The nonabelian Liouville-Arnold integrability by quadratures problem: a symplectic theory approach.

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ABSTRACT.There is proposed a symplectic theory approach to studying integrable via the nonabelian Liouville-Arnold theorem Hamiltonian systems on canonically symplectic phase spaces. A method of algebraic-analytical constructing the corresponding integral submanifold imbedding mappings is devised.

0.General setting.

0.1. As it is well known [1,4], the integrability by quadratures of a differential equation in space $\mathbb{R}^n$ is a method of seeking its solutions by means of finite number of algebraic operations (together with inversion of functions) and "quadratures"- calculations of integrals of known functions.

Assume that our differential equation is given as a Hamiltonian dynamical system on some appropriate symplectic manifold $(M^{2n}, \omega^{(2)})$, $n \in \mathbb{Z}_+$, in the form

$$du/dt = \{H, u\}, \qquad (0.1)$$

where $u \in M^{2n}$, $H : M^{2n} \to \mathbb{R}$ - a smooth enough Hamiltonian function [1,4] with respect to the Poisson bracket $\{\cdot,\cdot\}$ on $\mathcal{D}(M^{2n})$, dual to the symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$, and $t \in \mathbb{R}$ is the evolution parameter.

More than one hundred and fifty years ago french mathematicians and physicists, first E. Bour and next J. Liouville, proved the first "integrability by quadratures" theorem which in modern terms can be formulated as follows.

Theorem 0.1. *Let $M^{2n} \simeq T^*(\mathbb{R}^n)$ be a canonically symplectic phase space and there be given a dynamical system(0.1) with a Hamilton function H: $M^{2n} \times \mathbb{R}_t \to \mathbb{R}$, possessing a Poissonian Lie algebra $\mathcal{G}$ of $n \in \mathbb{Z}_+$ invariants $H_j : M^{2n} \times \mathbb{R}_t \to \mathbb{R}$, $j = \overline{1,n}$, such that*

$$\{H_i, H_j\} = \sum_{s=1}^{n} c_{ij}^s H_s, \tag{0.2}$$

and for all i,j,k $= \overline{1,n}$ values $c_{ij}^s \in \mathbb{R}$ are constants on $M^{2n} \times \mathbb{R}_t$. Put further

$$M_h^{n+1} =:= \{(u,t) \in M \times \mathbb{R}_t : h(H_j) = h_j,\ j = \overline{1,n},\ h \in \mathcal{G}^*\} \tag{0.2}$$

the integral submanifold of the set $\mathcal{G}$ of invariants at a regular element $h \in \mathcal{G}^*$, for which the set (0.3) will be a good defined connected submanifold of $M \times \mathbb{R}_t$. Then, if :

i) all functions of $\mathcal{G}$ are functionally independent on M_h^{n+1};

ii) $\sum_{s=1}^{n} c_{ij}^s h_s = 0$ for all $i,j = \overline{1,n}$;

iii) the Lie algebra $\mathcal{G} = span_{\mathbb{R}}\ \{H_j : M^{2n} \times \mathbb{R}_t \to \mathbb{R}$: $j = \overline{1,n}\}$ is solvable, the solutions to the Hamiltonian system (0.1) on M^{2n} can be found in quadratures.

As a simple corollary of the Bour-Liouville theorem one gets the following corollary.

Corollary 0.2. *If a Hamiltonian system on* $M^{2n} = T^*(\mathbb{R}^n)$ *possesses just* $n \in \mathbb{Z}_+$ *functionally independent invariants in involution, that is a Lie algebra* $\mathcal{G}$ *is abelian, then it is integrable by quadratures.*

In the autonomous case when a Hamiltonian $H = H_1$, and invariants $H_j : M^{2n} \to \mathbb{R}$, $j = \overline{1,n}$, don't depend on the evolution parameter $t \in \mathbb{R}$, the involutivity condition $\{H_i, H_j\} = 0$, $i, j = \overline{1,n}$, can be replaced by the more weak one $\{H, H_j\} = c_j H$ for some constants $c_j \in \mathbb{R}$, $j = \overline{1,n}$.

The first proof of the Theorem 0.1. was based on a one S. Lie's result, which could be formulated as follows.

Theorem 0.3 *(S. Lie) Let vector fields* $K_j \in \Gamma(M^{2n})$, $j = \overline{1,n}$, *be independent in some open neighborhood* $U_h \in M^{2n}$, *generate a solvable Lie algebra* $\mathcal{G}$ *with respect to the usual commutator* $[\cdot,\cdot]$ *on* $\Gamma(M^{2n})$ *and* $[K_j, K] = c_j K$ *for all* $j = \overline{1,n}$, *where* $c_j \in \mathbb{R}$, $j = \overline{1,n}$, *are constants. Then the dynamical system*

$$du/dt = K(u), \tag{0.1}$$

where $u \in U_h \subset M^{2n}$, *is integrable by quadratures.*

Example 0.4 *Motion of three particles on line* $\mathbb{R}$ *under uniform potential field.*

The motion of three particles on the axis $\mathbb{R}$ pairwise interacting via a uniform potential field $Q(\|\cdot\|)$ is described as a Hamiltonian system on the canonically symplectic phase space $M = T^*(\mathbb{R}^3)$ with the following Lie algebra $\mathcal{G}$ of invariants on M^{2n}:

$$H = H_1 = \sum_{j=1}^{3} p_j^2/2m_j + \sum_{i<j=1}^{3} Q(\|q_i - q_j\|), \tag{0.4}$$

$$H_2 = \sum_{j=1}^{3} q_j p_j, \qquad\qquad H_3 = \sum_{j=1}^{3} p_j,$$

where $(q_j, p_j) \in T^*(\mathbb{R}),\ j = \overline{1,3}$, -coordinates and impulses of particles on the axis $\mathbb{R}$. The commutation relations for the Lie algebra $\mathcal{G}$ are read as

$$\{H_1, H_3\} = 0, \qquad \{H_2, H_3\} = H_3, \quad \{H_1, H_2\} = 2H_1, \tag{0.5}$$

meaning evidently that it is solvable. Having taken now a regular element $h \in \mathcal{G}^*$, such that $h(H_j) = h_j = 0$, for $j = 1$ and 3, and $h(H_2) = h_2 \in \mathbb{R}$ being arbitrary, one obtains the integrability of the problem above in quadratures.

0.2. In 1974 V. Arnold proved [4] the following important theorem known as the commutative (abelian) Liouville-Arnold one.

Theorem 0.5 *(J.Liouville-V. Arnold). suppose a set $\mathcal{G}$ of functions $H_j : M^{2n} \to \mathbb{R},\ j = \overline{1,n}$, on a symplectic manifold M^{2n} is abelian, that is*

$$\{H_i, H_j\} = 0 \tag{0.6}$$

for all $i, j = \overline{1,n}$. If *on the compact and connected integral submanifold $M_h^n = \{u \in M^{2n}: h(H_j) = h_j \in \mathbb{R},\ j = \overline{1,n},\ h \in \mathcal{G}^*\}$ with $h \in \mathcal{G}$ being regular, and all functions $H : M^{2n} \in \mathbb{R},\ \ j = \overline{1,n}$, being functionally independent, then it is diffeomorphic to the n-dimensional torus $\mathbb{T}^n \simeq M^{2n}$, and the motion on it with respect to the Hamiltonian $H = H_1 \in \mathcal{G}$ will be a quasi-periodic function of the evolution parameter $t \in \mathbb{R}$.*

In the case of Theorem 0.3 a dynamical system is called completely integrable.

More than twenty years ago, in 1978 Mishchenko and Fomenko [2] generalized essentially the Liouville-Arnold theorem 0.3, having proved the following theorem.

Theorem 0.6. *(A. Mishchenko-A. Fomenko) Assume that on a symplectic manifold* $(M^{2n},\omega^{(2)})$ *there is given a nonabelian Lie algebra* $\mathcal{G}$ *of invariants* $H_j : M \in \mathbb{R}, \quad j=\overline{1,k},$ *with respect to the dual Poisson bracket on* M^{2n}, *that is*

$$\{H_i, H_j\} = \sum_{s=1}^{k} c_{ij}^s H_s, \tag{0.7}$$

where all values $c_{ij}^s \in \mathbb{R}$, $i,j,s =\overline{1,k}$, *are constants, and satisfying the following conditions:*

i) the integral submanifold $M_h^r := \{u \in M^{2n} : h(H_j) = h_j \in \mathbb{R}, \ j=\overline{1,k}, \ h \in \mathcal{G}^*\}$ *is compact and connected at a regular element* $h \in \mathcal{G}^*$;

ii) all functions $H_j : M^{2n} \to \mathbb{R}$, $j=\overline{1,k}$, *are functionally independent on* M^{2n};

iii)the Lie algebra $\mathcal{G}$ *of invariants satisfies the following relationship:*

$$\dim \mathcal{G} + rank\mathcal{G} = \dim M^{2n}, \tag{0.9}$$

where $rank\mathcal{G} = dim\ \mathcal{G}_h -$ *the dimension of a Cartan subalgebra* $\mathcal{G}_h \in \mathcal{G}$. *Then the submanifold* $M_h^r \subset M^{2n}$ *is* $r = rank\mathcal{G}$ *-dimensional, invariant with respect each vector field* $K \in \Gamma(M^{2n})$, *generated by an element* $H \in \mathcal{G}_h$, *and diffeomorphic to the r-dimensional torus* $\mathbb{T}^r \simeq M_h^r$, *the motion on which being a quasiperiodic function of the evolution parameter* $t \in \mathbb{R}$.

0.3. The most simple proof of the Mishchenko -Fomenko Theorem 0.4 can be obtained from the well known [3,16] classical Lie-Cartan theorem.

Theorem 0.7 *(S. Lie-E. Cartan) Suppose that a point* $h \in \mathcal{G}^*$ *for a given Lie algebra* $\mathcal{G}$ *of invariants* $H_j : M^{2n} \to \mathbf{R}$, $j=\overline{1,k}$, *is not critical, and the rank* $||\{H_i, H_j\} : i,j = \overline{1,k}|| = 2(n-r)$ *is constant in an open neighborhood* $U_h \in \mathbb{R}^n$ *of the point* $\{h(H_j) = h_j \in \mathbb{R}: j = \overline{1,k}\} \subset \mathbb{R}^k$. *Then in the neighborhood* $(h \circ H)^{-1} : U_h \subset M^{2n}$ *there exist some* $k \in \mathbb{Z}_+$ *independent functions*

$f_s : \mathcal{G} \to \mathbb{R}$, $s = \overline{1,k}$, such that functions $F_s := (f_s \circ H) : M^{2n} \in \mathbb{R}$, $s = \overline{1,k}$, *satisfy the following relationships:*

$$\{F_1, F_2\} = \{F_3, F_4\} = ... = \{F_{2(n-r)-1}, F_{2(n-r)}\} = 1, \tag{0.9}$$

with all other brackets $\{F_i, F_j\} = 0$, *where* $(i,j) \neq (2s-1, 2s)$, $s = \overline{1, n-r}$. *In particular,* $(k+r-n) \in \mathbb{Z}_+$ *functions* $F_j : M^{2n} \to \mathbb{R}$, $j = \overline{1, n-r}$, and $F_s : M^{2n} \to \mathbb{R}$, $s = \overline{1, k-2(n-r)}$, *compose an abelian algebra* $\mathcal{G}_\tau$ *of new invariants on* M^{2n}, *independent on* $(h \circ H)^{-1}(U_h) \subset M^{2n}$.

As a simple corollary of the Lie-Cartan Theorem 0.5 on gets the following : in the case of the Mishchenko-Fomenko theorem when $rank\mathcal{G} + dim\mathcal{G} = dimM^{2n}$, that is $r+k = 2n$, the abelian algebra $\mathcal{G}_\tau$ (it is not a subalgebra of $\mathcal{G}$!) of invariants on M^{2n} is just $n = 1/2 dimM^{2n}$-dimensional, giving rise to its local complete integrability in $(h \circ H)^{-1}(U_h) \subset \mathrm{M}^{2n}$ via the abelian Liouville-Arnold theorem 0.3. It is also evident that Mishchenko-Fomenko nonabelian integrability theorem 0.4 reduces to the commutative (abelian) Liouville-Arnold case when a Lie algebra $\mathcal{G}$ of invariants is just abelian, since then $rank\mathcal{G} = dim\mathcal{G} = 1/2 \dim M^{2n} = n \in \mathbf{Z}_+$, that is the standard complete integrability condition.

All the cases of integrability by quadratures described above pose the following fundamental problem: How to construct effectively by means of the algebraic-analytical methods the corresponding integral submanifold imbedding mappings

$$\pi_h : M_h^r \to M^{2n}, \tag{0.10}$$

where $r = dim\ rank\mathcal{G}$, making it possible to present searched solutions of an integrable flow on M_h^r as some exact quasiperiodic functions on the torus $\mathbb{T}^r \simeq M_h^r$.

Below we shall describe an algebraic-analytical algorithm of solving the problem above in the case when a symplectic manifold M^{2n} is diffeomorphic to the canonically symplectic cotangent phase space $T^*(\mathbb{R}) \simeq M^{2n\cdot}$.

1 General setting.

1.1. Our main object of studying will be differential systems of vector fields on the cotangent phase space $M^{2n} = T^*(\mathbb{R}^n)$, $n \in \mathbb{Z}_+$, endowed with the canonical symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$, where by definition, $\omega^{(2)} = d(pr^*\alpha^{(1)})$, and

$$\alpha^{(1)} :=< p, dq >= \sum_{j=1}^{n} p_j dq_j, \tag{1.1}$$

is the canonical 1-form on the base space $\Lambda^1(\mathbb{R}^n)$, lifted naturally to the space $\Lambda^1(M^{2n})$, $(q,p) \in M^{2n}$ are canonical coordinates on $T^*(\mathbb{R}^n)$, $pr : T^*(\mathbb{R}^n) \to \mathbb{R}$ - the canonical projection , and $< \cdot,\cdot > -$ the usual scalar product in R^n.

Assume further that there is also given a Lie subgroup G (not necessary compact), acting via mapping $\varphi : G \times M^{2n} \to M^{2n}$ symplectically on M^{2n}, that is the following diagram is commutative:

$$\begin{array}{ccccc} \mathcal{G} & \simeq & T(\mathcal{G}) & \overset{\varphi_*(u)}{\rightarrow} & T(M^{2n}) \\ & & \downarrow & & \downarrow \\ & & \mathcal{G} & \overset{\varphi(u)}{\rightarrow} & M^{2n} \end{array} \tag{1.2}$$

for any $u \in M^{2n}$, where by definition, the mapping $\varphi_* : T(\mathcal{G}) \to \Gamma(M^{2n})$ is a Lie algebra homomorphism of $\mathcal{G}$ and $\Gamma(M^{2n})$. Thus, for any $a \in \mathcal{G}$ one

can define a vector field $K_a \in \Gamma(M^{2n})$ as follows:

$$K_a = \varphi_* \cdot a. \tag{1.3}$$

Since the manyfold M^{2n} is symplectic, one can naturally define for any $a \in \mathcal{G}$ a function $H_a \in \mathcal{D}(M^{2n})$ as follows:

$$- i_{K_a}\omega^{(2)} = dH_a, \tag{1.4}$$

existing due to the invariance property

$$L_{K_a}\omega^{(2)} = 0 \tag{1.5}$$

for all $a \in \mathcal{G}$. The following lemma [1] is useful in applications.

Lemma 1.1. *If the first homology class $H_1(\mathcal{G};\mathbb{R})$ of the Lie algebra $\mathcal{G}$ vanishes, then the mapping $\Phi : \mathcal{G} \to \mathcal{D}(M^{2n})$* defined as

$$\Phi(a) := H_a \tag{1.6}$$

for any $a \in \mathcal{G}$, is a homomorphism of a Lie algebras $\mathcal{G}$ and $\mathcal{D}(M^{2n})$ (endowed with the Lie structure induced by the symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$), is called Poissonian.

As the mapping $\Phi : \mathcal{G} \to \mathcal{D}(M^{2n})$ is evidently linear in $\mathcal{G}$, the expression (1.6) defines naturally a so called momentum mapping $l : M^{2n} \to \mathcal{G}^*$ as follows: for any $u \in M^{2n}$ and all $a \in \mathcal{G}$

$$(l(u), a)_{\mathcal{G}} := H_a(u), \tag{1.7}$$

where $(\cdot,\cdot)_{\mathcal{G}}$ some scalar product on dual pair $\mathcal{G}^* \times \mathcal{G}$. The following characteristic equivariance [1] lemma holds.

Lemma 1.2. *The diagram*

$$\begin{array}{ccc} M^{2n} & \xrightarrow{l} & \mathcal{G}^* \\ \varphi_g \downarrow & & \downarrow Ad^*_{g^{-1}} \\ M^{2n} & \xrightarrow{l} & \mathcal{G}^* \end{array} \tag{1.8}$$

is commuting for all $g \in G$, where $Ad^*_{g^{-1}} : \mathcal{G}^* \to \mathcal{G}^*$ -the corresponding co-adjoint action of the Lie group G on the dual space $\mathcal{G}^*$.

Take now any vector $h \in \mathcal{G}^*$ and consider a subspace $\mathcal{G}_h \subset \mathcal{G}$, consisting of elements $a \in \mathcal{G}$, such that $ad^*_a h = 0$, where $ad^*_a : \mathcal{G}^* \to \mathcal{G}^*$-the corresponding Lie algebra $\mathcal{G}$ representation in the dual space $\mathcal{G}^*$.

The following lemmas hold.

Lemma 1.3. *The subspace $\mathcal{G}_h \subset \mathcal{G}$ is a Lie subalgebra of $\mathcal{G}$, called here a Cartan subalgebra.*

Lemma 1.4. *Assume a vector $h \subset \mathcal{G}^*$ is chosen in such a way that $r = dim \mathcal{G}_h$ is the least as possible. Then the Cartan Lie subalgebra $\mathcal{G}_h \subset \mathcal{G}$ is abelian.*

If there is the case of Lemma1.4, the corresponding element $h \subset \mathcal{G}^*$ is called regular and the number $r = \dim\ \mathcal{G}_h$ is called the $rank\mathcal{G}$ of the Lie algebra $\mathcal{G}$.

1.2. Some twenty years ago there was proved by Mishchenko and Fomenko [2] the following important noncommutative (nonabelian) Liouville-Arnold theorem.

Theorem 1.5. *Let on a symplectic space $(M^{2n}, \omega^{(2)})$ there be given a set of smooth functions $H_j \in \mathcal{D}(M^{2n})$, $j = \overline{1,k}$, whose linear span over $\mathbb{R}$ composes a Lie algebra $\mathcal{G}$ with respect to the corresponding Poisson bracket on M^{2n}. Suppose also that the set*

$$M_h^{2n-k} := \{u \in M^{2n} : h(H_j) = h_j \in \mathbb{R},\ \ j = \overline{1,k},\ \ \ h \in \mathcal{G}^*\}$$

with $h \in \mathcal{G}^*$ being regular, is a submanifold of M^{2n}, and on M_h^{2n-k} all the functions $H_j \in \mathcal{D}(M^{2n})$, $j = \overline{1,k}$, are functionally independent. Assume also that the Lie algebra $\mathcal{G}$ built above satisfies the following condition:

$$\dim \mathcal{G} + rank\mathcal{G} = \dim M^{2n}. \tag{1.9}$$

then the submanifold $M_h^r := M_h^{2n-k}$ *is rank*$\mathcal{G} = r-$*dimensional and invariant with respect to each vector field* $K_{\overline{a}} \in \Gamma(M^{2n})$ *with* $\overline{a} \in \mathcal{G}_h \subset \mathcal{G}$ *.Given a vector field* $K = K_{\overline{a}} \in \Gamma(M^{2n})$ *with* $\overline{a} \in \mathcal{G}_h$ *or* $K \in \Gamma(M^{2n})$ such that $[K, K_a] = 0$ *for all* $a \in \mathcal{G}$, *then, if the submanifold* M_h^r *is connected and compact, it is diffeomorphic to the* $r-$*dimensional torus* $\mathbb{T}^r \simeq M_h^r$ *and the motion of the vector field* $K \in \Gamma(M^{2n})$ *on it is a quasiperiodic function of the evolution parameter* $t \in \mathbb{R}$.

The most simple proof of this theorem can be obtained from the well known [3] classical Lie -Cartan theorem, mentioned in Introduction. Below we shall only make a sketch of the original Mishchenko-Fomenko proof based heavily on the symplectic theory techniques, drawn in part above.

◀*A sketch of the proof.*Define a Lie group G naturally as $G = \exp \mathcal{G}$, where $\mathcal{G}$- the Lie algebra of functions $H_j \in \mathcal{D}(M^{2n})$, $j = \overline{1,2}$, built in the Theorem , with respect to the Poisson bracket $\{\cdot,\cdot\}$ on M^{2n}. Then for an element $h \in \mathcal{G}^*$ and any $a = \sum_{j=1}^k c_j H_j \in \mathcal{G}$, where $c_j \in \mathbb{R}$, $j = \overline{1,k}$, the following equality

$$(h, a)_{\mathcal{G}} := \sum_{j=1}^{k} c_j h(H_j) = \sum_{j=1}^{k} c_j h_j \tag{1.10}$$

holds. Since upon the level submanifold $M_h^r \subset M^{2n}$ all functions $H_j \in \mathcal{D}(M^{2n})$, $j = \overline{1,2}$, are functionally independent, this evidently means that the element $h \in \mathcal{G}^*$ is regular for the Lie algebra $\mathcal{G}$. As a result one gets that the Cartan Lie subalgebra $\mathcal{G}_h \subset \mathcal{G}$ is abelian. The latter is proved by means of simple straightforward calculations. Moreover, the corresponding momentum mapping $l : M^{2n} \to \mathcal{G}^*$ is constant on M_h^r, satisfying the following relation:

$$l(M_h^r) = h \in \mathcal{G}^*. \tag{1.11}$$

This makes it possible to state that all vector fields $K_{\overline{a}} \in \Gamma(M^{2n})$, $\overline{a} \in \mathcal{G}_h$, are tangent to the submanifold $M_h^r \subset M^{2n}$. Thus the corresponding Lie subgroup $G_h := \exp \mathcal{G}_h$ acts naturally on M_h^r, persisting it invariant. If the submanifold $M_h^r \subset M^{2n}$ is connected and compact, due to (1.9) $\dim M_h^r = \dim M^{2n} - \dim \mathcal{G} = rank \mathcal{G} = r$, one obtains via the Arnold theorem [4] , that $M_h^r \simeq \mathbb{T}^r$ and the motion of the mentioned in Theorem vector field $K \in \Gamma(M^{2n})$ is a quasiperiodic function of the evolution parameter $t \in \mathbb{R}$, that proves the theorem. ▶

As a nontrivial consequence of the Lie-Cartan theorem mentioned before and of the theorem 1.5, one can prove the following dual theorem about the abelian Liouville-Arnold integrability.

Theorem 1.6. *Let a vector field* $K \in \Gamma(M^{2n})$ *is completely integrable via the nonabelian scheme of theorem 1.5. Then it is also Liouville-Arnold integrable on* M^{2n} *, possessing at some additional conditions another yet abelian Lie algebra* $\mathcal{G}_h$ *of functionally independent invariants on* M^{2n}*, for which* $\dim \mathcal{G}_h = n = 1/2 \dim M^{2n}$.

The available proof of the theorem above is rather complicated and nontrivial , on which we shall in detail further. Mention here only , that some analogs of the reduction theorem 1.5 in case when on the manifold $M^{2n} \simeq \mathcal{G}^*$ there is acting symplectically an arbitrary Lie group G, were proved also in [6-10]. Notice here, that in case when the equality (1.9) is not satisfied , one can then construct in a usual way the reduced manifold $\overline{M}_h^{2n-k-r} := M_h^{2n-k}/G_h$ on which there exists a symplectic structure $\overline{\omega}_h^{(2)} \in \Lambda^2(\overline{M}_h^{2n-k-r})$, defined as

$$r_h^* \overline{\omega}_h^{(2)} = \pi_h^* \omega^{(2)} \tag{1.12}$$

with respect to the following compatible reduction-imbedding diagram:

$$\overline{M}_h^{2n-k-r} \overset{r_h}{\longleftarrow} M_h^{2n-k} \overset{\pi_h}{\rightarrow} M^{2n}, \tag{1.13}$$

where $r_h : M_h^{2n-k} \rightarrow \overline{M}_h^{2n-k-r}$ and $\pi_h : M_h^{2n-k} \rightarrow M^{2n}$ - the corresponding reductions and imbedding mappings. The nondegeneracy of the 2-form $\overline{\omega}_h^{(2)} \in \Lambda^2(\overline{M}_h)$ defined by (1.12), follows simply from the expression

$$\ker(\pi_h^* \omega^{(2)}(u)) = T_u(M_h^{2n-k}) \cap T_u^{\perp}(M_h^{2n-k}) = span_{\mathbb{R}}\{K_{\overline{a}}(u) \in T_u(\overline{M}_h^{2n-k-r} := M_h^{2n-k}/G_h) : \overline{a} \in \mathcal{G}_h\} \tag{1.14}$$

for any $u \in M_h^{2n-k}$, since all vector fields $K_{\overline{a}} \in \Gamma(M^{2n})$, $\overline{a} \in \mathcal{G}_h$, are tangent to $\overline{M}_h^{2n-k-r} := M_h^{2n-k}/G_h \subset M^{2n}$. Thus, the reduced space $\overline{M}_h^{2n-k-r} := M_h^{2n-k}/G_h$ with respect to the orbits of the Lie subgroup G_h action on M_h^{2n-k} will be a $(2n-k-r)$-dimensional symplectic manifold. The latter evidently means that the number $2n-k-r = 2s \in \mathbb{Z}_+$ is even as there no symplectic structure on odd-dimensional manifolds. This obviously is closely connected with the problem of existence a symplectic group action of a Lie group G on a given symplectic manifold $(M^{2n}, \omega^{(2)})$ with a symplectic structure $\omega^{(2)} \in \Lambda^{(2)}(M^{2n})$ being *a′priori* fixed. From this point of view one can consider the inverse problem of constructing symplectic structures on a manifold M^{2n} , admitting a Lie group G action. Namely, owing to the momentum mapping $l : M^{2n} \to \mathcal{G}^*$ equivariance property (1.8), one can obtain the induced symplectic structure $l^*\Omega_h^{(2)} \in \Lambda^2(\overline{M}_h^{2n-k-r})$ on $\overline{M}_h^{2n-k-r}$ from the canonical symplectic structure $\Omega_h^{(2)} \in \Lambda^{(2)}(Or(h;G))$ on the orbit $Or(h;G) \subset \mathcal{G}^*$ of a regular element $h \in \mathcal{G}^*$. Since the symplectic structure $l^*\Omega_h^{(2)} \in \Lambda^2(\overline{M}_h)$ can be naturally lifted to 2-form $\widetilde{\omega}^{(2)} = (r_h^* \circ l^*)\Omega_h^{(2)} \in \Lambda^2(M_h^{2n-k})$, the latter being on M_h^{2n-k} degenerate, due to the imbedding $\pi_h : M_h^{2n-k} \to M^{2n}$ can be further apparently nonuniquely extended on the whole manifold M^{2n} to a symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$, with the action of the Lie group G on which being *a′priori* symplectic. Thus, many properties of a given dynamical system with a Lie algebra $\mathcal{G}$ of invariants on a manifold M^{2n} are deeply connected with a symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$ it is endowed, and in particular, with the corresponding integral submanifold imbedding mapping $\pi_h : M_h^{2n-k} \to M^{2n}$ at a regular element $h \in \mathcal{G}^*$. The problem of direct algebraic-analytical constructing this mapping was in part solved in [11] at $n = 2$ for an abelian algebra $\mathcal{G}$ on the manifold $M^4 = T^*(\mathbb{R}^2)$. The treatment of this problem in [11] has been heavily based both on

the classical Cartan studies of integral submanifolds of ideals in Grassmann algebras and on the modern Galisau-Reeb-Francoise results subject the symplectic manifold $(M^{2n}, \omega^{(2)})$ structure, on which there exists an involutive set $\mathcal{G}$ of functionally independent invariants $H_j \in \mathcal{D}(M^{2n})$, $j = \overline{1,n}$. In what will follow below we succeeded in generalizing the important Galisau-Reeb-Francoise results for the case of a non-abelian set of functionally independent functions $H_j \in \mathcal{D}(M^{2n})$, $j = \overline{1,k}$, composing a Lie algebra $\mathcal{G}$ and satisfying the sufficient Mishchenko-Fomenko condition (1.9): $\dim \mathcal{G} + rank \mathcal{G} = \dim M^{2n}$. The latter made it possible to devise an effective algebraic-analytical method of constructing the corresponding integral submanifold imbedding and reduction mappings, giving rise to a wide class of integrable by quadratures exact solutions of a given integrable vector field on M^{2n}.

2 Integral submanifold imbedding mapping problem: the case of an abelian Lie algebra of invariants

2.1. We shall consider here only a set $\mathcal{G}$ of commuting polynomial functions $H_j \in \mathcal{D}(M^{2n})$, $j = \overline{1,n}$, on the canonically symplectic phase space $M^{2n} = T^*(\mathbb{R}^n)$. Due to the Liouville -Arnold theorem[4] any dynamical system

$K \in \Gamma(M^{2n})$, commuting with corresponding Hamiltonian vector fields K_a for all $a \in \mathcal{G}$, will be integrable by quadratures in case of a regular element $h \in \mathcal{G}^*$, which defines the corresponding integral submanifold $M_h^n := \{u \in M^{2n} : h(H_j) = h_j \in \mathbb{R}, \ \ j = \overline{1,n}\}$ being diffeomorphic (when compact and connected) to the $n-$dimensional torus $\mathbb{T}^n \simeq M_h^n$. This in particular means that there exists some algebraic-analytical expression for the integral submanifold imbedding mapping $\pi_h : M_h^n \rightarrow M^{2n}$ into the ambient phase space M^{2n},which one should find for the proper integrability by quadratures.

The problem formulated above was posed and in part solved as was mentioned before, for $n = 2$ in [11] and in [13] in the case of a Henon-Heiles dynamical system integrated before in [14,15] by other tools. We shall generalize in this chapter the approach from [11] for the general case $n \in \mathbb{Z}_+$ and proceed further in chapter 3 to solving this problem in the case of a nonabelian Lie algebra $\mathcal{G}$ of polynomial invariants on $M^{2n} = T^*(\mathbb{R}^n)$, satisfying all the conditions of Mishchenko-Fomenko theorem 1.5.

2.2. Define now the basic vector fields $K_j \in \Gamma(M^{2n})$, $j = \overline{1,n}$, generated by basic elements $H_j \in \mathcal{G}$ of an abelian Lie algebra $\mathcal{G}$ of invariants on M^{2n}, as follows:

$$- i_{K_j}\omega^{(2)} = dH_j \tag{2.1}$$

for all $j = \overline{1,n}$. it is easy to see that the condition $\{H_j, H_i\} = 0$ for all $i,j = \overline{1,n}$, yields also $[K_i, Kj] = 0$ for all $i,j = \overline{1,n}$. Taking into account now that $\dim M^{2n} = 2n$ one obtains the equality $(\omega^{(2)})^n = 0$ identically on M^{2n}. This makes it possible to formulate the following Galisau-Reeb result.

Theorem 2.1. *Assume that an element $h \in \mathcal{G}^*$ chosen above is regular an a Lie algebra $\mathcal{G}$ of invariants on M^{2n} is abelian. Then there exist differential 1-forms $h_j^{(1)} \in \Lambda^1(U(M_h^n))$, $j = \overline{1,n}$, where $U(M_h^n)-$ some open* *neighborhood of the integral submanifold* $M_h^n \subset M^{2n}$, *satisfying the following properties:*

i) $\omega^{(2)}\big|_{U(M_h^n)} = \sum_{j=1}^n dH_j \wedge h_j^{(1)}$;

ii) external differentials $dh_j^{(1)} \in \Lambda^2(U(M_h^n))$ *belong to the ideal* $\mathcal{I}(\mathcal{G})$ *in the Grassmann algebra* $\Lambda(U(M_h^n))$, *generated by* 1*-forms* $dH_j \in \Lambda^1(U(M_h^n))$,

$j=\overline{1,n}$.

◀*Proof.* Consider the following identity on M^{2n} :

$$(\otimes_{j=1}^n i_{K_j})(\omega^{(2)})^{n+1} = 0 = \pm(n+1)!(\wedge_{j=1}^n dH_j) \wedge \ \omega^{(2)}, \tag{2.2}$$

meaning evidently that 2-form $\omega^{(2)} \in \mathcal{I}(\mathcal{G})$. The latter makes it possible to find some 1-forms $h_j^{(1)} \in \Lambda^1(U(M_h^n))$, $j=\overline{1,n}$, satisfying the

condition

$$\omega^{(2)}\big|_{U(M_h^n)} = \sum_{j=1}^n dH_j \wedge h_j^{(1)}. \tag{2.3}$$

Since 2-form $\omega^{(2)} \in \Lambda^2(U(M_h^n))$ is everywhere on M^{2n} nondegenerate, it yields that all 1-forms $h_j^{(1)}$, $j=\overline{1,n}$, in (2.3) are independent on $U(M_h^n)$, proving part *i)* of the theorem. Having made use of the next identity $d\omega^{(2)} = 0$ on the entire space M^{2n}, from (2.3) one gets that

$$\sum_{j=1}^n dH_j \wedge dh_j^{(1)} = 0 \tag{2.4}$$

on $U(M_h^n)$, meaning obviously that $dh_j^{(1)} \in \mathcal{I}(\mathcal{G}) \subset \Lambda(U(M_h^n))$ for all $j=\overline{1,n}$, proving part *ii)* of the theorem.▶

Now we proceed to studying properties of the integral submanifold $M_h^n \subset M^{2n}$ of the ideal $\mathcal{I}(\mathcal{G})$ in the Grassmann algebra $\Lambda(U(M_h^n))$. In general, integral submanifold M_h^n is completely described [16] by means of the imbedding mapping

$$\pi_h : M_h^n \to M^{2n} \tag{2.5}$$

based on which one can naturally reduce all vector fields $K_j \in \Gamma(M^{2n})$, $j = \overline{1,n}$, upon the submanifold $M_h^n \subset M^{2n}$, since all they are evidently tangent to it: if $\overline{K}_j \in \Gamma(M_h^n)$, $j = \overline{1,n}$, are corresponding pullbacked vector fields $K_j \in \Gamma(M^{2n})$, $j = \overline{1,n}$, then by definition, the equality

$$\pi_h^* \circ \overline{K}_j = K_j \circ \pi_h \tag{2.6}$$

holds for all $j = \overline{1,n}$. Similarly one can construct 1-forms $\overline{h}_j^{(1)} := \pi_h^* \circ h_j^{(1)} \in \Lambda^1(M_h^n)$, $j = \overline{1,n}$, which are characterized by the following Cartan-Jost [16] theorem.

Theorem 2.2. *The following assertions are true:*

i) *1-forms* $\overline{h}_j^{(1)} \in \Lambda^1(M_h^n)$, $j = \overline{1,n}$, *are independent on* M_h^n;

ii) *1-forms* $\overline{h}_j^{(1)} \in \Lambda^1(M_h^n)$, $j = \overline{1,n}$, *are exact on* M_h^n, *satisfying the equalities* $\overline{h}_j^{(1)}(\overline{K}_j) = \delta_{ij}$, $i, j = \overline{1,n}$

◀*Proof.* As the ideal $\mathcal{I}(\mathcal{G})$ is by definition vanishing on $M_h^n \subset M^{2n}$ and closed on $U(M_h^n)$, the integral submanifold M_h^n is well defined in the case of an element $h \in \mathcal{G}^*$ being chosen regular. This yields that the imbedding mapping (2.5) is not degenerate on $M_h^n \subset M^{2n}$, or the 1-forms $\overline{h}_j^{(1)} := \pi_h^* \circ h_j^{(1)}$, $j = \overline{1,n}$, will persist to be independent if such are 1-forms $h_j^{(1)} \in \Lambda^1(U(M_h^n))$, $j = \overline{1,n}$, proving part i) of the theorem. Based now on the property ii) of the theorem 2.1, one can obtain that on the integral submanifold $M_h^n \subset M^{2n}$ all 2-forms $d\overline{h}_j^{(1)} = 0$, $j = \overline{1,n}$. This means due to the Poincare lemma [1,16], that 1-forms $\overline{h}_j^{(1)} = d\bar{t}_j \in \Lambda^1(M_h^n)$, $j = \overline{1,n}$, for some mappings $\bar{t}_j : M_h^n \to \mathbb{R}$, $j = \overline{1,n}$, defining some global coordinates on an appropriate universal covering of M_h^n. Consider now based on the representation (2.3), the following identity:

$$i_{K_j}\ \omega^{(2)}\big|_{U(M_h^n)} = -\sum_{i=1}^{n} h_i^{(1)}(K_j) \quad dH_i := -dH_j, \tag{2.7}$$

being true for any $j = \overline{1,n}$. As all $dH_j \in \Lambda^1(U(M_h^n))$, $j = \overline{1,n}$, are independent, from (2.7) one follows that $h_i^{(1)}(K_j) = \delta_{ij}$ for all $i,j = \overline{1,n}.$ Recalling now that for any $i = \overline{1,n}$, $K_i \circ \pi_h = \pi_h^{(*)} \circ K_i$, one obtains simply that $\overline{h}_i^{(1)}(\overline{K}_j) = \pi_h^* h_i^{(1)}(\overline{K}_j) := h_i^{(1)}(\pi_h^{(*)} \circ K_j) := h_i^{(1)}(K_j \circ \pi_h) = \delta_{ij}$

for all $i,j = \overline{1,n}$, proving part ii) of the theorem.▶

As a result of the theorem 2.2 one can formulate the following corollary.

Corollary 2.3. *Let vector fields* $K_j \in \Gamma(M^{2n})$, $j = \overline{1,n}$, *are parametrized globally along their trajectories by means of the corresponding parameters* $t_j : M^{2n} \to \mathbb{R}$, $j = \overline{1,n}$, *that is on the entire phase space* M^{2n}

$$d/dt_j := K_j \tag{2.8}$$

for all $j = \overline{1,n}$. *Then on the integral submanifold* $M_h^n \subset M^{2n}$ *there hold the following important equalities*

$$t_j|_{M_h^n} = \overline{t}_j, \tag{2.9}$$

where $j = \overline{1,n}$, *up to the constant normalizations.*

2.3. To proceed with studying the algebraic-analytical properties of the imbedding mapping (2.5), we shall summarize some needed facts about the canonical transformations [1, 4] of symplectic manifolds and their generating functions.

Suppose there is given a symplectomorphism $\Phi : M^{2n} \to \tilde{M}^{2n}$, satisfying the condition

$$\Phi^* \tilde{\omega}^{(2)} = \omega^{(2)}, \tag{2.10}$$

where $\tilde{\omega}^{(2)} \in \Lambda^2(\tilde{M}^{2n})$ - a symplectic structure on $\tilde{M}^{2n} = T(\tilde{M}^n)$. Since by assumption $\omega^{(2)} = d \ (pr^*\alpha^{(1)})$, where $\alpha^{(1)} \in \Lambda^1(\mathbb{R}^n)$ is defined by (1.1), and there exists locally a 1-form $\tilde{\alpha}^{(1)} \in \Lambda^1(\tilde{M}^n)$, such that

$$pr^*\alpha^{(1)} - pr^*\tilde{\alpha}^{(1)} + d(\widetilde{pq}) = dS, \tag{2.11}$$

where locally, $S : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ - some differentiable function called generating. Having put on $\tilde{M}^n$

$$\tilde{\alpha}^{(1)} = \sum_{j=1}^{nj} \tilde{p}_j d\tilde{q}_j, \tag{2.12}$$

where $\tilde{p} \in T^*(\tilde{M}^n)$ - canonical local coordinates, from (2.11), (2.12) and (1.1) one gets readily that

$$p_j = \partial S(q,\tilde{p})/\partial q_j, \quad \tilde{q}_j = \partial S(q,\tilde{p})/\partial \tilde{p}_j \tag{2.13}$$

for any $j = \overline{1,n}$, the mapping $\Phi : M^{2n} \to \tilde{M}^{2n}$ should in local coordinates satisfy the following condition:

$$\det(\partial\widetilde{p}(q,p)/\partial p) \neq 0 \tag{2.14}$$

almost everywhere (a.e.) on M^{2n}. Since due to (2.14) one can define a.e. the mapping $p : \mathbb{R}^n \times \tilde{M}^n \to \mathbb{R}^n$, the equations (2.13) give rise to determining the generating function $S : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ up to some constant. And conversely [4], if there is given a generating function $S : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, satisfying a.e. the condition

$$\det(\partial^2 S(q,\widetilde{p})/\partial q \partial p) \neq 0 \tag{2.15}$$

on $\mathbb{R}^n \times \mathbb{R}^n$, then one can determine a canonical transformation $\Phi : M^{2n} \to \tilde{M}^{2n}$ of symplectic manifolds M^{2n} and $\tilde{M}^{2n}$, satisfying a.e. the condition (2.14).

Assume now additionally that there holds the case when a submanifold $\tilde{M}^n \subset \tilde{M}^{2n}$ coincides up to a diffeomorphism with the integral submanifold

M_h^n of the ideal $\mathcal{I}(\mathcal{G})$ considered before , and thereby the corresponding symplectic manifold $\tilde{M}^{2n}_{(\mathcal{G}^*)} := \cup_{h\in\mathcal{G}^*} T^*(M_h^n)-$ the usual topological sum of cotangent spaces, giving rise to a natural fibration of the symplectic manifold $M^{2n} = T^*(\mathbb{R}^n)$:

$$M^{2n} = T^*(\mathbb{R}^n) \simeq \cup_{h\in\mathcal{G}^*} T^*(M_h^n) = \tilde{M}^{2n}_{(\mathcal{G}^*)}. \tag{2.16}$$

The representation (2.16) appears to be very useful for treating the imbedding mapping (2.5). Namely, assume for further that the integral submanifold $M_h^n \subset M^{2n}$ admits a.e. on M_h^n coordinate charts from the base space $\mathbb{R}^n$

of the entire phase space $M^{2n} = T^*(\mathbb{R}^n)$. This evidently means that the set of annihilating on M_h^n 1-forms $dH_j \in \Lambda^1(M^{2n})$, $j = \overline{1,n}$, must be

solvable with respect to the cotangent differentials $dp_j \in \Lambda^1(M_h^n)$, $j = \overline{1,n}$:

$$\{dH_j|_{M_h^n} = 0, \quad j = \overline{1,n}\} \Rightarrow \{dp_j = \sum_{k=1}^{n} Q_{jk}(q,p)dq_j : (q,p) \in T^*(M_h^n)\}, \tag{2.17}$$

where $Q : T^*(M_h^n) \to Hom(\mathbb{R}^n)$ - some invertible a.e. mapping. The

implication (2.27) makes it possible to incorporate existing due to the Arnold theorem [4] on the integral submanifold M_h^n some special cyclic coordinates realizing the isomorphisms $M_h^n \simeq \mathbb{T}^n \simeq \otimes_{j=1}^n \mathbb{S}_j^1$ (in case when it is compact

and connected that we shall assume for further). Since the entire phase space M^{2n} can be evidently represented due to (2.16) up to a symplectomorphism as

$$M^{2n} = T^*(\mathbb{R}^n) \simeq \cup_{h\in\mathcal{G}^*} T^*(\otimes_{j=1}^n \mathbb{S}_j^1) \simeq \tilde{M}^{2n}_{(\mathcal{G}^*)} \tag{2.18}$$

and the integral submanifold $M_h^n \subset M^{2n}$ can be covered by charts with images in the base space $\mathbb{R}^n \subset T^*(\mathbb{R}^n)$ of the phase space M^{2n}, on the integral submanifold M_h^n there is induced a canonical 1-form $\alpha_h^{(1)} \in \Lambda^1(M_h^n)$ with respect to the imbedding mapping (2.5), projected upon the base space $\mathbb{R}^n : \alpha_h^{(1)} := \pi_h^* \circ pr^* \alpha^{(1)}$. This means that the following diagram is commutative:

$$\begin{array}{ccccccc} T^*(T^*(\mathbb{R}^n)) & \xrightarrow{\pi_h^*} & T^*(M_h^n) & \simeq|_{loc} & T^*(\mathbb{R}^n) & \xrightarrow{pr^*} & T^*(T^*(\mathbb{R}^n)) \\ pr^{'}\downarrow & & \downarrow pr_h & & \downarrow pr & & \downarrow pr^{'} \\ M^{2n} = T^*(\mathbb{R}^n) & \xleftarrow{\pi_h} & M_h^n & \simeq|_{loc} & \mathbb{R}^n & \xleftarrow{pr} & M^{2n} = T^*(\mathbb{R}^n) \end{array} ,$$

where $pr^{'} : T^*(T^*(\mathbb{R}^n)) \to T^*(\mathbb{R}^n)$ is the standard projection mapping . The representation (2.19) together with the isomorphism $M_h^n \simeq \otimes_{j=1}^n \mathbb{S}_j^1$ means in particular, that in own circle-like coordinates $\mu \in \otimes_{j=1}^n \mathbb{S}_j^1$ the canonical 1-forms $\alpha_h^{(1)} \in \Lambda^1(U(M_h^n))$ can be written down as

$$\alpha_h^{(1)} = \sum_{j=1}^n w_j d\mu_j = \pi_h^* \circ pr^* \alpha^{(1)}, \qquad (2.19)$$

which is naturally lifted to a canonical 1-form $pr_h^* \circ \alpha_h^{(1)} \in \Lambda^1(T^*(U(M_h^n)))$, where $pr_h : T^*(M_h^n) \to M_h^n$ - the natural projection upon the base M_h^n, $w : \otimes_{j=1}^n \mathbb{S}_j^1 \to \mathbb{R}^n$ - some parametrized by $h \in \mathcal{G}^*$ smooth a.e. mapping. Thus, in local coordinates on $T^*(\otimes_{j=1}^n \mathbb{S}_j^1)$ the imbedding mapping (2.5) parametrized by regular elements $h \in \mathcal{G}^*$, has the following form:

$$q_j = q_j(\mu; h), \qquad p_j = p_j(\mu; h), \qquad (2.20)$$

where in virtue of (2.20) for any $j = \overline{1,n}$

$$p_j = \sum_{i=1}^n w_i(\mu; h) \partial \mu_i(q; h)/\partial q_j \Big|_{M_h^n}, \qquad (2.21)$$

with the mapping $q := pr \circ \pi_h : M_h^n \to \mathbb{R}^n$ being invertible owing to the implication (2.17), $\mu : \mathbb{R}^n \to \otimes_{j=1}^n \mathbb{S}_j^1$ - its inverse, and some mapping $w : (\otimes_{j=1}^n \mathbb{S}_j^1) \times \mathcal{G}^* \to \mathbb{R}^n$, still not defined. The above analysis of the imbedding mapping problem for the integral submanifold $M_h^n \subset M^{2n}$ in the case when the implication (2.17) is solvable, tells us that the Liouville-Arnold foliation (2.18) can be described effectively by means of choosing such a special parametrization by elements $h \in \mathcal{G}^*$, for which the mapping $w : (\otimes_{j=1}^n \mathbb{S}_j^1) \times \mathcal{G}^* \to \mathbb{R}^n$ will be separable in the variables $\mu \in \otimes_{j=1}^n \mathbb{S}_j^1$, that is on M_h^n the expressions

$$w_j = w_j(\mu_j; h) \tag{2.22}$$

should hold for all $j = \overline{1,n}$. Such a case is called [1, 4] the Hamilton-Jacobi separation of variables method which can be now naturally applied to our problem of finding the embedding mapping (2.5) algebraic -analytically.

To proceed with, apply first to our manifold $\tilde{M}^{2n}$ as defined in (2.18), such a canonical transformation into a new fibration $\tilde{M}^{2n}_{(\mathbb{R}^n)}$ of the phase space $M^{2n} = T^*(\mathbb{R}^n)$, built as follows:

$$M^{2n} = T^*(\mathbb{R}^n) \simeq \cup_{\tau \in \mathbb{R}^n} T^*(M_\tau^n) := \tilde{M}^{2n}_{(\mathbb{R}^n)}, \tag{2.23}$$

where by definition, M_τ^n is an integral submanifold of of a dual integrable ideal $\mathcal{I}(h^{(1)}) \subset \Lambda(U(M_h^n))/\mathcal{I}(\mathcal{G}^*)$, being generated by closed 1-forms $\tilde{h}_j^{(1)} \in \Lambda^1(U(M_h^n))/\mathcal{I}(\mathcal{G}^*)$, $j = \overline{1,n}$, obtained via some continuation of the closed on M_h^n 1-forms $\bar{h}_j^{(1)} \in \Lambda^1(M_h^n)$, $j = \overline{1,n}$, persisting their closedness. The latter obviously means that the so constructed ideal $\mathcal{I}(h^{(1)})$ is integrable too on $U(M_h^n) \subset M^{2n}$, possessing integral submanifolds $M_\tau^n \subset U(M_h^n)$ parametrized by a constant vector parameter $\tau := (t_1, t_2, ..., t_n) \in \mathbb{R}^n$, composed of evolution parameters $t_j \in \mathbb{R}$, $j = \overline{1,n}$, of corresponding vector fields $K_j = d/dt_j$, $j = \overline{1,n}$, on the neighborhood $U(M_h^n) \subset M^{2n}$. As a result we have stated that two fibrations $\tilde{M}^{2n}_{(\mathcal{G}^*)}$ and $\tilde{M}^{2n}_{(\mathbb{R}^n)}$ are locally diffeomorphic in an open vicinity $U(M^n_h)$ of the integral submanifold $M^n_h \subset M^{2n}$. Having forced them to be symplectomorphic, one can write down that

$$\alpha^{(1)}_{\tau} = -\sum_{j=1}^{n} t_j \, dh_j = \pi^*_{\tau} pr^* \alpha^{(1)}, \tag{2.24}$$

holds on the integral submanifold M^n_{τ} for the canonical 1-form $\alpha^{(1)}_{\tau} \in \Lambda^1(M^n_{\tau})$, where $\pi_{\tau} : M^n_{\tau} \to M^{2n}$ - the corresponding imbedding mapping. Recall now the fact, that each symplectomorphism of the manifold M^{2n} is described [1, 4] by means of the corresponding generating function $S : M^n_h \times \mathbb{R}^n \to \mathbb{R}$, where by definition,

$$\alpha^{(1)}_{h} = \alpha^{(1)}_{\tau} + dS. \tag{2.25}$$

Making use now of the expression (2.20) and (2.25) , one obtains the local relationship

$$\sum_{j=1}^{n} w_j d\mu_j + \sum_{j=1}^{n} t_j dh_j = dS(\mu; h) \tag{2.26}$$

on some open neighborhood $M^{2n}_{h,\tau} \subset U(M^n_h) \cap U(M^n_{\tau})$, where $U(M^n_{\tau}) \subset M^{2n}$ - an open neighborhood of the integral submanifold $M^n_{\tau} \subset M^{2n}$.

Thus involving now into (2.27) the Hamilton-Jacobi separability condition (2.23), one gets

$$S(\mu; h) = \sum_{j=1}^{n} \int_{\mu^0_j}^{\mu_j} w_j(\lambda; h) d\lambda \tag{2.27}$$

for all $(\mu; h) \in M^{2n}_{h,\tau} \subset M^{2n}$, where $\mu^0 \in M^n_h \cap M^{2n}_{h,\tau}$ - some fixed point.

Whence one obtains the following searched expression for a vector parameter $\tau := (t_1, t_2, ..., t_n) \in \mathbb{R}^n$:

$$t_j = \partial S(\mu; h) / \partial h_j, \tag{2.28}$$

where $j=\overline{1,n}$ and $(\mu;h)\in M^{2n}_{h,\tau}$. On the other hand, the set of parameters (2.29) in virtue of the equality (2.9) can be represented dually as differential forms

$$\bar{h}^{(1)}_j = dt_j = \partial dS\,(\mu;h)\,/\partial h_j = \sum_{i=1}^{n} (\partial w_i(\mu_i;h)/\partial h_j)\ \ d\mu_i \qquad (2.29)$$

for all $j=\overline{1,n}$ on $M^{2n}_{h,\tau}$. Since 1-forms $\bar{h}^{(1)}_j\in\Lambda^1(M^{2n}_{h,\tau})$, $j=\overline{1,n}$, are assumed here to be known explicitly from the characteristic equation (2.3), we can them write down as

$$\bar{h}^{(1)}_j = \sum_{i=1}^{n} \bar{h}_{ji}(q,p)dq_i, \qquad (2.30)$$

where $\bar{h}_{ji}:M^{2n}_{h,\tau}\to\mathbb{R}$, $i,j=\overline{1,n}$ - some algebraic expressions. Making use further of the representations (2.21) and (2.28), the set of 1-forms (2.30) due to (2.31) is reduced to the following purely differential- algebraic relationships on $M^{2n}_{h,\tau}$:

$$\partial w_i(\mu_i;h)/\partial h_j = \mathbf{P}_{ji}(\mu,w;h), \qquad (2.31)$$

generalizing similar ones of [31,18], where characteristic functions $\mathbf{P}_{ji}:T^*(M^n_h)\to\mathbb{R}$, $i,j=\overline{1,n}$, are defined as follows:

$$\mathbf{P}_{ji}(\mu,w;h) := \sum_{i=1}^{n} \bar{h}_{js}(q(\mu;h), w\ \partial\mu/\partial q(\mu;h))\ \partial q_s/\partial\mu_i\ . \qquad (2.32)$$

A simple analysis of the relationships (2.32) and (2.33) tells us that for all $j=\overline{1,n}$ and $i\neq s\overline{=1,n}$ the following algebraic relations

$$\partial\mathbf{P}_{ji}(\mu;w;h)/\partial w_s = 0 \qquad (2.33)$$

if $i\neq s$, must be satisfied identically. It is rather clear that the produced above set of purely differential - algebraic relationships (2.33) and

(2.34) makes it possible to write down explicitly some first order compatible differential-algebraic equations, whose solution yields the first half of the sought imbedding mapping (2.5) for the integral submanifold $M_h^n \subset M^{2n}$ in an open neighborhood $M_{h,\tau}^{2n} \subset M^{2n}$ introduced before.

Let a mapping $q = \bar{q}(\mu; h)$, $(\mu; h) \in M_{h,\tau}^{2n}$, be an appropriate algebraic solution to equations (2.34). Whence having substituted it into the characteristic equations (2.32), one obtains due to (2.34) the next set of main characteristic Picard-Fuchs type [18,19] equations on $T^*(M_h^n)$:

$$\partial w_i(\mu_i; h)/\partial h_j = \bar{\mathbf{P}}_{ji}(\mu_i, w_i; h) \tag{2.34}$$

for all $i, j = \overline{1, n}$, where by definition,

$$\bar{\mathbf{P}}_{ji}(\mu_i, w_i; h) = \mathbf{P}_{ji}(\mu; w; h)|_{M_h^n \simeq \otimes_j^n \mathbb{S}_j^1} = \tag{2.35}$$

$$\sum_{i=1}^{n} \bar{h}_{js}(q(\mu; h), w\partial\mu/\partial q(\mu; h))\partial q_s/\partial\mu_i \ .$$

As a result of computations produced above one can formulate the following main theorem.

Theorem 2.4. *The imbedding mapping (2.5) for the integral submanifold $M_h^n \subset M^{2n}$ (compact and connected), parametrized by a regular parameter $h \in \mathcal{G}^*$, is an algebraic solution (up to diffeomorphism) to the set of characteristic Picard-Fuchs type equations (2.35) on $T^*(M_h^n)$, being representable in general case [19] in the following algebraic-geometric form:*

$$w_j^{n_j} + \sum_{s=1}^{n} c_{js}(\lambda; h) w_j^{n_j - s} = 0, \tag{2.36}$$

where $c_{js} : \mathbb{R} \times \mathcal{G}^* \rightarrow \mathbb{R}$, $s, j = \overline{1, n}$ - some algebraic expressions, depending only on the functional structure of the origin abelian Lie a algebra $\mathcal{G}$ of invariants on M^{2n}. In particular if the right hand side of the characteristic *equations (3.5) doesn't depend on* $h \in \mathcal{G}^*$, *then this dependence will be linear in* $h \in \mathcal{G}^*$.

It is necessary to notice here that some ten years ago there was made in [18,19] an attempt to describe the explicit algebraic form of Picard-Fuchs type equations (2.35) by means of straightforward calculations for the well known completely integrable Kowalewskaya top Hamiltonian system. The idea suggested in [18,19] was in some aspects very close to that devised independently in [11] and thoroughly treated in this article, but saying nothing about the explicit obtaining of algebraic curves (2.37) starting from an abelian Lie algebra $\mathcal{G}$ of invariants on a canonically symplectic phase space M^{2n}.

A set of algebraic curves (2.31), prescribed via the devised above algorithm, to a given *a priori* abelian Lie algebra $\mathcal{G}$ of invariants on the canonically symplectic phase space $M^{2n} = T^*(\mathbb{R}^n)$, as well known is realized by means of a set of n_j–sheeted Riemannian surfaces $\Gamma_h^{n_j}$, $j = \overline{1,n}$, covering the corresponding real valued cycles $\mathbb{S}_j^1$, $j = \overline{1,n}$, generate the corresponding homology group $H_1(\mathbb{T}^n;\mathbb{Z})$ of the Arnold torus $\mathbb{T}^n \simeq \otimes_{j=1}^n \mathbb{S}_j^1$, diffeomorphic to the integral submanifold $M_h^n \subset M^{2n}$.

Thus, having solved the set of algebraic equations (2.37) with respect to functions $w_j : \mathbb{S}_j^1 \times \mathcal{G}^* \to \mathbb{R}$, $j = \overline{1,n}$, from (2.29) one obtains that a vector parameter $\tau \in \mathbb{R}^n$ on M_h^n explicitly described by means of the following of Abelian type equations:

$$t_j = \sum_{s=1}^{n} \int_{\mu_s^0}^{\mu_s} d\lambda \ \ \partial w_s(\lambda;h)/\partial h_j = \tag{2.37}$$

$$= \sum_{s=1}^{n} \int_{\mu_s^0}^{\mu_s} d\lambda \ \bar{\mathbf{P}}_{js}(\lambda, w_s; h),$$

where $j = \overline{1,n}$, $(\mu^0; h) \in \Gamma_h^{n_j} \times \mathcal{G}^*$. Based now on the expression (2.28) and recalling that the generating function $S : M_h^n \times \mathbb{R}^n \to \mathbb{R}$ is a one-valued mapping on an appropriate covering space $(\bar{M}_h^n; H_1(M_h^n;\mathbb{Z}))$, one can build via Arnold [4] the so called action-angle coordinates on M_h^n. Denote $\sigma_j \subset M_h^n$, $j = \overline{1,n}$, - basic oriented cycles on M_h^n, generating with their duals its homology group its homology group $H_1(M_h^n;\mathbb{Z}) \simeq H_1(\mathbb{T}^n;\mathbb{Z}) = \oplus_{j=1}^n \mathbb{Z}_j$. In virtue of the diffeomorphism $M_h^n \simeq \otimes_{j=1}^n \mathbb{S}_j^1$, built above, there is a one-to-one correspondence between basic cycles of $H_1(M_h^n;\mathbb{Z})$ and those on the algebraic curves $\Gamma_h^{n_j}$, $j = \overline{1,n}$,given by (2.37):

$$\rho : H_1(M_h^n;\mathbb{Z}) \to \oplus_{j=1}^n \mathbb{Z}_j \sigma_{h,j} \ , \tag{2.38}$$

where $\sigma_{h,j} \subset \Gamma_h^{n_j}$, $j = \overline{1,n}$ - the corresponding real valued cycles on the Riemann surfaces $\Gamma_h^{n_j}$, $j = \overline{1,n}$.

Assume that the following meaning of the mapping (2.39) are prescribed:

$$\rho(\sigma_i) := \oplus_{j=1}^n n_{ij} \ \sigma_{h,j} \tag{2.39}$$

for each $i = -\overline{1,n}$, where $n_{ij} \in \mathbb{Z}$, $i,j = \overline{1,n}$ - some fixed integers. Then following the Arnold construction [4,18], one obtains the next set of so called action- variables on $M_h^n \subset M^{2n}$:

$$\gamma_{j:} := \frac{1}{2\pi} \oint_{\sigma_j} dS = \sum_{s=1}^n n_{js} \oint_{\sigma_{h,s}} d\lambda \ w_s(\lambda; h), \tag{2.40}$$

where $j = \overline{1,n}$. It is easy to show [4,16], that expressions (2.41) naturally define an a.e. differentiable invertible mapping

$$\xi : \mathcal{G}^* \to \mathbb{R}^n, \tag{2.41}$$

making it possible to re-coordinate the integral submanifold M_h^n as a submanifold $M_\gamma^n \subset M^{2n}$, where

$$M_\gamma^n := \{u \in M^{2n} : \xi(h) = \gamma \in \mathbb{R}^n\}. \tag{2.42}$$

But, as it was demonstrated in [18,32], in the general case sets (2.43) don't generate a global foliation of the phase space M^{2n},being connected with both topological and analytical constarints. Since the values (2.41) are evidently also commuting invariants on M^{2n}, one can define a next canonical transformation of the phase space M^{2n},generated by the following relationship on $M^{2n}_{h,\tau}$:

$$\sum_{j=1}^{n} w_j \ d\mu_j + \sum_{j=1}^{n} \varphi_j \ d\gamma_j = dS(\mu;\gamma), \tag{2.43}$$

where $\varphi \in \mathbb{T}^n$ - the so called angle-variables on the torus $\mathbb{T}^n \simeq M^n_h$ and $S : M^n_\gamma \times \mathbb{R}^n \to \mathbb{R}$ the corresponding generating function. Whence one can easily obtain, based on (2.28) and (2.38), that

$$\varphi_j := \partial S(\mu;\gamma)/\partial \gamma_j = \sum_{s=1}^{n} \partial S(\mu;\gamma(h))/\partial h_s \ \partial h_s/\partial \gamma_j = \tag{2.44}$$

$$= \sum_{s=1}^{n} t_s \omega_{sj}(\gamma), \qquad \frac{1}{2\pi} \oint_{\sigma_j} d\varphi_k = \delta_{jk} \ ,$$

where $\Omega := \{\omega_{sj} : \mathbb{R}^n \to \mathbb{R}, \ s, j = \overline{1, n} \ \}-$ the so called [4] frequency matrix being a.e. invertible on the integral submanifold $M^n_\gamma \subset M^{2n}$. As an evident result of (2.45), we claim that the evolution of any vector field $K_a \in \Gamma(M^{2n})$ for $a \in \mathcal{G}$ on the integral submanifold $M^n_\gamma \subset M^{2n}$ will be quasi-periodical with a set of frequencies, generated by the matrix $\Omega \overset{a.e.}{\in} Aut(\mathbb{R}^n)$, defined above. As examples showing the effectivity of the method of constructing integral submanifold imbedding mappings for abelian integrable Hamiltonian systems, one can verify the Liouville-Arnold integrability of all Henon-Heiles and Neumann type systems described in detail in [21,22], on what we shall here not stop.

3 Integral submanifold imbedding problem:the case of nonabelian Lie algebra of invariants.

3.1. from the very beginning we shall assume below that there is given a Hamiltonian vector field $K \in \Gamma(M^{2n})$ on the canonically symplectic phase space $M^{2n} = T^*(\mathbb{R}^n)$, $n \in \mathbb{Z}_+$, being endowed with a nonabelian Lie algebra $\mathcal{G}$ off invariants, satisfying all conditions of the Mishchenko-Fomenko theorem 1.5, that is

$$\dim \mathcal{G} + rank\mathcal{G} = \dim M^{2n} . \tag{3.1}$$

Then, as there was proved before, an integral submanifold $M_h^r \subset M^{2n}$ at a regular element $h \in \mathcal{G}^*$ is $rank\mathcal{G} = r-$dimensional and diffeomorphic (when compact and connected) to the standard r-dimensional torus $\mathbb{T}^r \simeq \otimes_{j=1}^r \mathbb{S}_j^1$. The following problem is natural: how to construct the corresponding integral submanifold imbedding mapping

$$\pi_h : M_h^r \to M^{2n} \quad , \tag{3.2}$$

describing evidently all possible orbits of the dynamical system $K \in \Gamma(M^{2n})$?

Having gained some experience in constructing the imbedding mapping (3.2) in the case of the abelian Liouville-Arnold theorem on the integrability by quadratures, we proceed below with studying the integral submanifold $M_h^r \subset M^{2n}$ by means of the Cartan theory [3,12,16,22] of the integrable ideals in the Grassmann algebra $\Lambda(M^{2n})$. Let $\mathcal{I}(\mathcal{G}^*)$ be an ideal in $\Lambda(M^{2n})$, generated by independent on some open neighborhood $U(M_h^r)$ differentials $dH_j \in \Lambda^1(M^{2n})$, $j = \overline{1,k}$, where by definition, $r = \dim \mathcal{G}$. The ideal $\mathcal{I}(\mathcal{G}^*)$ is obviously Cartan's integrable [23,16] with the integral submanifold $M_h^r \subset M^{2n}$ (at a regular element $h \in \mathcal{G}^*$) , on which it vanishes, that is π_h^*

$\mathcal{I}(\mathcal{G}^*)=0$. The dimension $\dim M_h^r=\dim M^{2n}$ -$dim\mathcal{G}=r=rank\mathcal{G}$ due to the condition (3.1) imposed on the Lie algebra $\mathcal{G}$ .It is useful to note here that owing to the nonequality $r\leq k$ for the rank $\mathcal{G}$, one obtains from(3.1) easily that the dimension $dim\mathcal{G}$ =$k\geq n$. Since each base element $H_j\in\mathcal{G},\ j=\overline{1,k}$, generates a symplectically dual vector field $K_j\in\Gamma(M^{2n}),\ j=\overline{1,k}$, one can try to study the corresponding differential system $K(\mathcal{G})$, being Cartan's integrable too on the entire open neighborhood $U(M_h^r)\subset M^{2n}$ .Denote the corresponding integral submanifold of the $\dim M_h^k=\dim K(\mathcal{G})$ =k. Consider now an abelian differential system $K(\mathcal{G}_h)\subset K(\mathcal{G})$, generated by the Cartan Lie subalgebra $\mathcal{G}_h\subset\mathcal{G}$ and its integral submanifold $\bar{M}_h^r\subset U(M_h^r)$. Since the Lie subgroup $G_h=\exp\mathcal{G}_h$ acts on the integral submanifold M_h^r invariantly (see chapter 1) and dim $\bar{M}_h^r$ $=rank\,\mathcal{G}$ =r, one obtains easily that $\bar{M}_h^r=M_h^r$. On the other hand, the system $K(\mathcal{G}_h)\subset K(\mathcal{G})$ by definition, meaning that the integral submanifold M_h^r is an invariant part of the integral submanifold $M_h^k\subset U(M_h^r)$ with respect to the Lie group $G=\exp\mathcal{G}$ - action on M_h^k. In this case the following lemma is true.

Lemma 3.1. *There exist just (n-r)$\in\mathbb{Z}_+$ vector fields $\tilde{F}_j\in K(\mathcal{G})/K(\mathcal{G}_h)$, $j=\overline{1,n-r}$, for which*

$$\omega^{(2)}(\tilde{F}_i,\tilde{F}_j)=0 \tag{3.3}$$

on $U(M_h^r)$ for all $i,\,j=\overline{1,n-r}$.

◀*Proof.* Indeed, the matrix $\omega(\tilde{K}):=\{\omega^{(2)}(\tilde{K}_i,\tilde{K}_j):\ \ i,j=\overline{1,k}\}$ has on $U(M_h^r)$ the $rank\,\omega(\tilde{K})=k-r$, since $\dim_{\mathbb{R}} ker(\pi_h^*\omega^{(2)})=\dim_{\mathbb{R}}(\pi_{h*}\,K(\mathcal{G}_h))=r$ on M_h^r at $h\in\mathcal{G}^*$ being regular. Let us now complexify naturally the tangent space $T(U(M_h^r))$ due to its even dimensionality. Whence one can deduce readily enough that on $U(M_h^r)$ there exist just $(n-r)\in\mathbb{Z}_+$ vectors (not vector fields!) $\tilde{K}_j^{\mathbb{C}}\in K^{\mathbb{C}}(\mathcal{G})/K^{\mathbb{C}}(\mathcal{G}_h),\ j=\overline{1,n-r},$ from the complexified [24] factor space $K^{\mathbb{C}}(\mathcal{G})/K^{\mathbb{C}}(\mathcal{G}_h)$.To show this, let us reduce the skew-symmetric matrix $\omega(\tilde{K})\in Hom(\mathbb{R}^{k-r})$ to its selfadjoint metric equivalence $\omega(\tilde{K}^{\mathbb{C}}) \in Hom(\mathbb{C}^{n-r})$, having taken into account that $\dim_{\mathbb{R}} \mathbb{R}^{k-r} = \dim_{\mathbb{R}} \mathbb{R}^{k+r-2r} = \dim_{\mathbb{R}} \mathbb{R}^{2(n-r)} = \dim_{\mathbb{C}} \mathbb{C}^{n-r}$. Let now $f_j^{\mathbb{C}} \in \mathbb{C}^{n-r}$, $j = \overline{1, n-r}$, be eigenvectors of the nondegenerate selfadjoint matrix $\omega(\tilde{K}^{\mathbb{C}}) \in Hom(\mathbb{C}^{n-r})$, that is

$$\omega(\tilde{K}^{\mathbb{C}})\ f_j^{\mathbb{C}} = \tilde{\lambda}_j f_j^{\mathbb{C}}, \tag{3.4}$$

where $\tilde{\lambda}_j \in \mathbb{R}$, $j = \overline{1, n-r}$, and for all $i, j = \overline{1, n-r}$, $< f_i^{\mathbb{C}}, f_j^{\mathbb{C}} >= \delta_{i,j}$. The above obviously means that in the basis $\{f_j^{\mathbb{C}} \in K^{\mathbb{C}}(\mathcal{G})/K^{\mathbb{C}}(\mathcal{G}_h) : j = \overline{1, n-r}\}$ the matrix $\omega(\tilde{K}^{\mathbb{C}}) \in Hom(\mathbb{C}^{n-r})$ is strictly diagonal being representable as follows:

$$\omega(\tilde{K}^{\mathbb{C}}) = \sum_{j=1}^{n-r} \tilde{\lambda}_j f_j^{\mathbb{C}} \otimes_{\mathbb{C}} f_j^{\mathbb{C}} \quad , \tag{3.5}$$

where $\otimes_{\mathbb{C}}$ $-$ the usual Kronecker tensor product of vectors from $\mathbb{C}^{n-r.}$ Owing to the construction of the complexified matrix $\omega(\tilde{K}^{\mathbb{C}}) \in Hom(\mathbb{C}^{n-r})$, one sees that the space $K^{\mathbb{C}}(\mathcal{G})/K^{\mathbb{C}}(\mathcal{G}_h) \simeq \mathbb{C}^{n-r}$ carries a Kähler structure [24] with respect to which the following expressions

$$\omega(\tilde{K}) = \operatorname{Im} \omega(\tilde{K}), \quad < \cdot, \cdot >_{\mathbb{R}} = \operatorname{Re} < \cdot, \cdot > \tag{3.6}$$

hold. Making use now of the representation (3.5) and expressions (3.6), one can find vector fields $\tilde{F}_j \in K(\mathcal{G})/K(\mathcal{G}_h)$, $j = \overline{1, n-r}$, such that

$$\omega(\tilde{F}) = \operatorname{Im} \omega(\tilde{F}^{\mathbb{C}}) = J, \tag{3.7}$$

holds on $U(M_h^r)$, where $J \in Sp(\mathbb{C}^{n-r})$- the standard symplectic matrix, satisfying the complex structure [24] identity $J^2 = -I.$. In virtue of the normalization conditions $< f_j^{\mathbb{C}}, f_j^{\mathbb{C}} >= \delta_{i,j}$, for all $i, j = \overline{1, n-r},$ one easily follows from (3.7) that $\omega^{(2)}(\tilde{F}_i, \tilde{F}_j) = 0$ for all $i, j = \overline{1, n-r},$ where by definition

$$\tilde{F}_j := \operatorname{Re} \tilde{F}_j^{\mathbb{C}} \tag{3.8}$$

for all $j = \overline{1, n-r}$, that proves the lemma.▶

Reminding now that the Lie algebra $\mathcal{G}$ of invariants on M^{2n} was before split into a direct sum of subspaces as

$$\mathcal{G} = \mathcal{G}_h \oplus \widetilde{\mathcal{G}}_h \quad , \tag{3.9}$$

where $\mathcal{G}_h$ is the Cartan subalgebra at a regular element $h \in \mathcal{G}^*$ (being abelian) and $\widetilde{\mathcal{G}}_h \simeq \mathcal{G} / \mathcal{G}_h$ is the corresponding complement to $\mathcal{G}_h$. Denote a basis of $\mathcal{G}_h$ as $\{\bar{H}_i \in \mathcal{G}_h : i = \overline{1, r}\}$, where $\dim \mathcal{G}_h = rank\mathcal{G} = k \in \mathbb{Z}_+$, and correspondingly, a basis of $\widetilde{\mathcal{G}}_h$ as $\{\tilde{H}_j \in \widetilde{\mathcal{G}}_h \simeq \mathcal{G}/\mathcal{G}_h : j = \overline{1, k-r}\}$. Then, owing to the results of chapter 1 the following relationships hold:

$$\{\bar{H}_i, \bar{H}_j\} = 0, \quad h(\{\bar{H}_i, \tilde{H}_s\}) = 0 \tag{3.10}$$

on the open neighborhood $U(M_h^r) \subset M^{2n}$ for all $i, j = \overline{1, r}$ and $s = \overline{1, k-r}$, having nothing still to say of expressions $h(\{\tilde{H}_s, \tilde{H}_m\})$ for $s, m = \overline{1, k-r}$. Making use now of the representation (3.8), for our vector fields (if they exist) $\tilde{F}_j \in K(\mathcal{G})/K(\mathcal{G}_h)$, $j = \overline{1, n-r}$,one can write down the following expansion:

$$\tilde{F}_i = \sum_{j=1}^{k-r} c_{ji}(h) \tilde{K}_j \quad , \tag{3.11}$$

where $i_{\tilde{K}_j} \omega^{(2)} := -d\tilde{H}_j$, $c_{ji} : \mathcal{G}^* \to \mathbb{R}$, $i = \overline{1, n-r}$, $j = \overline{1, k-r}$, - real valued functions on $\mathcal{G}^*$,being defined uniquely as a result of (3.11) one easily obtains, that there exist invariants $\tilde{f}_s : U(M_h^r) \to \mathbb{R}$, $s = \overline{1, n-r}$, such that

$$-\, i_{\tilde{F}_s} \omega^{(2)} = \sum_{j=1}^{k-r} c_{js}(h)\ d\tilde{H}_j := d\tilde{f}_s \quad , \tag{3.12}$$

where $\tilde{f}_s = \sum_{j=1}^{k-r} c_{js}(h) \tilde{H}_j$, $s = \overline{1, n-r}$, holds on $U(M_h^r)$.

3.2. To proceed with further , let us look at the following identity similar to (2.2):

$$(\otimes_{j=1}^{r} i_{\bar{K}_j})(\otimes_{s=1}^{n-r} i_{\tilde{F}_s})(\omega^{(2)})^{n+1} = 0 = \pm(n+1)!(\wedge_{j=1}^{r} d\bar{H}_j)(\wedge_{s=1}^{n-r} d\ \tilde{f}_s) \wedge \omega^{(2)}, \tag{3.13}$$

being true on $U(M_h^r)$. Whence, the following lemma due to Cartan's theory [3,16], holds.

Lemma 3.2. *The symplectic structure* $\omega^{(2)} \in \Lambda^2(U(M_h^r))$ *has the following canonical representation.:*

$$\omega^{(2)}\big|_{U(M_h^r)} = \sum_{j=1}^{r} d\bar{H}_j \Lambda\ \bar{h}_j^{(1)} + \sum_{s=1}^{n-r} d\tilde{f}_s \Lambda\ \tilde{h}_s^{(1)} \quad , \tag{3.14}$$

where $\bar{h}_j^{(1)}, \tilde{h}_s^{(1)} \in \Lambda^1(U(M_h^r)),\ \ j = \overline{1,r},\ \ s = \overline{1,n-r}.$

The expression (3.14) obviously means, that on $U(M_h^r) \subset M^{2n}$ differential 1-forms $\bar{h}_j^{(1)}, \tilde{h}_s^{(1)} \in \Lambda^1(U(M_h^r)),\ j = \overline{1,r},\ s = \overline{1,n-r}$, are independent together with exact 1-forms $d\bar{H}_j,\ j = \overline{1,r}$, and $d\tilde{f}_s,\ \ s = \overline{1,n-r}$. Since $d\omega^{(2)} = 0$ on the entire M^{2n} identically, from (3.14) one follows that differentials $d\bar{h}_j^{(1)}, d\tilde{h}_s^{(1)} \in \Lambda^2(U(M_h^r)),\ j = \overline{1,r},\ \ s = \overline{1,n-r},$ belong to the ideal $\mathcal{I}(\ \widetilde{\mathcal{G}}_h) \subset \mathcal{I}(\mathcal{G}^*)$, generated by exact forms $d\tilde{f}_s,\ \ s = \overline{1,n-r}$, and $d\bar{H}_j,\ \ j = \overline{1,r}$, for all regular $h \in \mathcal{G}^*$. Thereby the following analog of the Galisau-Reeb theorem 2.1 is true.

Theorem 3.3. *Let a Lie algebra* $\mathcal{G}$ *of invariants on the symplectic space* M^{2n} *be not abelian, and to satisfy the Mishchenko-Fomenko condition (3.1). At a regular element* $h \in \mathcal{G}^*$ *on some open neighborhood* $U(M_h^r)$ *of the integral submanifold* $M_h^r \subset M^{2n}$ *there exist differential 1-forms* $\bar{h}_j^{(1)}$, $j = \overline{1,n}$, *and* $\tilde{h}_s^{(1)}$, $s = \overline{1,n-r}$, *fulfilling the following properties:*

i) $\omega^{(2)}\big|_{U(M_h^r)} = \sum_{j=1}^{r} d\bar{H}_j \wedge \bar{h}_j^{(1)} + \sum_{s=1}^{n-r} d\tilde{f}_s \wedge \tilde{h}_s^{(1)}$,
$\bar{H}_j \in \mathcal{G}$, $j = \overline{1,r}$,– *a basis of the Cartan subalgebra* $\mathcal{G}_h \subset \mathcal{G}$ *(being abelian),*

and $\tilde{f}_s \in \mathcal{G}$, $s=\overline{1,n-r}$, - *some invariants from the complement space* $\widetilde{\mathcal{G}}_h \simeq \mathcal{G}/\mathcal{G}_h$;

ii) 1-forms $\bar{h}_j^{(1)} \in \Lambda^1(U(M_h^r))$, $j=\overline{1,r}$, *and* $\tilde{h}_s^{(1)} \in \Lambda^1(U(M_h^r))$, $s=\overline{1,n-r}$, *are exact on* $M_{h,}^r$ *satisfying the equations:* $\bar{h}_j^{(1)}(\bar{K}_i)=\delta_{i,j}$ *for all* $i,j=\overline{1,r}$, $\bar{h}_j^{(1)}(\tilde{F}_s)=0$ *and* $\tilde{h}_s^{(1)}(\bar{K}_j)=0$ *for all* $j=\overline{1,r}$, $s=\overline{1,n-r}$, *and* $\tilde{h}_s^{(1)}(\tilde{F}_m)=\delta_{s,m}$ *for all* $s,m=\overline{1,n-r}$.

◀*Proof.* It is obviously needed to prove only the last statement ii). Making use of the assertions of theorem 3.3. , one obtains that on the integral submanifold $M_h^r \subset M^{2n}$ differential 2-forms $d\bar{h}_j^{(1)} \in \Lambda^2(U(M_h^r))$, $j=\overline{1,r}$, and $d\tilde{h}_s^{(1)} \in \Lambda^2(U(M_h^r))$, $s=\overline{1,n-r}$, are identically vanishing. This means in particular owing to the classical Poincare lemma [1,4,16], existence of some exact 1-forms $d\bar{t}_{h,j} \in \Lambda^1(U(M_h^r))$, $j=\overline{1,r}$, and $d\tilde{t}_{h,s} \in \Lambda^1(U(M_h^r))$, $s=\overline{1,n-r}$, where $\bar{t}_{h,j}: M_h^r \to \mathbb{R}$, $j=\overline{1,r}$, and $\tilde{t}_{h,s}: M_h^r \to \mathbb{R}$, $s=\overline{1,n-r}$, - smooth independent a.e. functions on M_h^r, being one-valued on an appropriate covering of the manifold $M_h^r \subset M^{2n}$, supplying global coordinates on the integral submanifold M_h^r. Based now on the representation (3.14) , one can easily obtain that

$$-i_{\bar{K}_i}\omega^{(2)}\big|_{U(M_h^r)} = \sum_{j=1}^{r} d\bar{H}_j\, \bar{h}_j^{(1)}(\bar{K}_i) + \sum_{s=1}^{n-r} d\tilde{f}_s\, \tilde{h}_s^{(1)}(\bar{K}_i) = d\bar{H}_i \tag{3.15}$$

for all $i=\overline{1,r}$ and

$$-i_{\tilde{F}_m}\omega^{(2)}\big|_{U(M_h^r)} = \sum_{j=1}^{r} d\bar{H}_j\, \bar{h}_j^{(1)}(\tilde{F}_m) + \sum_{s=1}^{n-r} d\tilde{f}_s\, \tilde{h}_s^{(1)}(\tilde{F}_m) = d\tilde{f}_m \tag{3.16}$$

for all $m=\overline{1,n-r}$. Whence, from (3.15) one follows on $U(M_h^r)$, that

$$\bar{h}_j^{(1)}(\bar{K}_i)=\delta_{i,j}, \qquad \tilde{h}_s^{(1)}(\bar{K}_i)=0 \tag{3.17}$$

for all $i,j=\overline{1,r}$ and $s=\overline{1,n-r}$, and similarly, from (3.16) one follows on $U(M_h^r)$, that

$$\bar{h}_j^{(1)}(\tilde{F}_m)=0, \qquad \tilde{h}_s^{(1)}(\tilde{F}_m)=0 \tag{3.18}$$

for all $j=\overline{1,r}$ and $s,m=\overline{1,n-r}$, that proves the theorem.▶

Having defined now global evolution parameters $t_j : M^{2n}\rightarrow\mathbb{R},\ j=\overline{1,r},$ of the corresponding vector fields $\bar{K}_j = d/dt_j,\ j=\overline{1,r},$ and local evolution parameters $\tilde{t}_s : M^{2n}\cap U(M_h^r)\rightarrow\mathbb{R},\ s=\overline{1,n-r},$ of the corresponding vector fields $\tilde{F}_s\Big|_{U(M_h^r)} := d/d\tilde{t}_s,\ s=\overline{1,n-r},$ one can state easily from (3.18) that equalities

$$t_j|_{U(M_h^r)} = \bar{t}_j, \qquad \tilde{t}_s\big|_{U(M_h^r)} = \tilde{t}_{h,s} \tag{3.19}$$

hold for all $j=\overline{1,r}$, $s=\overline{1,n-r}$, up to constant normalizations. Thereby, one can develop a new approach, similar to that of chapter 2, to studying the integral submanifold imbedding problem in the case of the nonabelian Liouville-Arnold integrability theorem.

Before starting with, it is interesting to note, that the system of invariants

$$\mathcal{G}_\tau := \mathcal{G}_h \oplus span_{\mathbb{R}}\{\tilde{f}_s \in \mathcal{G}/\mathcal{G}_h : s=\overline{1,n-r}\}$$

constructed above, compose a new involutive (abelian) complete algebra $\mathcal{G}_\tau$, to which evidently can be applied the abelian Liouville-Arnold theorem on integrability by quadratures and the integral submanifold imbedding theory devised in chapter 2, for producing theory exact solutions by means of algebraic-analytical expressions. Namely, the following corollary holds.

Corollary 3.5. *Assume that a not abelian Lie algebra $\mathcal{G}$ satisfies the Mishchenko-Fomenko condition (3.1) and $M_h^r \subset M^{2n}$ is its integral submanifold (compact and connected) at a regular element $h\in\mathcal{G}^*$, being diffeomorphic to the standard torus $\mathbb{T}^r \simeq M_{h,\tau}^n$. Assume also that the dual complete abelian algebra $\mathcal{G}_\tau$ $(\dim\mathcal{G}_\tau = n = 1/2\dim M^{2n})$ of independent invariants built above, is globally defined . Then its integral submanifold $M_{h,\tau}^n \subset M^{2n}$ is diffeomorphic to the standard torus $\mathbb{T}^n \simeq M_{h,\tau}^n$, containing the torus $\mathbb{T}^r \simeq M_h^r$ as a direct product with some completely degenerate torus $\mathbb{T}^{n-r}$, that is $M_{h,\tau}^n \simeq M_h^r \times \mathbb{T}^{n-r}$.*

Thus, having applied successively the algorithm of chapter 2 to algebraic-analytical treating integral submanifolds of a given nonabelian Liouville - Arnold integrable Lie algebra $\mathcal{G}$ of invariants on the canonically symplectic manifold $M^{2n} \simeq T^*(\mathbb{R}^n)$, one can produce a wide class of its exact solutions represented by quadratures as it was wanted from the very beginning. At this place it is necessary to notice that up to now the dual to $\mathcal{G}$ abelian complete algebra $\mathcal{G}_\tau$ of invariants at a regular $h \in \mathcal{G}^*$ was constructed only on some open neighborhood $U(M_h^r)$ of the integral submanifold $M_h^r \subset M^{2n}$. As it was mentioned before, the global existence of the algebra $\mathcal{G}_\tau$ strongly depends on the possibility of extending these invariants on the entire manifold M^{2n}. The latter is one-to-one tied with existence of some global complex structure [24] on the reduced integral submanifold $\tilde{M}_{h,\tau}^{2(n-r)} := M_h^k/G_h$, induced by the reduced symplectic structure $\pi_\tau^* \, \omega^{(2)} \in \Lambda^2(M_h^k/G_h)$, where $\pi_\tau : M_h^k \to M^{2n}$ - the imbedding mapping for the integrable differential system $K(\mathcal{G}) \subset \Gamma(M^{2n})$, introduced before. If this is the case , the resulting complexified manifold ${}^{\mathbb{C}}\tilde{M}_{h,\tau}^{n-r} \simeq \tilde{M}_{h,\tau}^{2(n-r)}$ will be endowed with a Kahlerian structure, which makes it possible to produce the dual abelian algebra $\mathcal{G}_\tau$ as a globally defined set of invariants on M^{2n} . This problem will be analyzed in more detail in the forthcoming chapter 5.

4 Examples

4.1.Below we shall attach some examples of the nonabelian Liouville-Arnold integrability by quadratures, fulfilling the conditions of theorem 1.5.

Example 4.1. *Point vortices on plane.*

Consider $n \in \mathbb{Z}_+$ point vortices on the plane $\mathbb{R}^2$, described by the Hamiltonian function

$$H = -\frac{1}{2\pi} \sum_{i \neq j=1}^{n} \xi_i \xi_j \ln \|q_i - p_j\| \tag{4.20}$$

with respect to the following not-completely canonical symplectic structure on $M^{2n} \simeq T^*(\mathbb{R}^n)$:

$$\omega^{(2)} = \sum_{j=1}^{n} \xi_j dp_j \wedge dq_j, \tag{4.21}$$

where $(p_j, q_j) \in \mathbb{R}^2$, $j = \overline{1,n}$, - coordinates of vortices on the plane $\mathbb{R}^2$. There exist three additional invariants

$$P_1 = \sum_{j=1}^{n} \xi_j q_j, \qquad P_2 = \sum_{j=1}^{n} \xi_j p_j, \tag{4.22}$$

$$P = \frac{1}{2} \sum_{j=1}^{n} \xi_j (q_j^2 + p_j^2),$$

satisfying the following Poisson brackets:

$$\{P_1, P_2\} = -\sum_{j=1}^{n} \xi_j \ , \quad \{P_1, P\} = -P_2 \ \ , \{P_2, P\} = P_1 \ , \tag{4.23}$$

$$\{P, H\} = \ 0 \ = \ \{P_j, H\} \ .$$

It is evident, that invariants (4.1) and (4.3) compose at $\sum_{j=1}^{n} \xi_j = 0$ a four-dimensional Lie algebra $\mathcal{G}$, whose $rank\ \mathcal{G} = 2$. Indeed, assume a vector $h \in \mathcal{G}^*$ is chosen regular, being parametrized by real values $h_j \in \mathbb{R}$, $j = \overline{1,4}$, where

$$h(P_i) = h_i, \ \ h(P) = h_3 \ , \ \ h(H) = h_4 \ , \tag{4.24}$$

and $i = \overline{1,2}$. Then , one can easily verify that the element

$$Q_h = (\sum_{j=1}^{n} \xi_j)P - \sum_{i=1}^{n} h_i P_i \tag{4.25}$$

belongs to the Cartan Lie subalgebra $\mathcal{G}_h \subset \mathcal{G}$, that is by definition

$$h(\{Q_h, P_i\}) = 0, \quad h(\{Q_h, P\}) = 0 \ . \tag{4.26}$$

Since $\{Q_h, H\} = 0$ for all values $h \in \mathcal{G}^*$, we claim that $\mathcal{G}_h = span_{\mathbb{R}}\{H, Q_h\}$ - the Cartan subalgebra of $\mathcal{G}$. Thus, $rank\mathcal{G} = \dim \mathcal{G}_h = 2$, and one comes right away that the condition (3.1)

$$\dim M^{2n} = 2n = rank\mathcal{G} + \dim \mathcal{G} = 6 \tag{4.27}$$

holds only if $n = 3$. Thereby, the following theorem is proved.

Theorem 4.1. *The three -vortex problem (4.1) on the plane* $\mathbb{R}^2$ *is nonabelian Liouville-Arnold integrable by quadratures on the phase space* $M^6 \simeq T^*(\mathbb{R}^3)$ *with the symplectic structure (4.2).*

As a result, the corresponding integral submanifold $M_h^2 \subset M^6$ is two-dimensional and diffeomorphic (when compact and connected) to the torus $\mathbb{T}^2 \simeq M_h^2$, on which the motions are quasiperiodic functions of evolution parameters.

Concerning the corollary 3.5 , the dynamical system (4.1) is also abelian Liouville-Arnold integrable with an extended integral submanifold $M_{h,\tau}^3 \subset M^6$, which can be found via the scheme suggested above in chapter 3. As a result of simple enough calculations, one obtains an additional invariant $Q = (\sum_{j=1}^{3} \xi_j)P - \sum_{i=1}^{3} P_i^2 \notin \mathcal{G}$, which commutes with H and P of $\mathcal{G}_h$. Therefore, there exists a new complete dual abelian algebra $\mathcal{G}_\tau = span_{\mathbb{R}}$ $\{Q, P, H\}$ of independent invariants on M^6 with $\dim \mathcal{G}_\tau = 3 = 1/2 \dim M^6$, whose integral submanifold $M^3_{h,\tau} \subset M^6$ (when compact and connected) is diffeomorphic to the torus $\mathbb{T}^3 \simeq M^2_h \times \mathbb{S}^1$.

Note also here, that the above additional invariant $Q \in \mathcal{G}_\tau$ is naturally extended to the case of arbitrary $n \in \mathbb{Z}_+$ vortices as follows: $Q = (\sum_{j=1}^n \xi_j)P - \sum_{i=1}^n P_i^2 \in \mathcal{G}_\tau$, which obviously, commutes too with invariants (4.1) and (4.3) on the entire phase space M^{2n}.

Example 4.2. *A material point motion in central field.*

Consider the motion of a material point in the space $\mathbb{R}^3$ under a central potential field whose Hamiltonian

$$H = \frac{1}{2}\sum_{j=1}^{3} p_j^2 + Q(\|q\|), \tag{4.28}$$

contains a central field $Q : \mathbb{R}_+ \to \mathbb{R}$. The motion is carried out on the canonically phase space $M^6 = T^*(\mathbb{R}^3\ )$, possessing three additional invariants:

$$P_1 = p_2q_3 - pq, \qquad P_2 = p_3q_1 - p_1q_3, \quad P_3 = p_1q_2 - p_2q_1\ \ , \tag{4.29}$$

satisfying the following Poisson brackets:

$$\{P_1, P_2\} = P_3\ , \quad \{P_3, P_1\} = P_2\ , \ \{P_2, P_3\} = P_1\ . \tag{4.30}$$

Since $\{H, P_j\} = 0$ for all $j = \overline{1, 3}$, one sees that the problem under consideration possesses a four-dimensional Lie algebra $\mathcal{G}$ of invariants, isomorphic to the classical rotation Lie algebra $so(3) \times \mathbb{R} \simeq \mathcal{G}$. Let us show that at a regular element $h \in \mathcal{G}^*$ the Cartan subalgebra $\mathcal{G}_h \subset \mathcal{G}$ has the dimension $\dim \mathcal{G}_h = 2 = rank \mathcal{G}$. Indeed, one easily verifies that the following invariant

$$P_h = \sum_{j=1}^{3} h_j P_j \tag{4.31}$$

belongs to the Cartan subalgebra $\mathcal{G}_h$, that is

$$\{H, P_h\} = 0, \qquad h(\{P_h, P_j\}) = 0 \tag{4.32}$$

for all $j = \overline{1,3}$. Thus, as the Cartan subalgebra $\mathcal{G}_h = span_{\mathbb{R}}\{H \ \ and \ P_h \subset \mathcal{G}\}$, one gets $\dim \mathcal{G}_h = 2 = rank\mathcal{G}_h$, that is the Mishchenko-Fomenko condition 3.1

$$\dim M^6 = 6 = rank\mathcal{G} + \dim \mathcal{G} = 4 + 2 \tag{4.33}$$

holds. The latter makes it possible to prove its integrability by quadratures via the nonabelian Liouville Liouville-Arnold theorem 1.5. Thereby the following theorem is true.

Theorem 4.3. *The free materialpoint motion in* $\mathbb{R}^3$ *is a completely integrable by quadratures dynamical system on the canonically symplectic phase space* $M^6 = T^*(\mathbb{R}^3)$ *within the nonabelian Liouville-Arnold theorem 1.5. The corresponding integral submanifold* $M_h^2 \subset M^6$ *at a regular element* $h \in \mathcal{G}^*$ *(if compact and connected) is two dimensional and diffeomorphic to the standard torus* $\mathbb{T}^2 \simeq M_h^2$.

Making use of the integration algorithm devised in chapters1 and 2, one can regularly obtain the corresponding integral submanifold imbedding mapping $\pi_h : M_h^2 \to M^6$ by means of algebraic-analytical expressions and transformations.

There exist evidently many other interesting for applications nonabelian Liouville-Arnold integrable Hamiltonian systems on canonically symplectic phase spaces, whose algebraic -analytical integration can be now accomplished. On such algebraic-analytical calculations we hope to stay in detail elsewhere.

5 Existence problem for a global set of invariants

5.1 There was stated above in chapter 3, that locally, in some open neighborhood $U(M_h^r) \subset M^{2n}$ of the integral submanifold $M_h^r \subset M^{2n}$ one can find by means of algebraic-analytical tools just $n-r \in \mathbb{Z}_+$ independent vector fields $\tilde{F}_j \in K(\mathcal{G})/K(\mathcal{G}_h) \cap \Gamma(U(M_h^r)$, $j=\overline{1,n-r}$, satisfying the condition (3.3). Since each vector field $\tilde{F}_j \in K(\mathcal{G})/K(\mathcal{G}_h)$, $j=\overline{1,n-r}$, is generated by invariant $\tilde{H}_j \in \mathcal{D}(\ U(M_h^r))$, $j=\overline{1,n-r}$, from (3.3) one follows evidently that

$$\{\tilde{H}_i, \tilde{H}_j\} = 0 \tag{5.1}$$

for all $i,j=\overline{1,n-r}$. Thus, on an open neighborhood $U(M_h^r)$ there exist just $n-r \in \mathbb{Z}_+$ additional to $\mathcal{G}_h$ invariants $\tilde{H}_j \in \mathcal{D}(\ U(M_h^r))$, $j=\overline{1,n-r}$, being together in involution. Denote as before this new set of invariants as $\mathcal{G}_\tau$, keeping in mind that $\dim \mathcal{G}_\tau = r+(n-r) = n \in \mathbb{Z}_+$. Thereby, on an open neighborhood $U(M_h^r) \subset M^{2n}$ we have constructed the set $\mathcal{G}_\tau$ of just $n = 1/2 \dim M^{2n}$ invariants commuting with each other, supplying within the abelian Liouville-Arnold theorem its local complete integrability by quadratures. this means that there exists locally such a mapping $\pi_\tau : M_{h,\tau}^k \to M^{2n}$, where $M_{h,\tau}^k := U(M_h^r) \cap M_\tau^k$- the integral submanifold of the differential system $K(\mathcal{G})$, making it possible to describe the behavior of integrable vector fields on the reduced manifold $\bar{M}_{h,\tau}^{2(n-r)} := M_{h,\tau}^{k-r}/G_h$. Being interested in the global integrability properties of a given set $\mathcal{G}$ of invariants on $(M^{2n},\ \omega^{(2)})$, satisfying the Mishchenko-Fomenko condition (3.1), it is needed to have the additional set of invariants $\tilde{H}_j \in \mathcal{D}(\ U(M_h^r))$, $j=\overline{1,n-r}$, extended from $U(M_h^r)$ to the entire phase space M^{2n}. this problem evidently depends on the existence of continuation of vector fields $\tilde{F}_j \in \Gamma(U(M_h^r))$, $j = \overline{1, n-r}$, from the neighborhood $U(M_h^r) \subset M^{2n}$ on the whole phase space M^{2n} . On the other hand , there was stated before, that the existence of such a continuation depends deeply on the properties of the complexified differential system $K^{\mathbb{C}}(\mathcal{G})/K^{\mathbb{C}}(\mathcal{G}_h)$, possessing a nondegenerate complex metric $\omega(\tilde{K}^{\mathbb{C}}) : T(\bar{M}_{h,\tau}^{2(n-r)})^{\mathbb{C}} \times T(\bar{M}_{h,\tau}^{2(n-r)})^{\mathbb{C}} \rightarrow \mathbb{C}$, induced by the symplectic structure $\omega^{(2)} \in \Lambda^2(M^{2n})$. This point can be clarified more exactly having introduced the notion [24-27] of Kahler's manifold, pertinently connected with the constructions presented above. Namely, consider the local isomorphism $T(\bar{M}_{h,\tau}^{2(n-r)})^{\mathbb{C}} \simeq T(^{\mathbb{C}}\bar{M}_{h,\tau}^{n-r})$, where $^{\mathbb{C}}\bar{M}_{h,\tau}^{n-r}$ - the complex $(n-r)$-dimensional local integral submanifold of the complexified differential system $K^{\mathbb{C}}(\mathcal{G})/K^{\mathbb{C}}(\mathcal{G}_h)$. This means that the space $T(\bar{M}_{h,\tau}^{2(n-r)})$ we endowed with the standard almost complex structure

$$J : T(\bar{M}_{h,\tau}^{2(n-r)}) \rightarrow T(\bar{M}_{h,\tau}^{2(n-r)}), \qquad J^2 = -1, \tag{5.2}$$

such , that the 2-form $\omega(\tilde{K}) := \operatorname{Im} \omega(\tilde{K}^{\mathbb{C}}) \in \Lambda^2(\bar{M}_{h,\tau}^{2(n-r)})$ induced from the defined above metric on $T(^{\mathbb{C}}\bar{M}_{h,\tau}^{n-r})$, should be closed, that is $d\omega(\tilde{K}) = 0$. If this is the case, the almost complex structure on the manifold $T(\bar{M}_{h,\tau}^{2(n-r)})$ is called integrable, defining the proper complex manifold $^{\mathbb{C}}\bar{M}_{h,\tau}^{n-r}$, on which one can then define globally vector fields $\tilde{F}_j \in K(\mathcal{G})/K(\mathcal{G}_h)$, $j = \overline{1, n-r}$, searched before for the involutive algebra $\mathcal{G}_\tau$ of invariants on M^{2n} to be integrable by quadratures within the abelian Liouville-Arnold theorem. Thus the following theorem is stated.

Theorem 5.1. *A nonabelian set $\mathcal{G}$ of invariants on the symplectic space $M^{2n} \simeq T^*(R^n)$, satisfying the Mishchenko-Fomenko condition 3.1, admits the algebraic-analytical integration by quadratures for the integral submanifold imbedding manifold mapping $\pi_h : M_h^r \rightarrow M^{2n}$, if the corresponding complexified reduced manifold $^{\mathbb{C}}\bar{M}_{h,\tau}^{n-r} \simeq \bar{M}_{h,\tau}^{2(n-r)} = M_{h,\tau}^{k-r}/G_h$*

of the differential system $K^{\mathbb{C}}(\mathcal{G})/K^{\mathbb{C}}(\mathcal{G}_h)$ *is Kahlerian with respect to the standard almost complex structure (5.1) and nondegenerate complex metric* $\omega(\tilde{K}^{\mathbb{C}}) : T(\bar{M}_{h,\tau}^{2(n-r)})^{\mathbb{C}} \times T(\bar{M}_{h,\tau}^{2(n-r)})^{\mathbb{C}} \to \mathbb{C}$, *induced by the symplectic structure* $\omega^{(2)} \in \Lambda^2(M^{2n})$ *is integrable, that is* $d\,\mathrm{Im}\,\omega(\tilde{K}^{\mathbb{C}}) = 0$ identically.

The theorem 5.1 means in particular, that the nonabelian Liouville-Arnold integrability by quadratures doesn't in general imply the integrability within the abelian Liouville-Arnold theorem, that is connected with some topological obstacles associated with the Lie algebra structure of invariants $\mathcal{G}$ on the phase space M^{2n}. We hope to stay on these intriguing problems in another place.

6 Supplement

Here we shall attach some examples of searching integral submanifold imbedding mappings for abelian Liouville-Arnold integrable Hamiltonian systems on $T^*(\mathbb{R}^2)$.

6.1.The Henon-Heiles system.

This Hamiltonian flow is governed by the Hamiltonian

$$H_1 = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + q_1 q_2^2 + \frac{1}{3}q_1^3 \tag{6.1}$$

on the canonically symplectic phase space $M^4 = T^*(\mathbb{R}^2)$ with the symplectic structure

$$\omega^{(2)} = \sum_{j=1}^{2} dp_j \wedge dq_{j.} \tag{6.2}$$

As it is well known, there exists the second commuting with (6.1) invariant

$$H_2 = p_1 p_2 + 1/3 q_2^3 + q_1^2 q_2, \tag{6.3}$$

that is $\{H_1, H_2\} = 0$ on the entire space M^4.

Take an element $h \in \mathcal{G} := \{H_j : \ M^4 \to \mathbb{R}: \quad j = \overline{1,2}\}$, being chosen regular with fixed values $h(H_j) = h_j \in \mathbb{R}, \ \ j = \overline{1,2}$. Then the integral submanifold

$$M_h^2 := \{(q,p) \in M^4 : h(H_j) = h_j \in \mathbb{R}, \ j = \overline{1,2}\} \tag{6.4}$$

if compact and connected will be due to the Liouville-Arnold theorem diffeomorphic to the standard torus $\mathbb{T}^2 \simeq \mathbb{S}^1 \times \mathbb{S}^1$, on which one can find cyclic (separable) coordinates $\mu_j \in \mathbb{S}^1, \ j = \overline{1,2}$, such that the symplectic structure (6.2) will take the form:

$$\omega^{(2)} = \sum_{j=1}^{2} dw_j \wedge d\mu_j \ , \tag{6.5}$$

where the conjugate variables $w_j \in T^*(\mathbb{S}^1)$, $j = \overline{1,2}$, are upon M_h^2 depending only on the corresponding variables $\mu_j \in \mathbb{S}_j^1, \ j = \overline{1,2}$, In this case evidently, the evolution along M_h^2 will be separable and representable by means of quasi- periodic functions of evolution parameters.

To show this, recall that the fundamental determining equations (2.34) are based on the 1-forms $\bar{h}_j^{(1)} \in \Lambda(M_h^2), \ j = \overline{1,2}$, satisfy the identity

$$\sum_{j=1}^{2} dH_j \wedge_j \bar{h}_j^{(1)} = \sum_{j=1}^{2} dp_j \wedge dq_j \ . \tag{6.6}$$

Having put by definition

$$\bar{h}_j^{(1)} = \sum_{k=1}^{2} \bar{h}_{jk}(q,p) dq_k, \tag{6.7}$$

where $j = \overline{1,2}$, and substituted (6.7) into (6.6), one obtains that

$$\bar{h}_1^{(1)} = \frac{p_1 dq_1}{p_1^2 - p_2^2} + \frac{p_2 dq_2}{p_1^2 - p_2^2}, \quad \bar{h}_2^{(1)} = \frac{p_2 dq_1}{p_{21}^2 - p_1^2} + \frac{p1 dq_2}{p_1^2 - p_2^2} \ . \tag{6.8}$$

On the other hand, the following implication holds on $M_h^2 \subset M^4$:

$$\alpha_h^{(1)} = \sum_{j=1}^{2} w_j(\mu_j; h) d\mu_j \Rightarrow \sum_{j=1}^{2} p_j dq_j \ := \alpha^{(1)}, \tag{6.9}$$

where we have assumed that the integral submanifold M_h^2 admits the local coordinates in the base manifold $\mathbb{R}^2$ endowed with the canonical 1-form $\alpha_h^{(1)} \in \Lambda(M_h^2)$ as given in (6.9). Thus, making use of the imbedding manifold mapping $\pi_h : M_h^2 \ \to T^*(\mathbb{R}^2)$ in the form

$$q_j = q_j(\mu; h) \ , \quad p_j = p_j(\mu; h) \ , \tag{6.10}$$

$j = \overline{1, 2}$, one easily obtains that the equalities

$$p_j = \sum_{k=1}^{2} w_k(\mu_k; h) \ \ \partial\mu_k / \partial q_j \tag{6.11}$$

hold for $j = \overline{1, 2}$ on the entire integral submanifold M_h^2.

Having now substituted (6.11) into (6.8) and based on the characteristic relationships (2.34), one obtains after simple but cumbersome calculations the following differential-algebraic expressions:

$$\partial q_1 / \partial \mu_1 - \partial q_2 / \partial \mu_1 = 0, \quad \partial q_1 / \partial \mu_2 + \partial q_2 / \partial \mu_2 = 0, \tag{6.12}$$

whose the simplest solution can be written down as

$$q_1 = (\mu_1 + \mu_2)/2 \ , \quad q_2 = (\mu_1 - \mu_2)/2 \ . \tag{6.13}$$

Being extended naturally with the expressions (6.11), namely

$$p_1 = w_1 + w_2 \ , \quad p_2 = w_1 - w_2 \quad , \tag{6.14}$$

where

$$w_1 = \sqrt{h_1 + h_2 - 4/3\mu_1^3}, \qquad w_2 = \sqrt{h_1 - h_2 - 4/3\mu_1^3}, \tag{6.15}$$

one obtains the separable [15] Hamiltonian functions (6.1) and (6.3) in a vicinity of the cotangent space $T^*(M_h^2)$:

$$h_1 = \frac{1}{2}w_1^2 + \frac{1}{2}w_2^2 + \frac{2}{3}(\mu_1^3 + \mu_2^3), \quad h_2 = \frac{1}{2}w_1^2 - \frac{1}{2}w_2^2 + \frac{2}{3}(\mu_1^3 - \mu_2^3), \tag{6.16}$$

generating the following separable motions on $M_h^2 \subset T^*(\mathbb{R}^2)$:

$$d\mu_1/dt := \partial h_1/\partial w_1 = \sqrt{h_1 + h_2 - 4/3\mu_1^3}, \tag{6.17}$$

$$d\mu_2/dt := \partial h_1/\partial w_2 = \sqrt{h_1 - h_2 - 4/3\mu_1^3}$$

for the Hamiltonian (6.1), and

$$d\mu_1/dx := \partial h_2/\partial w_1 = \sqrt{h_1 + h_2 - 4/3\mu_1^3}, \tag{6.18}$$

$$d\mu_2/dt := \partial h_1/\partial w_2 = -\sqrt{h_1 - h_2 - 4/3\mu_1^3}$$

for the Hamiltonian (6.3), where $x, t \in \mathbb{R}$ are the corresponding evolution parameters.

As can be shown analogously, there exists [28, 29] a similar to (6.13) and (6.14) integral submanifold imbedding mapping for the following also integrable modified Henon-Heiles involutive system:

$$H_1 = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + q_1 q_2^2 + \frac{16}{3}q_1^3, \tag{6.19}$$

$$H_2 = 9p_2^4 + 36q_1p_2^2q_2^2 - 12p_1p_2q_2^3 - 2q_2^4(q_2^2 + 6q_1^2) \quad ,$$

where $\{H_1, H_1\}$ on the entire phase space $M^4 = T^*(\mathbb{R}^2)$.

Based on the consideration similar to the above, one can deduce the following [29] expressions:

$$q_1 = -\frac{1}{4}(\mu_1 + \mu_2) - \frac{3}{8}(\frac{w_1 + w_2}{\mu_1 - \mu_2})^2, \tag{6.20}$$

$$q_2^2 = -2\sqrt{h_2}/(\mu_1 - \mu_2), \quad w_1 = \sqrt{2/3\mu_1^3 - 4/3\sqrt{h_2} - 8h_1}\ ,$$

$$p_1 = \frac{1}{2\sqrt{-6(\mu_1 + \mu_2 + 4q_{1})}}[\frac{-2\sqrt{h_2}}{\mu_1 - \mu_2} - \mu_1\mu_2 + 4(\mu_1 + \mu_2)q_1 + 32q_1^2],$$

$$p_2 = \sqrt{h_2}(\mu_{1+}\mu_2 + 4q_1)/(3(\mu_{1-}\mu_2))\ , \ \ w_2 = \sqrt{2/3\mu_2^3 + 4/3\sqrt{h_2} - 8h_1}\ ,$$

solving explicitly the problem of finding the corresponding integral submanifold imbedding mapping $\pi_h : M_h^2 \to T^*(\mathbb{R}^2)$, generating separable flows in the variables $(\mu\ , w) \in T^*(M_h^2)$.

6.2.The reduced Focker-Plank system on $T^*(\mathbb{R}^2)$.

This system is generated by the Hamiltonian

$$H_1 = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + p_1q_1q_2 - p_2q_2^2 \tag{6.21}$$

on the canonically symplectic phase space $M^4 = T^*(\mathbb{R}^2)$, being Liouville-Arnold integrable . Its additional invariant has the following [30] form:

$$H_2 = p_1^2(2p_2 + q_1^2), \tag{6.22}$$

that is $\{H_{1,}H_2\} = 0$ on the entire M^4. Applying as above the method devised in chapter 2, one can find the following characteristic 1-forms $\bar{h}_j^{(1)} \in \Lambda^{(1)}(M_h^2),\ \ j = \overline{1,2}$, on the integral submanifold $M_h^2 \subset T^*(\mathbb{R}^2)$ defined by expression(6.4) :

$$\bar{h}_1^{(1)} = \frac{p_1dq_1}{p_1(p_1 + q_1q_2) + (2p_2 + q_1^2)(q_2^2 - p_2)} \tag{6.23}$$

$$-\frac{(2p_2 + q_1^2)dq_2}{p_1(p_1 + q_1q_2) + (2p_2 + q_1^2)(q_2^2 - p_2)},$$

$$\bar{h}_2^{(1)} = \frac{(q_2^2 - p_2)dq_1}{2p_1[p_1(p_1 + q_1q_2) + (2p_2 + q_1^2)(q_2^2 - p_2)]}$$

$$+\frac{(p_1 + q_1q_2)dq_2}{2p_1[p_1(p_1 + q_1q_2) + (2p_2 + q_1^2)(q_2^2 - p_2)]}\ ,$$

which should be compatible with the relations (6.11). Whence, having substituted (6.11) into (6.25), one derives the characteristic functions (2.35) satisfying the determining relationships (2.34). As a result of simple but a little cumbersome handling the resulting expressions one obtains the following :

$$q_1^2 = h_2^{1/2}(\mu_1\mu_2 - 1)^2/\mu_1\mu_2, \quad q_2 = (\mu_1\mu_2 - 1)/(2(\mu_1\mu_2 + 1)), \tag{6.24}$$
$$h_2^{1/2} = w_1(\mu_1; h)\ \mu_1 + 2h_1, \qquad h_2^{1/2} = w_2(\mu_2; h)\mu_2\ + 2h_1,$$
$$p_1 = h_2^{1/4}(\mu_1\mu_2)^{1/2}/(\mu_1\mu_2 + 1), \quad p_2 = (h_2 - (p_1q_1)^2)/(2p_1^2),$$
$$p_1q_1 = h_2^{1/2}((\mu_1\mu_2 - 1)/(\mu_1\mu_2 + 1)\ ,$$

taking place on the integral submanifold $M_h^2 \subset T^*(\mathbb{R}^2)$. In new canonically conjugated variables $(\mu, w) \in T^*(\mathbb{S}^1 \times \mathbb{S}^1) \simeq T^*(M_h^2)$ the evolution of the flows generated by Hamiltonians (6.21) and (6.22) will be separable, that is integrable by quadratures.

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