $z$, then $H$ is the unique representation of $B(H)$. It follows that $H$ is the graph in Fig. 5(b). \qed

Lemma 4.9 below yields a slight extension of Lemmas 4.6 and 4.7. Lemma 4.8 is needed in the proof of Lemma 4.9.

**Lemma 4.8.** Let $G$ and $G'$ be $b$-equivalent graphs such that $B(G)$ is connected and $E(G)$ is not a bicycle. Then $S \subseteq E(G)$ is a line or balloon of $G$ if and only if it is a line or balloon of $G'$.
Proof. The lemma will follow from the following characterization: \( S \) is contained in a line or balloon of \( G \) if and only if every bicycle of \( G \) either contains \( S \) or has empty intersection with \( S \).

The “only if” part is trivial. To show the “if” part let \( S \subseteq E(G) \) be such that every bicycle of \( G \) either contains \( S \) or has empty intersection with \( S \). Let \( e, f \in S \). Since \( B(G) \) has no single-element cocircuit, the pair \( \{e, f\} \) is a cocircuit. Now using the graphic characterization of the cocircuits of \( B(G) \) and the fact that \( E(G) \) is not a bicycle, it is easily seen that \( e \) and \( f \) are in the same line or balloon. \( \square \)

A graph is *cosimple* if each line is an edge and each balloon is a loop. If \( G \) is a graph such that \( B(G) \) is connected and \( E(G) \) is not a bicycle, then associated with \( G \) is a unique (up to edge names) cosimple graph, denoted by \( \tilde{G} \), obtained by replacing each line by an edge and each balloon by a loop. The graph \( \tilde{G} \) is an *extension* of \( G \). If \( E(G) \) is a bicycle, then \( \tilde{G} \) is defined to be the graph on one vertex with two edges.

**Lemma 4.9.** Let \( G \) and \( H \) be b-equivalent graphs. Then \( G \) and \( H \) are r-equivalent if and only if \( \tilde{G} \) and \( \tilde{H} \) are r-equivalent.

**Proof.** (\( \Rightarrow \)) Straightforward.

(\( \Leftarrow \)) If \( E(G) \) is a bicycle, then the result is true by definition of replacement. Suppose otherwise. Since \( \tilde{G} \) and \( \tilde{H} \) are r-equivalent there exists a sequence of graphs \( \tilde{G}_1, \tilde{G}_2, \ldots, \tilde{G}_t \) such that \( \tilde{G} = \tilde{G}_1, \tilde{H} = \tilde{G}_t \), and \( \tilde{G}_{i+1} \) is obtained from \( \tilde{G}_i \) by a legitimate operation, for \( 1 \leq i \leq t-1 \).

Suppose \( \tilde{G}_2 \) is obtained from \( \tilde{G}_1 \) by a rolling of the edge \( e \) away from the vertex \( v \). The edge \( e \) corresponds to a unique line \( L \) of \( G \). Define \( G_1 = \tilde{G} \) and \( G_2 = G \) to be the graph obtained by a rolling of \( L \) away from \( v \). Clearly this rolling is legitimate. Moreover, \( G_2 \) is an extension of \( G_2 \). In the case that \( \tilde{G}_2 \) is obtained from \( \tilde{G}_1 \) by an unrolling or rotation, \( G_2 \) is similarly defined.

By repeating the above procedure there exists a sequence of graphs \( G_1, \ldots, G_t \) such that \( G_{i+1} \) is obtained from \( G_i \) by a legitimate operation and \( G_i \) is an extension of \( G_{i-1} \). Evidently \( G_i \), and \( H \) are b-equivalent and are both extensions of \( \tilde{H} \). Moreover, by Lemma 4.8, \( H \) is obtained from \( G_t \) by a sequence of replacements. \( \square \)
Define $\mathcal{G}$ to be the class of graphs, each of which is an extension of one of the graphs in Fig. 4 or Fig. 5. Then Lemmas 4.6-4.9 imply that if $G$ and $G'$ are b-equivalent graphs from $\mathcal{G}$, then $G$ and $G'$ are r-equivalent if and only if $G'$ is obtained from $G$ by a sequence of replacements.

4.3. The main theorem

In this section it is proved that two b-equivalent graphs are either r-equivalent or they are in $\mathcal{G}$. The following special case of this result was proved by Wagner [13].

**Theorem 4.10.** Let $G$ and $G'$ be b-equivalent graphs such that $B(G)$ is 3-connected and $|V(G)| = |V(G')| \geq 5$. Then $G$ and $G'$ are r-equivalent.

The main theorem is the following.

**Theorem 4.11.** Let $G$ and $G'$ be b-equivalent graphs such that $B(G)$ is connected. Then either $G$ and $G'$ are r-equivalent or $G$ and $G'$ are in $\mathcal{G}$.

The proof will be postponed. By Lemmas 4.6, 4.7 and 4.9 and Theorem 4.10 the above theorem is true when the cosimplification of $G$ has a 3-connected bicircular matroid. In the case that the cosimplification of $G$ is not 3-connected, Proposition 3.1 implies that there exists a partition $\{E_1, E_2\}$ of $E(G)$ such that neither $E_1$ nor $E_2$ is contained in a line or balloon, and $G[E_1]$ has exactly one vertex of attachment.

A natural step at this point would be to decompose $G$ and $G'$ using the partition $\{E_1, E_2\}$ and apply induction. However, $\{E_1, E_2\}$ may not be nice in that $G'[E_1]$ may have more than one vertex of attachment. Thus, the first step is to show that there exists a partition $\{E_1, E_2\}$ such that $G'[E_1]$ and $G'[E_1]$ have exactly one vertex of attachment.

**Lemma 4.12.** Let $G$ and $G'$ be b-equivalent graphs such that $B(G)$ is not 3-connected. Then there exists a partition $\{E_1, E_2\}$ of $E(G)$ such that $|E_1| \geq 2 \leq |E_2|$ and both $G[E_1]$ and $G'[E_1]$ have exactly one vertex of attachment.

**Proof.** Since $B(G)$ is not 3-connected, by Proposition 3.1 there exists a partition $\{E_1, E_2\}$ of $E(G)$ such that $|E_1| \geq 2 \leq |E_2|$, $G[E_1]$ has exactly one vertex of attachment and neither $E_1$ nor $E_2$ is contained in a line or balloon. Among all such partitions assume $\{E_1, E_2\}$ is chosen so that $|E_1|$ is minimum. By Lemma 4.8 neither $E_1$ nor $E_2$ is contained in a line or balloon of $G'$.

Since $E_1$ is not contained in a balloon, $G[E_1]$ is not a polygon. Since $|E_1|$ is minimum, $G[E_1]$ cannot have a degree-1 vertex. Thus, by Proposition 3.1, $B(G[E_1])$ is connected implying $B(G'[E_1])$ is connected.

Now it is straightforward to check that $\{E_1, E_2\}$ is a 2-separation of $B(G) = B(G')$ implying
\[ |V(G'[E_1]) \cap V(G'[E_2])| = |T(G'[E_1])| + |T(G'[E_2])| + 1. \]

Since \( B(G'[E_1]) \) is connected, \( |T(G'[E_1])| = 0 \).

Every component of \( T(G'[E_2]) \) must have at least two vertices in \( V(G'[E_1]) \cap V(G'[E_2]) \) since \( G' \) has no degree-1 vertices. Thus,
\[ |T(G'[E_2])| + 1 \geq 2|T(G'[E_2])| \]

which implies \( |T(G'[E_2])| \leq 1 \). If \( |T(G'[E_2])| = 1 \), then \( G'[E_2] \) is a path with \( V(G'[E_1]) \cap V(G'[E_2]) \) equal to the vertices of attachment of \( G'[E_7] \). Therefore, \( E_7 \) is contained in a line or balloon of \( G' \), a contradiction. Hence, \( |T(G'[E_2])| = 0 \) implying \( \{E_1, E_2\} \) is the desired partition. \( \square \)

One final lemma before proving the main theorem.

**Lemma 4.13.** Let \( G \) and \( G' \) be \( r \)-equivalent graphs such that \( e \) is a loop of both \( G \) and \( G' \). Then \( G' \) is obtainable from \( G \) by a sequence of graphs each obtained from its predecessor by a legitimate operation such that \( e \) is a loop in every graph of the sequence.

**Proof.** If \( E(G) \) is a bicycle, then the result is true, suppose not. Since \( G \) and \( G' \) are \( r \)-equivalent there exists a sequence of graphs, \( G_1, \ldots, G_t \), such that \( G = G_1 \), \( G' = G_t \), and \( G_{i+1} \) is obtained from \( G_i \) by a legitimate operation, for \( 1 \leq i \leq t - 1 \). Suppose that for all such sequences, there is a \( k \) such that in \( G_k \), \( e \) is not a loop. Choose the sequence so that \( k \) is maximum, subject to \( t \) being minimum.

Let \( S \) be the unique balloon of \( G_{k-1} \) containing \( e \). Then \( G_k \) is obtained from \( G_{k-1} \) by unrolling \( S \) to a vertex \( v \). Let \( J \) be the end-block of \( G_k \) containing \( S \). By Theorem 4.1, every cycle of \( J \) contains \( u \), which is a tip of \( J \). By maximality of \( k \), each remaining operation in the sequence is a rolling or a rotation involving only edges in \( J \).

First suppose the next two operations are rollings and/or unrollings. Consider the rolling of a line \( P \), say, away from vertex \( x \), that takes \( G_k \) to \( G_{k+1} \). (This operation is in fact a rolling, since \( S \) has no balloons.) If \( x \neq v \), then both \( x \) and \( v \) are contained in every cycle of \( J \), implying \( J \) consists of internally disjoint \((x, v)\)-paths and the lemma is easily seen to be true. Thus, \( x = v \). Assume the ends of \( P \) in \( G_k \) are \( x \) and \( u \). Consider the next operation, taking \( G_{k+1} \) to \( G_{k+2} \). Suppose this operation is a rolling of a line away from some vertex \( w \). Then by Theorem 4.1, \( G_k = J, w = u \), and every cycle of \( G_{k+1} \setminus P \) contains \( u \). Since every cycle of \( G_{k+1} \) contains \( u \), \( G_{k+1} \setminus P \) consists of internally disjoint \((u, v)\)-paths, and the lemma follows. Assume, therefore, the operation taking \( G_k \) to \( G_{k+2} \) is an unrolling, necessarily of the line \( P \), to some vertex \( w \) (\( w \neq v \), by minimality of \( t \)). Then \( G_k = J \), and in \( G_{k+1} \setminus P \), every cycle contains \( w \) and \( v \), implying \( G_{k+1} \setminus P \) consists of internally disjoint \((v, w)\)-paths, one of which is \( S \). Now it can be seen that \( G_{k+2} \) can be obtained from \( G_k \) by a single rotation at \( u \), contradicting minimality of \( t \).
Now assume the next operation is a rolling of a line $P$ followed by a rotation at a vertex $w$. Let $L_1$, $L_2$, and $L_3$ be the lines of the rotation, where the ends of $L_1$ are $w$ and $z$ and the ends of $L_2$ and $L_3$ are $w$ and $y$. Let $L=L_1\cup L_2\cup L_3$, and define $J'$ to be the block of $G_{k+1}$ containing $L$. Note that $P$ contains an end-block and $J'$ (by Theorem 4.2) is an end-block of $G_{k+1}$. Thus, $G_k=J$. Also, every cycle of $J\setminus L$ contains $z$, and $z$ is a tip of $J'$. Since every cycle of $J$ contains $v$, every cycle of $J'$ contains $v$. Thus, $v\in V(J'[L_1\cup L_2])$. Since $v$ is the end of the line $S$ in $G_k$, either $v=y$ or $v=w$. If $v=w$, then $J\setminus L$ is a path joining $y$ and $z$. Since $L_1$ is a line with end $z$, the vertex of attachment of the balloon $P$ must be $z$. Thus, $G_{k-1}$ consists of four lines — two $(w,z)$-paths, a $(y,z)$-path, and a $(w,y)$-path — and the balloon $S$ having vertex of attachment $y$. On the other hand, if $v=y$, then every cycle of $J\setminus L$ contains both $y$ and $z$, implying $J\setminus L$ consists of internally disjoint $(y,z)$-paths. Now, in $G_{k-1}$, $P$ is a line with ends $y$ and $z$, and $S$ is a balloon at $w$ or $z$. It is now straightforward to check that $G_1$ can be obtained from $G_{k-1}$ by a sequence of rotations, a contradiction.

The final case is when the operation applied to $G_k$ is a rotation. The analysis here is very similar to the previous case, and thus omitted. \[\Box\]

**Proof of Theorem 4.11.** Let $\{E_1,E_2\}$ be a partition of $E(G)$ whose existence is provided by Lemma 4.12. Let $v$ (respectively $v'$) be the vertex of attachment of $G[E_1]$ (respectively $G'[E_1]$). Define $G_i$ to be the graph obtained from $G[E_i]$, $i=1,2$, by adding a loop $e\in E(G)$ incident to $v$. Define $G'_i$, $i=1,2$, analogously. Then $B(G_1)$ and $B(G_2)$ are connected. Moreover $B(G_1)=B(G'_1)$ and $B(G_2)=B(G'_2)$. Observe that $G_1,G_2,G'_1,G'_2\notin \emptyset$ since none of the graphs in $\emptyset$ have loops. Therefore, by induction, $G_1$ is $r$-equivalent to $G'_1$, and $G_2$ is $r$-equivalent to $G'_2$. Now by applying Lemma 4.13 to $G'_1$ and $G_1$, and the loop $e$, it is straightforward to check that any operation used in obtaining $G'_1$ from $G_1$ can be “lifted” to a legitimate operation to be used in obtaining $G'$ from $G$. Likewise for $G'_2$ and $G_2$. \[\Box\]

### 5. Matrix representations

By Propositions 2.1 and 2.2, bicircular matroids are representable over the real numbers. In particular, given b-equivalent graphs $G$ and $G'$, there exist generalized-incidence matrices $N$ and $N'$ of $G$ and $G'$, respectively, such that $M(N)=M(N')=B(G)$. If $B(G)$ is binary, then a well-known result (see e.g. Bixby and Cunningham [1]) says that $N'=TND$, where $T$ and $D$ are nonsingular and $D$ is diagonal. However, bicircular matroids are, in general, not binary since the matroid $U^2_2$ is bicircular. (A representation of $U^2_2$ is given by two vertices joined by four edges.)

The main result of this section is for any collection of $r$-equivalent graphs $G_1,...,G_t$ there exist row-equivalent matrices $N_1,...,N_t$ such that $N_i$ is a generalized-incidence matrix of $G_i$, for $1\leq i\leq t$. The proof will be via a sequence of lemmas and propositions.
Proposition 5.1. Let $A$ be a full-row-rank $m \times n$ matrix such that $M(A)$ is bicircular. Let $G$ be a graph representation of $M(A)$ such that the star of every vertex of $G$ is a cocircuit. Then there exists a unique (up to row scaling) matrix $T$ such that $TA$ is a generalized-incidence matrix of $G$.

Proof. Denote by $H_i$ the set of columns of $A$ corresponding to edges of $G$ not incident to vertex $i$. Since the star of every vertex is a cocircuit, $H_i$ is a hyperplane of $M(A)$, implying the linear rank of $H_i$ is $m - 1$.

Consider the system $t_iH_i = 0$. Since $H_i$ is a hyperplane, the system has a unique nonzero solution, up to scaling, say $i_i$. Then the support of $i_iA$ is precisely the set of edges incident to vertex $i$. Define $T$ to be the matrix whose $i$th row is $i_i$. Then $T$ is the desired matrix. \( \square \)

The above result has two drawbacks. First, not every graph $G$ satisfies the property that the star of every vertex is a cocircuit of $B(G)$. Second, the matrix $T$ may be singular.

Lemma 5.2. Let $A$ be a full-row-rank matrix such that $M(A)$ is bicircular. Assume $M(A)$ has a graph representation $G$ such that $G$ has a loop or a pair of vertex-disjoint cycles. If $T$ is a matrix such that $TA$ is a generalized-incidence matrix of $G$, then $T$ is nonsingular.

Proof. It suffices to show that $TA$ has full row rank, or equivalently, $M(TA) = M(A)$.

First suppose $G$ has a loop. Let $F$ be the set of columns of $TA$ corresponding to a spanning tree of $G$ plus a loop. Then $|F| = |V|$. Moreover, since the rows and columns of $F$ may be permuted to form a triangular matrix, each diagonal element of which is nonzero, $F$ is independent.

Now suppose $G$ has two vertex-disjoint cycles, $C_1$ and $C_2$. Extend $C_1$ to a spanning tree plus one edge, and let $F$ be the corresponding set of columns of $TA$. The rank of $F$ is either $|V|$ or $|V| - 1$; moreover, $F$ has rank $|V| - 1$ if and only if the columns corresponding to $C_1$ are dependent. Suppose $TA$ does not have full row rank. Then $TA$ has rank $|V| - 1$, and both $C_1$ and $C_2$ correspond to dependent sets of columns of $TA$. Since $TA$ has rank $|V| - 1$, one row may be added to obtain a matrix representing $M(A)$. But since $C_1$ and $C_2$ are vertex disjoint, the addition of any row leaves the columns corresponding to $C_1 \cup C_2$ dependent contradicting the fact that $M(A) = B(G)$. \( \square \)

Lemma 5.3. Let $A$ be a full-row-rank matrix such that $M(A)$ is bicircular. Let $G$ and $G'$ be graph representations of $M(A)$ such that $G'$ is obtained from $G$ by a line replacement. If $T$ is a nonsingular matrix such that $TA$ is a generalized-incidence matrix of $G$, then there exists a nonsingular matrix $T'$ such that $T'A$ is a generalized-incidence matrix of $G'$. 
Proof. Let \( e_1 \) and \( e_2 \) be edges incident to a degree-2 vertex \( v \) of \( G \). Then it suffices to assume that \( G' \) is obtained from \( G \) by interchanging \( e_1 \) and \( e_2 \).

Let \( v_1 \) and \( v_2 \) be the other ends, in \( G \), of \( e_1 \) and \( e_2 \). Let \( N = TA \). Then a full-row-rank matrix \( N' \) that is a generalized-incidence matrix of \( G' \) can be obtained from \( N \) by subtracting off appropriate multiples of row \( v \) from rows \( v_1 \) and \( v_2 \).

The following theorem is useful in the linear-programming application discussed in Section 2.

Theorem 5.4. Let \( A \) be a full-row-rank matrix such that \( M(A) \) is bicircular and connected. Let \( G_1 \) and \( G_2 \) be \( r \)-equivalent graph representations of \( M(A) \). If there exists a generalized-incidence matrix \( N_1 \) of \( G_1 \) that is row-equivalent to \( A \), then there exists a generalized-incidence matrix \( N_2 \) of \( G_2 \) that is row-equivalent to \( A \).

Proof. Suppose the star of every vertex of \( G_2 \) is a cocircuit. If \( G_2 \) has a loop or a pair of vertex-disjoint cycles, then by Proposition 5.1 and Lemma 5.2 the result follows. Assume \( G_2 \) is loopless and every pair of cycles intersect.

If a legitimate rolling can be performed on \( G_2 \), then \( G_2 \) has a vertex whose star is not a cocircuit, a contradiction. If a legitimate unrolling can be performed on \( G_2 \), then \( G_2 \) has a balloon \( B \). But the star of every vertex of \( B \) is a cocircuit of \( B(G_2) \) if and only if the unique cycle of \( B \) is a loop. Since \( G_2 \) is loopless, no legitimate unrollings can be performed. If a legitimate rotation can be performed, then there exists a vertex of \( G_2 \) that is contained in every cycle. But then the star of this vertex is not a cocircuit, a contradiction. Therefore \( G_2 \) is obtained from \( G_1 \) by a sequence of replacements. As observed \( G_2 \) can have no balloons. So the theorem follows from Lemma 5.3.

Now consider the case when the star of some vertex of \( G_2 \) is not a cocircuit. By Lemma 4.4 there exists a graph \( G'_3 \) such that \( B(G'_3) = B(G_2) \) and the star of every vertex of \( G'_3 \) is a cocircuit. Moreover, \( G'_3 \) is obtained \( G_2 \) by a sequence of rollings and contractions. Thus, \( G'_3 \) has a loop. Therefore by Proposition 5.1 and Lemma 5.2 there exists a matrix \( N'_2 \) that is row-equivalent to \( A \) such that \( N'_2 \) is a generalized-incidence matrix of \( G'_3 \).

Let \( e \) be an edge that is a loop of \( G'_3 \), but not of \( G_2 \). Suppose \( e \) is incident to \( u \) in \( G'_2 \), and \( u \) and \( v \) in \( G_2 \). By Lemma 4.4 (and its proof) there exists a cocircuit \( D \) contained in \( st(G'_3)(u) \cup \{e\} \) that contains \( e \). Therefore there exists a vector \( w \) in the row space of \( N'_2 \) having support \( D \). Let \( r_u \) be the row of \( N'_2 \) corresponding to \( u \). Clearly, there exists some scalar \( \lambda \) such that the matrix \( N'_2 \) obtained from \( N'_2 \) by replacing \( r_u \) by \( r_u + \lambda w \), has full row rank and is a generalized-incidence matrix of the graph obtained from \( G'_3 \) by redifining the incidence relation of \( e \) so that \( e \) joins \( u \) and \( v \). Continuing in this way, a full-row-rank matrix \( N_2 \) that is generalized-incidence matrix of \( G_2 \) and is row-equivalent to \( N'_2 \) (and thus to \( A \)) can be obtained. \( \square \)
By combining Theorem 5.4 and Propositions 2.1 and 2.2 the following result is obtained.

**Theorem 5.5.** Let $G_1, \ldots, G_r$ be $r$-equivalent graphs such that $B(G_1)$ is connected. Then there exist row-equivalent matrices $N_1, \ldots, N_r$ such that $N_i$ is a representation of $B(G_i)$ and a generalized-incidence matrix of $G_i$.

Theorem 5.4 is not true if "$r$-equivalent" is replaced by "$b$-equivalent" as shown by the following example.

**Example 5.6.** Let

\[
A = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & -3 & -3 & 4 & 0 & 0 \\
-3 & 4 & 0 & 0 & 7 & 3 \\
0 & 0 & -1 & 3 & 3 & 2
\end{pmatrix}.
\]

Then $M(A)$ is bicircular with graph representations $G_1$ and $G_2$ shown in Fig. 7. By choosing $N_1 = A$, the hypothesis of Theorem 5.4 are satisfied. The star of every vertex of $G_2$ is a cocircuit, and so by Proposition 5.1 there exists a unique $T$ such that $TA$ is a generalized-incidence matrix of $G_2$. The matrix $T$ is computed, as in the proof of Proposition 5.1, to be

\[
T = \begin{pmatrix}
4 & 3 & -7 \\
-1 & -2 & 3 \\
3 & 1 & -4
\end{pmatrix},
\]

which is singular. □

6. Remarks

(1) Let $A_1$ and $A_2$ be matrices such that $M(A_1) = M(A_2)$. Cunningham [2] conjectured the following: there exists a nonsingular matrix $T$ such that $TA_1$ has the same nonzero pattern as $A_2$. This conjecture is known to be true for binary matroids;
see, e.g., Bixby and Cunningham [1]. A counterexample for the non-binary case is given by Example 5.6. Choose $A_1 = A$ and $A_2$ to be the generalized-incidence matrix of $G_2$ given by Propositions 2.1 and 2.2. A counterexample has also been found by S. Halfin.

(2) In a subsequent paper we will give a polynomial-time algorithm that for a given bicircular matroid $M(A)$ produces a nonsingular matrix $T$ and a graph $G$ such that $TA$ is a generalized-incidence matrix for $G$ and $B(G) = M(A)$, assuming such a $T$ and $G$ exist. The algorithm is based on many of the results of this paper. A different algorithm for the same problem has been developed in the series of papers by Shull, Orlin, Shuchat and Gardner [8, 9], and Shull, Shuchat, Orlin and Gardner [10]. Their algorithm yields as a by-product Theorem 4.11. Moreover, they independently proved Theorem 5.4.

References