A Monadic Framework for Relational Verification
(Functional Pearl)

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Relational properties relate multiple runs of one or more programs. They characterize many useful notions of security, program refinement, and equivalence for programs with diverse computational effects, and have received much attention in the recent literature. Rather than designing and developing tools for special classes of relational properties, as typically proposed in the literature, we advocate the relational verification of effectful programs within general purpose proof assistants. The essence of our approach is to model effectful computations using monads and prove relational properties on their monadic representations, making the most of existing support for reasoning about pure programs.

We apply our method in F and evaluate it by encoding a variety of relational program analyses, including static information flow control, semantic declassification, provenance tracking, program equivalence at higher order, game-based cryptographic security, and various combinations thereof. By relying on SMT-based automation, unary weakest preconditions, user-defined effects, and monadic reification, we show that the task of verifying relational properties requires little additional effort from the F programmer.

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1 INTRODUCTION

Generalizing unary properties (which describe single runs of programs), relational properties describe multiple runs of one or more programs. Relational properties are useful when reasoning about program refinement, approximation, equivalence, as well as many forms of security. A great many relational program analyses have been proposed in the recent literature, including works by Antonopoulos et al. (2017); Asada et al. (2016); Banerjee et al. (2016); Barthe et al. (2012, 2013b, 2014, 2015); Benton et al. (2009); Ștefan Ciobăcă et al. (2016); Godlin and Strichman (2010); Hedin and Sabelfeld (2012); Kundu et al. (2009); Küsters et al. (2015); Yang (2007); Zaks and Pnueli (2008); and Çiçek et al. (2017). While some systems have been designed for verifying very specific relational properties of programs (notably information-flow type systems, e.g., Sabelfeld and Myers (2003a)), others support verifying larger classes of relational properties. This includes tools based on product program construction that aim at automatically proving relations between first-order imperative programs (e.g., SymDiff...
(Lahiri et al. 2012) and Descartes (Sousa and Dillig 2016)), as well as approaches based on relational program
logics (Benton 2004) that support interactive verification of relational properties within proof assistants (e.g.,
EasyCrypt (Barthe et al. 2012) and RHTT (Nanevski et al. 2013)).

To save the need for bespoke relational logics and other special-purpose tools, we propose a simple and flexible
method for relational reasoning within existing dependently typed proof assistants. The essence of our proposal is
simple: use monads to model and program effectful computations; and use Filinski’s (1994) monadic reification
to carefully control monadic abstraction and reveal the pure representations of a computation in support of speci-
fication and proof. As such, we reduce the problem of relating effectful computations to relating their reified pure
representations, support for reasoning about pure programs being already very advanced in proof assistants.

While our proposal should be usable in a variety of proof assistants, we choose to work in F⋆ (Swamy et al.
2016), a dependently typed programming language and proof assistant. By relying on its support for SMT-based
automation, unary weakest preconditions, and user-defined effects (Ahman et al. 2017), we demonstrate, through
several examples, that our approach enables the effective verification of relational properties with an effort com-
parable to proofs of unary properties in F⋆ and to proofs in relational logics with SMT-based automation.

Our approach of encoding a variety of relational proof techniques within a general-purpose dependently typed
language has many benefits, including minimizing the metatheoretic arguments that accompany each new rela-
tional logic. Furthermore, for systems based on interactive proof, our unified approach facilitates comparing and
composing various relational analyses within the same framework. Even if dedicated tools are eventually needed
to achieve fully automated relational analysis for a specific class of programs, our method still usefully applies for
initial prototyping and for certifying their usage within a proof assistant. While it would be premature to claim
that our work is an outright substitute for the growing body of work on relational verification, it at least provides
a versatile generic alternative.

1.1 Relational reasoning via monadic reification: A first example

We sketch the main elements of our method on a proof of equivalence of the two stateful, recursive functions
below:

```ocaml
let rec sum_up r lo hi =
  if lo < hi then
    (r := !r + lo; sum_up r (lo + 1) hi)
let rec sum_dn r lo hi =
  if lo > hi then
    (r := !r + hi - 1; sum_dn r lo (hi - 1))
```

Both functions sum all numbers between lo and hi into some accumulator r, the former by counting up from lo,
the latter by counting down from hi.

**Unary reasoning about monadic computations** Most prior relational analyses target reasoning about such
effectful computations. So, as a first step, we embed effectful computations within a dependently typed language.
There have been many proposals for how to do this—one straightforward approach is to encapsulate effectful
computations within a parameterized monad (Atkey 2009). In F⋆, as in Hoare Type Theory (Nanevski et al. 2008),
these monads are indexed by a computation’s pre- and postconditions and proofs are conducted using a unary
program logic (i.e., not relational), adapted for use with higher-order, dependently typed programs. Beyond state,
F⋆ supports reasoning about unary properties of a wide class of user-defined monadic effects, where the monad
used to model an effect can be chosen to best suit the intended style of unary proof.

**Relating reified effectful terms** Our goal is to conveniently state and prove properties that relate multiple ef-
fective terms, e.g., to prove sum_up and sum_dn equivalent. Our general recipe for doing so is inspired in part
by Filinski’s (1994) reify operator, a coercion that reveals the pure representation of an effectful computation.
Whereas Filinski’s use was to uniformly implement monads using continuations, we apply monadic reification to
reveal the pure representations of a computation in support of specification and proof only. Thus, in order to relate effectful terms, one simply reasons about their pure reifications. Turning to our example, we prove the following lemma, expressed as a total function in $F^\star$, stating that running $\text{sum}_\text{up}$ and $\text{sum}_\text{dn}$ in identical initial states produces equivalent final states. (A short proof is given in §2.4.)

\[
\text{r : ref} \rightarrow \text{lo : int} \rightarrow \text{hi : int}\{\text{hi} \geq \text{lo}\} \rightarrow \text{h : heap}\{r \in h\} \rightarrow \text{reify}\left(\text{sum}_\text{up}\ r \ \text{lo} \ \text{hi}\right) \ h \sim \text{reify}\left(\text{sum}_\text{dn}\ r \ \text{lo} \ \text{hi}\right) \ h
\]

**A flexible specification and proving style with SMT-backed automation** Although seemingly simple, proving $\text{sum}_\text{up}$ and $\text{sum}_\text{dn}$ equivalent is somewhat cumbersome, if at all possible, in most prior relational program logics. Prior relational logics rely on common syntactic structure and control flow between multiple programs to facilitate an analysis. To reason about transformations such as loop reversal, rules that exploit syntactic similarity are not very useful and instead a typical proof in prior systems may involve several indirections, e.g., first proving the correctness of each loop and then showing that both loops have equivalent functional specifications. Through monadic reification, effectful terms are *self-specifying*, removing the need for tedious rewriting of the same code in a pure style just to enable specification and reasoning.

Further, whereas many prior systems are specialized to proving binary relations, it is often convenient to structure proofs using relations of a higher arity, a style naturally supported by our method. For example, a key lemma in the proof of $\text{equiv}_\text{sum}_\text{up}_\text{dn}$ is an inductive proof of a ternary relation, which states that $\text{sum}_\text{up}$ is related to $\text{sum}_\text{up}$ on a prefix combined with $\text{sum}_\text{dn}$ on a suffix of the interval $[\text{lo}, \text{hi})$.

Finally, using the combination of typechecking, weakest precondition calculation, and SMT solving provided by $F^\star$, many relational proofs go through with a degree of automation comparable to existing proofs of unary properties, at least as highlighted by the examples in the paper.

1.2 Structure of the paper

This paper is inspired by a one-line example given by Ahman et al. (2017). They introduce monadic reification in $F^\star$ and sketch how it might be useful to prove relational properties related to information-flow control. Elaborating on their idea, we present a methodology for relational verification (section 2), covering both broadly applicable ingredients such as representing effects using monads and monadic reification, as well as our use of specific features in $F^\star$ that enable proof flexibility and automation.

The rest of the paper is structured as a series of “small pearls” illustrating our methodology at work. Through these examples we aim to show that our technique enables comparing and composing various styles of relational program proof in the same system, thus taking a step towards unifying several prior strands of research. Our examples are divided into four sections that can be read in any order, each being an independent case study:

**Transformations of effectful programs (section 3)** We develop an extensional, semantic characterization of a stateful program’s read and write effects, based on the relational approach of Benton et al. (2006). Using these semantic read and write effects, we derive commonly used rules used to prove the correctness of program transformations, e.g., swapping the order of two commands, eliminating redundant writes, etc.

**Cryptographic security (section 4)** We show how to model basic code-based, game-based cryptographic proofs (Bellare and Rogaway 2006) by proving equivalences between probabilistic programs. Our example proves the security of one-time pads, an elementary use of Barthe et al.’s (2009) probabilistic relational Hoare logic, used with a higher-order program.

**Information flow control (section 5)** We encode several styles of information flow control, while accounting for declassification. Highlighting the ability to compose various proof styles in a single framework, we combine automated, security-type based analysis of imperative programs with SMT-backed, semantic proofs of noninterference.
Proofs of algorithmic optimizations (section 6)  With a few exceptions, prior relational program logics apply to first-order programs, providing only incomplete proof rules that exploit similarities in syntactic structure between the programs in relation. Not being bound by syntax, we prove relations of higher arities (e.g., 4-ary and 6-ary relations) between higher-order, effectful programs with differing control flow by reasoning about the reified semantics of effectful terms. We present two larger examples. First, we show how to memoize a recursive function using McBride’s (2015) partiality monad and prove it equivalent to the original non-memoized version. Then, we implement a union-find data structure, adding the classic union-by-rank and path compression optimizations in several steps, proving each a refinement.

Through these case studies we conclude that the “relational reasoning about reified monadic computations” methodology is both conceptually simple and flexible. We are encouraged to continue research in this direction, aiming to place proofs of relational properties of effectful programs on an equal footing with proofs of pure programs in F\* as well as other proof assistants and verification tools.

The real code for the examples in this paper is available at https://github.com/FStarLang/FStar/tree/c_relati...
bind = \( \lambda (a \ b : \text{Type}) \ (f : \text{st mem} \ a) \ (g : a -> \text{st mem} \ b) \ (m : \text{mem}) \rightarrow \text{let} \ z, m' = f \ m \ \text{in} \ g \ z \ m' \);
get = \( \lambda () \ (m : \text{mem}) \rightarrow m \);
put = \( \lambda (m : \text{mem}) \rightarrow () , m \)

This code defines the return and bind of this monad, together with two actions: get for obtaining the current memory, and put for updating the memory. The new effect \( \text{STATE}_m \) is parameterized over a type of memories, which allows us to choose a memory model best suited to the programming and verification task at hand. We often instantiate mem to heap (a map from references to their values, as in ML), obtaining the \( \text{STATE} \) effect, as shown below—we use other types of memory in section 4, section 5 and section 6.

**total new effect \( \text{STATE} = \text{STATE}_m \) heap**

While such monad definitions could in principle be used to directly extend the implementation of any functional language with state, a practical language needs to allow keeping the representation of some effects abstract so that they are efficiently implemented primitives (PeytonJones 2010). \( F^* \) uses its simple module system to keep the monadic representation of the \( \text{STATE} \) effect abstract and implements it under the hood using the ML heap, rather than state passing (and similarly for the other primitive ML effects such as exceptions). Whether implemented, or not, the monadic definition of each effect is always the model used by \( F^* \) to reason about effectful code, both intrinsically using a (non-relational) weakest precondition calculus (subsection 2.2) and extrinsically using monadic reification (subsection 2.3).

For the purpose of verification, monads provide great flexibility in the modeling of effects, which enables expressing relational properties and conducting proofs at the right level of abstraction. In particular, since the difficulty of reasoning about effectful code is proportional to the complexity of the effect, we do not use a single very complex monad for all the code, but instead define monads for sub-effects and relate them using monadic lifts. For instance, we define a \( \text{READER} \) monad for computations that only read the store, lifting \( \text{READER} \) to \( \text{STATE} \) only where necessary (subsection 5.1 provides a detailed example of its use). While \( F^* \) code is always written in an ML-like direct style, the \( F^* \) typechecker automatically inserts binds, returns and lifts under the hood (Swamy et al. 2011).

### 2.2 Unary weakest preconditions for user-defined effects and intrinsic proof

For each user-defined effect, \( F^* \) derives a weakest precondition calculus for specifying unary properties and computing verification conditions for programs using that effect. Each effect definition induces a computation type indexed by a predicate transformer describing that computation’s effectful semantics.

In the case of \( \text{STATE} \), we obtain a computation type \( \text{‘STATE} \ a \ \text{wp} \) indexed by a result type \( a : \text{Type} \) and by \( \text{wp} \), a predicate transformer of type \( (a 

\text{get : unit \rightarrow \text{STATE} \ heap (} \lambda (post: (heap \rightarrow \text{heap} \rightarrow \text{Type}))(h: \text{heap}) \rightarrow post \ h \ h) \)

\text{put : h:heap \rightarrow \text{STATE} \ heap (} \lambda (post: (unit \rightarrow \text{heap} \rightarrow \text{Type}))(\_\text{heap}) \rightarrow post () \ h) \)

The type of get states that to prove any postcondition \( \rightarrow \) evaluated in state \( h \), it suffices to prove post \( h \ h \) whereas for put \( h \) it suffices to prove post \( () h \). Rather than indexing computations types with predicate transformers, it is idiomatic in \( F^* \) to index computations with pre- and postconditions (as in Hoare Type Theory (Nanevski et al. 2008), or sometimes not at all, using the following abbreviations:

\( \text{ST} \ a \ (\text{requires} \ p) \ (\text{ensures} \ q) = \text{STATE} \ a \ (\lambda \ \text{post} \ h_0 \rightarrow p \ h_0 \land (\forall \ (x:a) (h_1:\text{heap}). \ q \ h_0 \times h_1 \Rightarrow \text{post} \ x \ h_1)) \)

\( \text{St} \ a = \text{ST} \ a \ (\text{requires} \ (\lambda : \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow 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F* computes weakest preconditions generically for any effect. Intuitively, this works by putting the code into an explicit monadic form and then translating the binds, returns, and lifts from the expression level to the weakest precondition level. This enables a convenient form of *intrinsic* proof in F*, i.e., one annotates a term with a type capturing properties of interest; F* computes a weakest precondition for the term and compares it to the annotated type using a built-in subsumption rule, checked by an SMT solver. For example, in the code below, F* checks that the inferred computation type is sufficient to prove that noop leaves the memory unchanged.

\[
\text{let} \quad \text{noop}() : \text{ST}\ unit \ (\text{requires} \ (\lambda \ m \to \top)) \ (\text{ensures} \ (\lambda \ m \to m \equiv m')) = \text{put} \ (\text{get}())
\]

For a more interesting example, we turn to \text{sum\_up} from §1.1, which can be given the following intrinsic type:

\[
r : \text{ref} \ \text{int} \to \text{lo}\ \text{nat} \to \text{hi}\ \text{nat}\{\text{hi} \geq \text{lo}\} \to \text{ST}\ \text{unit} \ (\text{requires} \ \lambda \ h \to r \in h) \ (\text{ensures} \ \lambda \ h \to r \in h)
\]

This is a dependent function type, for a function with three arguments \(r\), \(lo\), and \(hi\) returning a terminating, stateful computation. The *refinement* type \(hi:\text{nat}\{hi \geq lo\}\) restricts \(hi\) to only those natural numbers greater than or equal to \(lo\). The computation run by `\text{sum\_up}\ r\ \text{lo} \ \text{hi}` simply requires and ensures that its reference argument \(r\) is present in the memory. F* computes a weakest precondition from the implementation of \text{sum\_up} (using the types of (\!) and (\Rightarrow)) provided by the heap memory model used by \text{STATE}) and proves that its inferred specification is subsumed by the user-provided annotation. The same type can also be given to \text{sum\_dn}.

### 2.3 Exposing effect definitions via monadic reification

Intrinsic proof of effectful programs in F* is inherently restricted to unary properties. Notably, the pre- and postcondition of a program are required to be pure terms, making it impossible for the specification of one effectful program to refer directly to another, e.g., \text{sum\_up} cannot directly use itself or \text{sum\_dn} in its specification. To break this restriction, we need a way to coerce an effectful computation to a pure term—Filinski’s (1994) monadic reification provides just that facility.

Each new user-defined effect in F* induces a \text{reify} operator that exposes the representation of a computation type in terms of its underlying total function.\(^1\) For instance for the \text{STATE} effect, F* provides the following (derived) rule for \text{reify}, to coerce a stateful computation to a total, explicitly state-passing function.

\[
S;\Gamma \vdash e : \text{ST}\ t \ (\text{requires} \ \text{pre}) \ (\text{ensures} \ \text{post})
\]

\[
S;\Gamma \vdash \text{reify}\ e : h:\text{heap}\{\text{pre} \ h\} \to \text{Tot} \ (r:\{\text{t} = \text{heap}\}\{\text{post} \ h \ (\text{fst} \ r) \ (\text{snd} \ r)\})
\]

Armed with \text{reify}, we can write an *extrinsic* proof of a lemma relating \text{sum\_up} and \text{sum\_dn} (discussed in detail in §2.4), i.e., a proof of a lemma, separate from the definition of \text{sum\_up} and \text{sum\_dn}, that relates their executions. The \text{reify} operator clearly breaks the abstraction of the underlying monad and needs to be used with care. Ahman et al. (2017) show that programs that do not use these operators can be compiled efficiently. Specifically, if the computationally relevant part of a program is free of \text{reify} then the \text{STATE} computations can be compiled to using primitive state with destructive updates.

To retain these benefits of abstraction, we rely on F*’s module system to control how this abstraction-breaking coercion can be used in client code. In particular, when abstraction violations cannot be tolerated, we use F*’s \text{Ghost} effect (explained further in §2.4) to mark \text{reify} as being usable only in computationally irrelevant code, limiting the use of monadic reification to specifications and proofs. This allows one to use reification even though state, exceptions, and some of the other effects are implemented primitives in F*.

\(^1\)Less frequently, we use \text{reify}'s dual, \text{reflect}, which packages a pure function as an effectful computation. Reflect is useful for effect handlers and for hiding unobservable effects.

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2.4 Extrinsic specification and proof, eased by SMT-based automation in F^⋆

Our basic idea of extrinsically proving relational properties via monadic reification is simple and should be applicable in a variety of settings. We now look at the proof relating sum_up and sum_dn in detail, explaining along the way several F^⋆-specific idioms that we find essential to making our method work well.

Computational irrelevance (aka the Ghost effect) A built-in computation type in F^⋆, Ghost, is used to track a form of computational irrelevance. The computation type Ghost t (requires pre) (ensures post) is the type of a computation that returns a value of type t satisfying post, provided pre is valid. However, the computation must be erased before running the program.

Adding proof irrelevance (aka Lemma) F^⋆ provides two closely related forms of proof irrelevance. First, when a term (e:t) can be given the refinement type x:t(φ), it validates the formula φ[e/x], although no proof of φ is materialized. For example, borrowing the terminology of Nogin (2002), the value () is a squashed proof of u:unit{0 ≤ 1}. Combining proof and computation irrelevance, e : Ghost unit pre (λ() → post) is a squashed proof of pre → post. This latter form is so common that we write it as Lemma (requires pre) (ensures post) and further abbreviate Lemma (requires ⊤) (ensures post) as Lemma post.

A proof by induction relating sum_up and sum_dn Spelling out the main lemma of §1.1, our goal is a value of the following type (where v r (_, h) = h,[r] and h,[r] selects the contents of the reference r from the heap h).

val equiv_sum_up_dn (r:ref int) (lo:int) (hi:int{hi ≥ lo}) (h:heap{r ∈ h})  : Lemma (v r (reify (sum_up r lo hi) h) = v r (reify (sum_dn r lo hi) h))

An attempt to give a trivial definition equiv_sum_up_dn = λ_._._. → () fails, because the SMT solver cannot automatically prove the strong postcondition of equiv_sum_up_dn. Instead our proof involves calling an auxiliary lemma sum_up_dn_aux, a ternary relation shown below:

let equiv_sum_up_dn r lo hi h = sum_up_dn_aux r lo hi hi h

While the statement of equiv_sum_up_dn above is different from the statement of sum_up_dn_aux below, the SMT-based automation fills in the gaps and accepts the proof sketch.

val sum_up_dn_aux (r:ref int) (lo:int) (mid:int{mid ≥ lo}) (hi:int{hi ≥ mid}) (h:heap{r ∈ h})  : Lemma (v r (reify (sum_up r lo hi) h) = v r (reify (sum_dn r lo mid) h) + v r (reify (sum_up r mid hi) h) − h,[r]) (decreases (mid − lo))

In particular, the SMT solver figures out that sum_up r hi hi is a no-op by looking at its reified definition. In other cases, the user has to provide more interesting proof sketches that include not only calls to lemmas that the SMT solver cannot automatically apply but also the cases of the proof and the recursive structure. This is well illustrated by the proof of sum_up_dn_aux:

let rec sum_up_dn_aux r lo mid hi h = if lo ≠ mid then (sum_up_dn_aux r lo (mid − 1) hi h;
sum_up_commute r mid hi (mid − 1) h;
sum_dn_commute r lo (mid − 1) (mid − 1) h)

This proof is by induction on the difference between mid and lo (as illustrated by the decreases clause of the lemma, this is needed because we are working with potentially-negative integers). If this difference is zero, then the property is trivial since the SMT solver can figure out that sum_dn r lo lo is a no-op, so we can simply prove this case by returning the unit value. Otherwise, we call sum_up_dn_aux recursively for mid – 1 as well as two further commutation lemmas (not shown) about sum_up and sum_dn and the SMT automation can take care of the rest.
Making this work well involved bringing together all the features discussed here and extending the SMT encoding of F* to deal well with reification: it basically uses (an extended version of) the F* normalizer to partially evaluate the term to reveal its monadic structure. Computationally, based on the semantics of Ahman et al. (2017), reify traverses the term and unfolds the monadic definitions of return, bind, actions and lifts. What is left to encode to SMT is very close to the monadic code one would write by hand, which F* could encode already (Aguirre et al. 2016).

3 CORRECTNESS OF PROGRAM TRANSFORMATIONS
Several researchers have devised custom program logics for proving the correctness of transformations of imperative programs (Barthe et al. 2009; Benton 2004; Carbin et al. 2012). We show how to derive similar rules justifying the correctness of generic program transformations within our monadic framework. We focus on stateful programs with a fixed, finite memory. We leave a treatment of proving transformations of commands that dynamically allocate memory to future work; in a somewhat different context, §6 contains examples that use dynamic allocation and local state.

3.1 An extensional specification of a command’s footprint
A command \( c \) is a function of type \( \text{unit} \rightarrow \text{St\ unit} \) that may read or write arbitrary references in memory.

\[
\text{type command = unit } \rightarrow \text{St\ unit}
\]

In trying to validate transformations of commands, it is traditional to employ an effect system to delimit the parts of memory that a command may read or write. Most effect systems are unary, syntactic analyses. For example, consider the classic frame rule from separation logic:

\[
\{P\}c\{Q\} \Rightarrow \{P \ast R\}c\{Q \ast R\}
\]

The command \( c \) requires ownership of a subset of the heap \( P \) in order to execute, then returns ownership of \( Q \) to its caller. Any distinct heap fragment \( R \) remains unaffected by the function. Reading this rule as an effect analysis, one may conclude that \( c \) may read or write the \( P \)-fragment of memory—however, this is just an approximation of \( c \)'s extensional behavior. Benton et al. (2006) observe that a more precise, semantic characterization of effects arises from a relational perspective. Adopting this perspective, one can define the footprint of a command extensionally, using two unary properties and one binary property.

Circumscribing the write effect of a command is straightforward with a unary property: ‘writes \( c \) \( \text{ws} \)’ states that the initial and final heaps agree on the contents of all their references, except for those in the set \( \text{ws} \).

\[
\text{type addr} = \text{S.set\ addr}
\]

\[
\text{let writes (c:\text{command}) (ws:addr)} = \forall (h:\text{heap}).
\]

\[
\begin{align*}
\text{let h' = snd (reify (c () h) in}
\end{align*}
\]

\[
(\forall r. r \in h \iff r \in h') \land \left( \star \text{no allocation} \star \right)
\]

\[
(\forall r. \text{addr.of } r \notin \text{ws} \implies h.[r] = h'.[r]) \left( \star \text{only references in ws are modified} \star \right)
\]

Stating that a command only reads references \( \text{rs} \) is similar in spirit to the statement of noninterference (to which we return in §5.1). Interestingly, it is impossible to describe the set of locations that a command may read without also speaking about the locations it may write. The relation ‘reads \( c \) \( \text{rs} \) \( \text{ws} \)’ states that if \( c \) writes at most the references in \( \text{ws} \), then executing \( c \) in heaps that agree on the references in \( \text{rs} \) produces heaps that agree on \( \text{ws} \), i.e., \( c \) does not depend on references outside \( \text{rs} \).

\[
\text{let equiv.on (rs:addr_set) (h0:heap) (h1:heap)} =
\]

\[
\forall (r:\text{ref a}). \text{addr.of } r \in \text{rs} \land r \in h_0 \land r \in h_1 \implies h_0.[r] == h_1.[r]
\]

\[
\text{let reads (c:\text{command}) (rs ws:addr)} =
\]

3.2 Several transformations on commands

Making use of relational footprints, we can prove other relations between commands, e.g., equivalences that justify program transformations. Command equivalence \( c_0 \sim c_1 \) states that running \( c_0 \) and \( c_1 \) in identical initial heaps produces (extensionally) equal final heaps.

\[
\forall (h_0, h_1: \text{heap}).
\begin{align*}
\text{let } h'_0, h'_1 &= \text{snd } (\text{reify } (c ()) h_0), \text{snd } (\text{reify } (c ()) h_1) \text{ in} \\
(\text{equiv}_\text{on } rs h_0 h_1 \land \text{writes } c \text{ ws}) \implies \text{equiv}_\text{on } ws h'_0 h'_1
\end{align*}
\]

Putting the pieces together, we define a read- and write-footprint-indexed type for commands:

\[
\text{type } \text{cmd } (\text{rs } \text{ws}: \text{addr}) = c: \text{command} \{ \text{writes } c \text{ ws } \land \text{reads } c \text{ rs } \text{ws}\}
\]

One can also define combinators to manipulate footprint-indexed commands. For example, here is a ‘\( \gg \)’ combinator for sequential composition. Its type proves that read and write-footprints compose by a pointwise union, a higher-order relational property; the proof requires an (omitted) auxiliary lemma \( \text{seq}_{\text{lem}} \) (recall that variables preceded by a \# are implicit arguments):

\[
\begin{align*}
\text{let } &\text{seq } (\#r1 \#w1 \#r2 \#w2 : \text{addr}) \text{ (c1:cmd r1 w1) (c2:cmd r2 w2) : command } = c1(); c2() \\
\text{let } & (\gg) \#r1 \#w1 \#r2 \#w2 \text{ (c1:cmd r1 w1) (c2:cmd r2 w2) : cmd } (r1 \cup r2) (w1 \cup w2) = \text{seq}_{\text{lem}} c1 c2; \text{ seq } c1 c2
\end{align*}
\]

Our first equivalence, listed below, shows that if a command’s read and write footprints are disjoint, then it is idempotent. The proofs of idem and the other lemmas below are perhaps peculiar to SMT-based proofs. In all cases, the proofs involve simply mentioning the terms \( \text{reify } (c ()) h \), which suffice to direct the SMT solver’s quantifier instantiation engine towards finding a proof. While more explicit proofs are certainly possible, with experience, concise SMT-based proofs can be easier to write.

\[
\begin{align*}
\text{let } & (\sim ) (c0: \text{command}) (c1: \text{command}) = \forall h. \\
\text{let } & h_0, h_1 = \text{snd } (\text{reify } (c0 ()) h), \text{snd } (\text{reify } (c1 ()) h) \text{ in} \\
& \forall (a: \text{Type}) (r: \text{ref } a). (r \in h_0 \iff r \in h_1) \land (r \in h_0 \implies h_0.[r] == h_1.[r])
\end{align*}
\]

Our next equivalence shows that two commands can be swapped if they write to disjoint sets, and if the read footprint of one does not overlap with the write footprint of the other—this lemma is identical to a rule for swapping commands in a logic presented by Barthe et al. (2009).

\[
\begin{align*}
\text{let } &\text{swap } \#rs \#ws1 \#ws2 \text{ (c1:cmd rs1 ws1) (c2:cmd rs2 ws2)} \\
& : \text{Lemma } (\text{requires } (\text{disjoint } rs \text{ ws})) (\text{ensures } ((c >> c) \sim c)) \\
& = \forall \text{intro } (\lambda h. h \to \text{let }, h_1 = \text{reify } (c ()) h \text{ in } \_ = \text{reify } (c ()) h_1 \text{ in } \_ <\: \text{Lemma } (\text{equiv}_\text{on } h (c >> c) c h))
\end{align*}
\]

Next, we show elimination of redundant writes by proving that \( c1 >> c2 \) is equivalent to \( c2 \) if \( c1 \)'s write footprint is (a) a subset of \( c2 \)'s write footprint, and (b) disjoint from \( c2 \)'s read footprint.

\[
\begin{align*}
\text{let } &\text{redundant writes } \#rs \#rs1 \#ws1 \#ws2 \text{ (c1:cmd rs1 ws1) (c2:cmd rs2 ws2)} \\
& : \text{Lemma } (\text{requires } (\text{disjoint } rs1 \text{ ws1 } \subseteq \text{ws1}) \text{ ws1} \subseteq \text{ws2}) (\text{ensures } ((c1 >> c2) \sim c2)) \\
& = \forall \text{intro } (\lambda h. h \to \text{let } \_ = \text{reify } (c1 ()) h, \text{reify } (c2 ()) h \text{ in } \_ <\: \text{Lemma } (\text{equiv}_\text{on } h (c1 >> c2) c2 h))
\end{align*}
\]

Finally, we consider conditional commands and show that if two branches of a conditional command are equivalent, then the conditional command itself is equivalent to both the branches. We first define the conditional combinator, and then show the equivalence.

(* guard does not change the heap *)

type guard = f : (unit → St bool) { ∀ (h:heap). h == snd (reify (f ()) h) }

let ite (c:guard) (c1:command) (c2:command) : command = λ () → if c () then c1 () else c2 ()

let dead_code (c:guard) (c1:command) (c2:command) :

Lemma (requires (c1 ~ c2)) (ensures (ite c c1 c2) ~ c1)

= ∀ _intro (λ h → let _, _ = reify (c1 ()) h, reify (c2 ()) h in () : Lemma (equiv_on_h (ite c c1 c2) c1 h))

4 CRYPTOGRAPHIC SECURITY

In this section, we show how to construct a simple model for reasoning about programs with random sampling. We focus, in particular, on proving equivalence relations between the distributions computed by such programs, relying on a methodology developed by Barthe et al.'s (2009) probabilistic relational Hoare logic (pRHL) to formalize standard cryptographic arguments. In comparison with the proofs carried out in tools based on pRHL (e.g., EasyCrypt), our example is extremely simple. Nevertheless, our encoding provides an easy way to prototype and explore proofs in logics like pRHL with a low entry cost.

4.1 A monad for random sampling

Code-based cryptographic proofs model cryptographic constructions as probabilistic programs. Proving the security of constructions often involves proving relations among their programmatic models, e.g., secrecy properties are often stated in terms of the indistinguishability of two runs of a program (a form of probabilistic noninterference).

To support such reasoning, we begin by defining a monad for random sampling, a hybrid between a reader and a writer monad. The RANDOM effect provides a single action, sample, which reads an infinite random tape at the current position and advances it.

type tape = int → byte

type random a = (int * tape) → Tot (a * int)

total new_effect { RANDOM : a:Type → Effect with repr = random a;

bind = λ (a b:Type) (x:random a) (f: a → random b) s → let z, n = x s in f z (n, snd s);

return = λ (a:Type) (x:a) s → (x, fst s);

sample = λ () s → let n, tape = s in tape n, n + 1 }

effect Rand a = RANDOM a (λ initial_tape post → ∀(x:(a * int)). post x)

4.2 Proving the perfect security of encryption based on one-time pads

Consider the following one-byte, one-time pad construction. The function mk otp samples a random key, and then returns two closures, one for encrypting a message (simply by XOR’ing it with the random key), and another for decrypting the message (also by XOR’ing it with the key).2

1In this case, encrypt and decrypt are identical; however, we use this style to evoke many other cryptographic constructions (e.g., El Gamal encryption) which have a similar structure and proof of security.

let mk otp () : Rand ((byte → byte) * (byte → byte)) =
let key = sample () in
let encrypt (msg:byte) = msg ⊕ key in
let decrypt (cipher:byte) = key ⊕ cipher in
encrypt, decrypt

Provided encryption is called on a single message, and decryption is called only on the resulting ciphertext, one-time pads are perfectly secure, inasmuch as the distribution of ciphers observed by the adversary are indistinguishable from the random distribution. Such a proof is possible in pRHL, a relational Hoare logic for while programs, similar in spirit to Benton’s (2004) except with support for random sampling. A significant novelty of pRHL is its rule for relating two uses of random sample, i.e., its (R-Rand) rule, a simplified variant of which we reproduce below, where \( f \) is a bijection on byte:

\[
\{ \forall y. \Phi[x_0 \mapsto y, x_1 \mapsto f y] \} \ x_0 \leftarrow \text{sample}() \sim x_1 \leftarrow \text{sample}() \{ \Phi \}
\]

The rule is a relational Hoare triple stating that two runs of sample() produce results \( x_0, x_1 \) satisfying \( \Phi \), if one can prove \( \Phi \) just for \( y, f y \). In a nutshell, (R-Rand) allows relating two values sampled uniformly at random by a chosen bijection \( f \). The rule formally captures the intuition that if a every run of a probabilistic program producing a given outcome can be matched with another run (provided by the bijection) producing the same outcome, then the distribution of outcomes is uniformly random. We encode the essence of (R-Rand) in the following proof of security and correctness of one-time pads.

First, we lift bijections on bytes to bijections on tapes, pointwise—two tapes are related by \( b \) if they are related at point \( i \) by \( b \ i \), where \( \text{bij} \) is the type of bijections on bytes (i.e., a refinement of byte → byte from F*’s library).

let related (b:int → bijection) (t_0 t_1:tape) = \forall i. b i (t_0 i) = (t_1 i)

The following lemma establishes the security and correctness of one-time pads by showing that two runs of one-time pad on related tapes produce indistinguishable plain texts, and that decryption correctly recovers the message from a cipher. (The lemma statement captures the correct “one time” usage of our construction by encrypting \( x_i \) and decrypting \( \text{enc}_{c_i} x_i \) with tape \( \text{tape}_i \), for a pair of arbitrary but fixed messages.)

let xor @ (pos:int) (x : byte) (i:int) : bijection = if i = pos then x xor x else id
let one time pad @ ok x_0 x_1 tape_0 tape_1 : Lemma
(requires (related (xor @ (pos:int) (x : byte) (i:int)) tape_0 tape_1))
(ensures (let (enc_0, dec_0), _, = reify (mk otp ()) (0, tape_0) in
let (enc_1, dec_1), _, = reify (mk otp ()) (0, tape_1) in
enc_0 x_0 = enc_1 x_1 \&\& \{ \text{ciphers are indistinguishable *}\}
dec_0 enc_0 x_0 = x_0 \&\& dec_1 enc_1 x_1 = x_1 \{ \text{decryption is an inverse of encryption *}\}) = ()

The main insight behind such proofs is in choosing the relation between the tapes wisely. For one-time pads, the choice is relatively easy. Our bijection relates the tapes such that when \( \text{key}_0 = \text{tape}_0 0 \), we have \( \text{key}_1 = \text{tape}_1 0 = (x_0 \oplus x_1) \oplus \text{key}_0 \). Thus, when encrypting \( \text{enc}_1 x_1 = \text{key}_1 \oplus x_1 \), we produce \( \text{key}_0 \oplus x_0 \), a value identical to the first cipher. Of course, two runs of enc do not actually produce identical ciphers; yet, using (our encoding of) pRHL’s (R-Rand) precondition, and carefully choosing bijections between random tapes, we reduce reasoning about indistinguishability of distributions to reasoning about equality of values.

As a final remark, by embedding this style of relational reasoning in a proof assistant, we work naturally with higher-order programs. In contrast, pRHL (and most of its descendants) are only for first-order programs. Of course, we simply follow pRHL’s (R-Rand) rule—we do not (yet) develop all of pRHL’s metatheory within our framework.
5 INFORMATION FLOW CONTROL

In this section, we present a case study examining various styles of information flow control (IFC), a security paradigm based on noninterference (Goguen and Meseguer 1982), a property that compares two runs of a program differing only in the program’s secret inputs and requires the non-secret outputs to be equal. Many special-purpose systems, including syntax-directed type systems, have been devised to enforce noninterference-like security properties (see, e.g., Hedin and Sabelfeld 2012; Sabelfeld and Myers 2006).

- We start our IFC case study by showing how a classic IFC type system (Volpano et al. 1996) for a small, interpreted, embedded, imperative language can be encoded using our method. We derive the type system by proving each rule as a relation between two runs of the interpreter, mechanizing a proof of the type system’s correctness directly, rather than resorting to commonly used syntactic proof techniques (e.g., Pottier and Simonet’s (2003) bracketed semantics). (§5.1)

- Being syntax-directed, IFC type systems provide only an imprecise, though automated, analysis. However, our IFC type system, composes well with precise, semantic proofs for the parts of a program where syntactic analysis does not suffice. Inspired by the approach of Küsters et al. (2015), we show how to compose our IFC type system with semantic noninterference proofs. (§5.2)

- Noninterference is a useful baseline property for IFC. However, on its own, it is often too strong for practical use. The final step in our IFC case study is a semantic treatment of declassification based on delimited release (Sabelfeld and Myers 2003b). We show how our technique is applicable beyond embedded imperative languages, carrying out noninterference modulo delimited release proofs for F* programs in general. (§5.3)

In summary, we conclude that our method for relational verification is flexible enough to accommodate various IFC disciplines, allowing them to be compared and composed within the same framework.

5.1 Deriving an IFC type system

Consider the following small while language consisting of expressions, which may only read from the heap, but not modify it, and commands, which may write to the heap and branch, depending on its contents. The definition of the language should be unsurprising, the only subtlety worth noting is the decr expression in the while command, a metric used to ensure loop termination.

\[
\begin{align*}
  e &::= i \mid r \mid e_1 \oplus e_2 \\
  c &::= \text{skip} \mid r := e \mid c_1; c_2 \mid \text{if } e = 0 \text{ then } c_1 \text{ else } c_2 \mid \text{while } e \neq 0 \text{ do } c \text{ (decr } e')
\end{align*}
\]
A classic IFC type system. Volpano et al. (1996) devise an IFC type system for a similar language to check that programs executing over a memory containing both secrets (stored in memory locations labeled High) and non-secrets (in locations labeled Low) never leak secrets into non-secret locations. The type system includes two judgments $\Gamma \vdash e : l$, which states that the expression $e$ (with free variables in $\Gamma$) depends only on locations labeled $l$ or lower; and $\Gamma, pc : l \vdash c$, which states that a command $c$ in a context that is control-dependent on the contents of memory locations labeled $l$, does not leak secrets. Their system, as adapted to our example language, is shown in Figure 1.

Our goal in this section is to embed this while language in $F^*$, to define an interpreter for it, and to derive Volpano et al.’s type system by relating multiple runs of the interpreter. In doing so, we highlight several distinctive features of our approach, including the use of multiple monads to structure our interpreter and simplify our proofs.

Multiple effects to structure an interpreter for while. We deeply embed the syntax of while in $F^*$ using data types exp and com, for expressions and commands, respectively. The expression interpreter interp$_{\text{exp}}$ only requires reading the value of the variables from the store whereas the command interpreter, interp$_{\text{com}}$, also requires writes to the store, where store is an integer store mapping a fixed set of integer references ($\text{ref}$ int) to int. Additionally, interp$_{\text{com}}$ may also raise an Out of fuel exception when it detects that a loop may not terminate (e.g., because the claimed metric is not actually decreasing). We could define both interpreters using a single effect, but this would require us to prove that interp$_{\text{exp}}$ does not change the store and does not raise exceptions. Avoiding the needless proof overhead, we use a $\text{READER}$ monad for interp$_{\text{exp}}$ and STEXN, a combined state and exceptions monad, for interp$_{\text{com}}$. By defining $\text{READER}$ as a sub-effect of STEXN, expression interpretation is transparently lifted by $F^*$ to the larger effect when interpreting commands.

type reader (a : Type) = store $\to$ Tot a

total new$_{\text{effect}}$ \{ READER : a : Type $\to$ Effect
  with repr = reader;
  return = $\lambda$ (a : Type) (x : a) (s : store) $\to$ x;
  bind = $\lambda$ (a b : Type) (f : reader a) (g : a $\to$ reader b) (s : store) $\to$ let z = f s in g z s;
  get = $\lambda$ (s : store) $\to$ s \}

type stexn (a : Type) = store $\to$ Tot (either a exn * store)

total new$_{\text{effect}}$ \{ STEXN . . . \}

sub$_{\text{effect}}$ READER $\to$ STEXN \{ lift = $\lambda$ (a : Type) (f : reader a) (s : store) $\to$ let x = f s in (Inl x, s) \}

Using these effects, interp$_{\text{exp}}$ and interp$_{\text{com}}$ form a standard, recursive, definitional interpreters for while, with the following trivial signatures. Just as we sometimes use St, the unindexed version of STATE, here we use Reader and StExn, unindexed versions of READER and STEXN with trivial pre- and postconditions.

val interp$_{\text{exp}}$ : exp $\to$ Reader int
val interp$_{\text{com}}$ : com $\to$ StExn unit

Deriving IFC typing for expressions. For starters, we use a store$_{\text{labeling}} = \text{ref}$ int $\to$ label, where label $\in \{\text{High}, \text{Low}\}$, to partition the store between secrets (High) and non-secrets (Low). An expression is noninterferent at level $l$ when its interpretation does not depend on locations labeled greater than $l$ in the store. To formalize this, we define a notion of low-equivalence on stores, relating stores that agree on the contents of all Low-labeled references, and noninterferent expressions (at level Low, i.e., ni$_{\text{exp}}$ env e Low) as those whose interpretation is identical in low-equivalent stores.

type low$_{\text{equiv}}$ (env : store$_{\text{labeling}}$) (s0 s1 : store) =
  $\forall$ (r : ref int). env x = Low $\Rightarrow$ s0.[r] = s1.[r]
let ni\_exp (env:\text{store\_labeling}) (e:exp) (l:\text{label}) =
∀(s0 | s1:store). (\text{low\_equiv env s0 s1 ∧ l = Low}) \implies \text{reify (interp\_exp e) s0 = reify (interp\_exp e) s1}

With this definition of noninterference for expressions we capture the semantic interpretation of the typing judgment $\Gamma \vdash e : l$: if the expression $e$ can be assigned the label $l$ and if $l$ is Low, then the computation of $e$ is only influenced by Low values. Using this definition, we can derive the expression rules of Figure 1; for instance here is a lemma for the EBINOP rule:

\text{let binop\_exp (env:\text{store\_labeling}) (op:binop) (e1:e2:exp) (l:\text{label}) =}
\text{Lemma (requires (ni\_exp env e1 l ∧ ni\_exp env e2 l)) (ensures (ni\_exp env (AOp op e1 e2) l)) = ()}

We construct a lemma from the inference rule in a straightforward manner: the premise of the inference rule forms the \text{requires} clause, while the conclusion of the rule forms the \text{ensures} clause of our lemma. The proof for this lemma is simple and can be discharged purely by SMT, without the need of any further annotations. The other rules for expressions can be shown in the same way and all of them can be discharged by SMT.

\text{Deriving IFC typing for commands.} As explained previously, the judgment $\Gamma, \text{pc : l} \vdash c$ deems $c$ noninterferent when run in context control-dependent only on locations whose label is at most $l$. More explicitly, the judgment establishes the following two properties:

1. locations labeled below $l$ are not modified by $c$—this is captured by $\text{no\_write\_down}$, a unary property.
2. the command $c$ does not leak the contents of a High location to Low location—this is captured by $\text{ni\_com'}$, a binary property.

The type-system is intentionally termination-insensitive, meaning that a program may diverge depending on the value of a secret. Consider, for instance, two runs of the program while $hi < 0$ do \{\text{skip}\}; $lo := 0$, one with $hi = 0$ and another with $hi = 1$. The first run terminates and writes to $lo$; the second run loops forever. As such, we do not expect to prove noninterference in case the program loops.

\text{let run c s =}
match reify (interp\_com c) s with
| Inr Out\_of\_fuel, _ \rightarrow Loops
| _ | s' \rightarrow \text{Returns s'}

\text{let no\_write\_down env c l s =}
match run c s with
| Loops \rightarrow ⊤
| Returns s' \rightarrow ∃(i:id). env i < l \implies s'[i] = s[i]

\text{let ni\_com' env c l s0 s1 =}
match run c 0, run c s1 with
| Returns s0', Returns s1' \rightarrow low\_equiv env s0 s1 \implies low\_equiv env s0' s1'
| Loops, _ \rightarrow_loops \rightarrow ⊤

Putting the pieces together, we define $\Gamma, \text{pc : l} \vdash c$ to be $\text{ni\_com} \Gamma, c l$.

\text{let ni\_com (env:\text{store\_labeling}) (c:com) (l:\text{label}) = (∀s0 s1. ni\_com' env c l s0 s1) ∧ (∀s. no\_write\_down env c l s)}

As in the case of expression typing, we derive each rule of the command-typing judgment as a lemma about $\text{ni\_com}$. For example, here is our statement for the CCOND rule:

\text{val cond\_com (env:\text{store\_labeling}) (e:exp) (ct:com) (cf:com) (l:\text{label}) =}
\text{Lemma (requires (ni\_exp env e l ∧ ni\_com env ct l ∧ ni\_com env cf l)) (ensures (ni\_com env (If e ct cf) l))}

Unlike expression typing, the proofs of many of these command-typing rules are only partially automated by SMT—they take about 250 lines of specification and proof in F⋆. But, once proven, it is easy to build a certified, syntax-directed typechecker for while programs that repeatedly applies these lemmas to prove that a program satisfies $ni_{com}$. This certified typechecker has the following type:

$$\text{val } tc_{\text{com}} : \text{env}\text{.store}\text{labeling} \to c\text{:com} \to \text{Exn label (requires } \top) \to ni_{com} \text{ env} c \mid \_ \to \mathcal{T}$$

5.2 Combining syntactic IFC analysis with semantic noninterference proofs

Building on subsection 5.1, we show how programs that fall outside the syntactic information-flow typing discipline can be proven secure using a combination of our syntactic IFC type system and semantic proofs of noninterference. This example is evocative (though at a smaller scale) of the work of Küsters et al. (2015), who combine automated information flow analysis in the Joana analyzer (Hammer and Snelting 2009) with semantic proofs in the KeY verifier for Java programs (Darvas et al. 2005; Scheben and Schmitt 2011). In contrast, we sketch a combination of syntactic and semantic proofs of relational properties in a single framework.

Consider the following while program, where the label of c and lo is Low and the label of hi is High.

```plaintext
while c ≠ 0 do
  hi := lo + 1;
  lo := hi + 1;
  c := c − 1 (decr c)
```

The assignment $lo := hi + 1$ is ill-typed in the type system of §5.1, since it directly assigns a High expression to a Low location. However, the previous command overwrites hi so that hi does not contain a High value anymore at line 3. As such, even though the IFC type system cannot prove it, the program is actually noninterferent.

To prove it, one could directly attempt to prove $ni_{com}$ for the entire program, which would require a strong enough (relational) inductive invariant for the loop. However, a simpler approach is to prove just the sub-program at lines 2-3 ($c_{2,3}$) noninterferent, while relying on the syntax-directed type system for the rest of the program. The sub-program can be automatically proven secure:

```plaintext
let $c_{2,3}ni () : \text{Lemma} (ni_{com} \text{ env} (\text{Seq } c_{2} c_{3}) \text{ Low}) = ()
```

This lemma has exactly the form of the other standard, typing rules proven previously, except it is specialized to the two commands in question. As such, $c_{2,3}ni$ can just be used in place of the standard sequence-typing rule (CSEQ) when proving the while loop noninterferent.

We can even modify our automatic typechecker from the previous subsection to take as input a list of commands that are already proved noninterferent (by whichever means), and simply look up the command it tries to typecheck in the list before trying to typecheck it syntactically. The type (and omitted implementation) of this typechecker is very similar to that of $tc_{\text{com}}$, the only difference is the extra list argument:

```plaintext
\text{val } tc_{com\text{,hybrid}} : \text{env}\text{.store}\text{labeling} \to c\text{:com} \to \text{list (cl:(com\text{*label})\{ni_{com} \text{ env} (fst cl) (snd cl)})} \to \\
\text{Exn label (requires } \top) \to \text{ensures } \lambda ol \to \text{Inl? ol} \Rightarrow ni_{com} \text{ env} c (\text{Inl? v ol})
```

We can complete the noninterference proof automatically by passing the ($\text{Seq } c_{2} c_{3}, \text{Low}$) pair proved in the $ni_{com}$ relation by lemma $c_{2,3}ni$ (or directly by SMT) to this hybrid IFC typechecker variant:

```plaintext
let $c_{1,4}ni () : \text{Lemma} (\text{ensures } ni_{com} \text{ env } c_{1,4} \text{ Low}) = \\
  c_{2,3}ni(); \text{ignore (reify } (tc_{com\text{,hybrid}} \text{ env } c_{1,4} [\text{Seq } c_{2} c_{3}, \text{Low}]) ())
```

Typechecking works by simply evaluating the invocation of $tc_{com\text{,hybrid}}$ (which is well-typed because of the preceding invocation of $c_{2,3}ni$); this reduces fully to $\text{Inl Low}$ and the intrinsic type of $tc_{com\text{,hybrid}}$ guarantees ($ni_{com} \text{ env } c_{1,4} \text{ Low}$).
5.3 Semantic declassification

Beyond noninterference, reasoning directly about relational properties allows us to understand various forms of declassification where programs intentionally reveal some information about secrets. For example, Sabelfeld and Myers (2003b) propose delimited release, a discipline in which programs are allowed to reveal the value of only certain pure expressions.

An example provided by Sabelfeld and Myers is the following, simple scenario where some constant amount of money (k) is transferred from one account (hi) to another (lo). Simply by virtue of observing whether or not the funds are received, the owner of the lo account gains some information about the other account, namely whether or not hi contains at least k units of currency—this is, however, by design.

\[
\text{let transfer} \ (\text{k:int}) \ (\text{hi:ref int}) \ (\text{lo:ref int}) = \text{if k < !hi then (hi := !hi - k; lo := !lo + k)}
\]

To characterize this kind of intentional release of information, delimited release is a relational property that relates two runs of a program in initial states where the secrets, instead of being arbitrary, are related in some manner, e.g., the initial states may agree on the value of the term being explicitly declassified. This is easily captured in our setting. For example, we can prove the following lemma for transfer, which proves that lo gains no more information than intended.

\[
\text{let transfer}_{\text{ok}} \ (\text{k:int}) \ (\text{hi:lo:ref int})\{\text{addr_of lo ≠ addr_of hi}\} \ (\text{s0 s1:heap}) \ (\text{lo ∈ s0} \land \text{hi ∈ s0} \land \text{lo ∈ s1} \land \text{hi ∈ s1})
\]

: Lemma \ (\text{requires} \ (\text{s0.[lo] = s1.[lo]} \land \text{(* initial memories agree on lo *)})

\[
(k < s0.[hi] \iff k < s1.[hi]]) \text{ (* and also agree on the declassified term *)}
\]

\[
\text{(ensures} \ (\text{snd (reify (transfer k hi lo) s0).[lo]} = (\text{snd (reify (transfer k hi lo st1).[lo]} = ()
\]

Delimited release was about the what dimension of declassification (Sabelfeld and Sands 2009). We also built a very simple model that is targeted at the when dimension, illustrating a customization of the monadic model to the target relational property. For instance, to track when information is declassified, we augment the state with a bit recording whether the secret component of the state was declassified and is thus allowed to be leaked.

\[
\text{type ifc_state = \{ secret:int; public:int; release:bool \}}
\]

\[
\text{new_effect STATE_{IFC} = STATE_{hi ifc_state}}
\]

In this case the noninterference property we prove of individual programs depends on the extra instrumentation bit we added to the state.

\[
\text{let ri (f:unit → St unit) = ∀s0 s1. let (s0, s0'), (s1, s1') = reify (f ()) s0, reify (f ()) s1 in}
\]

\[
\text{s0'.release ∨ s1'.release ∨ (low equivalence s0 s1 ⇒ low equivalence s0' s1')}
\]

6 ALGORITHMIC OPTIMIZATIONS AND PROGRAM REFINEMENT

This section presents two fully worked examples to prove a few, classic algorithmic optimizations correct. These properties are very specific to their application domains and a special-purpose logic for various classes of relational properties would not be suitable. Instead, we make use of the generality of our approach to prove application-specific relational properties (including 4- and 6-ary relations) of higher-order programs with local state. In contrast, most prior relational logics are specialized to proving binary relations, or, at best, properties of n runs of a single first-order program (Sousa and Dillig 2016).

6.1 Memoizing recursive functions with a custom effect

We begin by looking at memoizing total functions dom → codom, including memoizing the recursive calls of a function based on a technique due to McBride (2015). We prove that a memoized function is extensionally equal to the original.

Our monadic framework for relational verification encourages the use of new monadic effects (subsection 2.1) to add a tailor-made effect `Memo` supporting the operations that we need for the task at hand. Abstracting the concrete details of memory management, the effect `Memo` is a state monad where the state consists of a (partial, finite) mapping from `dom` to `codom`. This effect is characterized by the two operations it supports:

- `get : dom → Memo (option codom)`, which returns an already memoized value if it exists; and
- `put : dom → codom → Memo unit`, which adds a new memoization pair into the heap.

This abstract model could be instantiated in the future with an efficient hash-table with a specific memory-management policy.

**Take 1: Memoizing a total function.** Our goal is to turn a pure total function `g` into a memoized function `f` computing the same values as `g`. This relation between `f`'s reification and `g` is captured by the `computes` predicate below, depending on an invariant of the memoization state, `valid_memo`. A memoization state `(h:memost)` is valid for memoizing some total function `g : (dom → codom)` when `h` is a subset of the graph of `g`:

```
let valid_memo (h:memost) (g:dom → codom) = forall prop (λ (x,y) → y == g x) h
```

So we have `f` `computes` `g` when any state `h0` containing a subgraph of `g`, `f x` returns `g x` and maintains the invariant that the result state `h1` is a subgraph of `g`. It’s easy to program and prove that a simple memoizing function is correct.

```
let memoize (g : dom → codom) (x:dom) = match get x with Some y → y | None → let y = g x in put x y; y
```

The proof of this lemma is straightforward: we only need to prove that the value `y` we get back from the heap in the first branch is indeed `g x` which is enforced by the `valid_memo` in the precondition of `computes`.

**Take 2: Memoizing the recursive calls of a recursive function.** Now, what if we want to memoize a recursive function such as, for example, a function computing the Fibonacci sequence? We also want to memoize the intermediate recursive calls, and in order to achieve it we need an explicit representation of the recursive structure of the function. Following McBride (2015), we can represent it by a function `(x:dom) → partial_result x`, where a partial result is either a finished computation of type `codom` or a request for a recursive call together with a continuation of the computation.

```
type partial_result (x0:dom) =
  | Done : codom → partial_result x0
  | Need : x:dom{x < x0} → cont:(codom → partial_result x0) → partial_result x0
```

As we define the fixed point using `Need x f`, we crucially require `x < x0`, meaning that value of the function is requested at a point `x` where function’s definition already exists. For example encoding the Fibonacci sequence with this presentation amounts to the following code where the 2 recursive calls in the second branch have been replaced by applications of the `Need` constructor:

```
let fib_skel (x:dom) : partial_result x =
  if x ≤ 1 then Done 0 else Need (x - 1) (λ y1 → Need (x - 2) (λ y2 → Done (y1 + y2)))
```

Given a function `f : dom → codom` we can define its total fixpoint:

```
let rec fixp (f : x:dom → partial_result x) (x0:dom) : codom =
  let rec complete_fixp x = function
    | Done y → y
```
To obtain its memoized variant, we need to memoize functions defined only on part of the domain, \(x: \text{dom}\{p x\}\).

\[
\text{let} \ \text{partial_memoize} \ (p: \text{dom} \rightarrow \text{Type}) \ (f: x: \text{dom}\{p x\} \rightarrow \text{Memo} \ \text{codom}) \ (x: \text{dom}\{p x\}) =
\]

\[
\text{match} \ \text{get} \ x \ \text{with} \ \text{Some} \ y \rightarrow y | \ \text{None} \rightarrow \text{let} \ y = g x \ \text{in} \ \text{put} x \ y; \ y
\]

\[
\text{let rec} \ \text{memoize_rec} \ (f: x: \text{dom} \rightarrow \text{Tot} \ (\partial \text{result} \ x)) \ (x: \text{dom}) : \text{Memo} \ \text{codom} =
\]

\[
\text{let rec} \ \text{complete_memo_rec} \ x = \text{function}
\]

\[
| \text{Done} \ y \rightarrow y
\]

\[
| \text{Need} \ x' \ \text{cont} \rightarrow \text{let} \ y = \text{partial_memoize} \ (\lambda \ y' \rightarrow y' < x) \ (\text{memoize_rec} \ f) \ x' \ \text{in} \ \text{complete_memo_rec} \ (\text{cont} \ y)
\]

Since both functions are syntactically similar it is relatively easy to prove by induction on the structure of the code of \(\text{memoize_rec}\) that for any skeleton of a recursive function \(f: x: \text{dom} \rightarrow \text{Tot} \ (\partial \text{result} \ x)\) we have a lemma \(\text{memoize_rec}\) \_lemma \(f\) asserting that \((\text{memoize_rec} \ f) \ `\text{computes}` (\(\text{fixp} \ f\)). The harder part is proving that \(\text{fixp} \ f\_\text{skel}\) is extensionally equal to fibonacci, the natural recursive definition of the Fibonacci sequence, since these two functions are not syntactically similar—but at least this proof involves reasoning only about pure functions. The good news is that having already proven that \(\text{memoize_rec} \ f\_\text{skel}\) computes \(\text{fixp} \ f\_\text{skel}\), we gain a proof of the equivalence of \(\text{memoize_rec} \ f\_\text{skel}\) and fibonacci by transitivity.

Finally, we can encapsulate the \text{Memo} effect and provide a pure (albeit state-passing) interface to the user:

\[
\text{type} \ \text{memo_pack} \ (f: \text{dom} \rightarrow \text{codom}) =
\]

\[
| \text{MemoPack} : h0: \text{memo_set} \{\text{valid_memo} \ h0 \ f\} \rightarrow mf: (\text{dom} \rightarrow \text{Memo} \ \text{codom}) \{mf \ `\text{computes}` f\} \rightarrow \text{memo_pack} \ f
\]

\[
\text{let} \ \text{apply_memo} \ (ff: \text{dom} \rightarrow \text{codom}) \ (mp: \text{memo_pack} \ f) \ (x: \text{dom}) : (\text{codom} \times \text{memo_pack} \ f) =
\]

\[
\text{let} \ \text{MemoPack} \ h0 \ mf = \text{mp} \ \text{in} \ \text{let} \ y, h1 = \text{reify} \ (mf \ x) \ h0 \ \text{in} \ y, \ \text{MemoPack} \ h1 \ mf
\]

\[
\text{let mk_memo_pack} \ f : \text{memo_pack} \ (\text{fixp} \ f) = \text{memo lemma} \ f ; \ \text{MemoPack} \ [] \ (\text{memoize_rec} \ f)
\]

### 6.2 Stepwise refinement and relations of high arity: Union-find with two optimizations

In this section, we prove several classic optimizations of a union-find data structure introduced in several stages, each a refinement. For each refinement step, we employ relational verification to prove that the refinement preserves the canonical structure of union-find. We specify correctness using, in some cases, 4- and 6-ary relations, which are easily manipulated in our monadic framework.

**Basic union-find implementation** A union-find data structure maintains disjoint partitions of a set of elements, such that each element belongs to exactly one of the partitions. The data structure supports two operations: find, that identifies which partition an element belongs to, and union, that takes as input two elements and combines their partitions.

An efficient way to implement the union-find data structure is as a forest of disjoint trees, one tree for each element in the sequence is the \(i^{th}\) element of the sequence is the \(i^{th}\) set element, containing its parent and the list of all the nodes in the subtree rooted at that node. The list is erased, indicating that it is computationally irrelevant—we only use it to express the disjointness invariant and the termination metric for recursive functions (e.g. find).

\[
\text{type} \ \text{elt} \ (n: \mathbb{N}) = i: \mathbb{N}\{i < n\} \times \text{erased} \ (\text{list} \ \mathbb{N})
\]
The basic find and union operations are shown below, where set and get are (stateful) functions that read and write the $i$th index in the uf sequence. Reasoning about mutable, pointer structures requires maintaining invariants regarding the liveness and separation of the memory referenced by the pointers. While important, these are orthogonal to the relational refinement proofs we focus on—so we elide them from this presentation, although these invariants are proven intrinsically in our code.

We formally reason about the refinement by proving that the outputs of the find and union functions do not depend on the newly added rank field. The rank_independence lemma (a 4-ary relation) states that find and union when run on two heaps that differ only on the rank field, output same results and the resulting heaps also differ only on the rank field.

Next, we want to prove the refinement of union to union_by_rank sound. Suppose we run union on a heap $h_1$; and suppose we run union_by_rank in $h$ producing $h_2$. Clearly, we cannot prove that find for a node $j$ returns the same result in $h_1$ and $h_2$. But we prove that the canonical structure of the forest is same in $h_1$ and $h_2$, by showing that two nodes are in the same partition in $h_1$ iff they are in the same partition in $h_2$:

This property is 6-ary relation, relating 1 run each of union and union_by_rank to 4 runs of find—its proof is a relatively straightforward case analysis.

Path compression Finally, we optimize find to find_compress which, in addition to returning the root for an element, sets the root as the new parent of the element so that the subsequent find queries are faster.

To prove the refinement of find to find_compress sound, we prove a 4-ary relation showing that if running find on a heap $h$ results in the heap $h_1$, and running find_compress on $h$ results in the heap $h_2$, then the partition of a node $j$ is same in $h_1$ and $h_2$. This also implies that find_compress retains the canonical structure of the union-find forest.
val find_compress_refinement n uf i h j

: Lemma (let (r₁, h₁), (r₂, h₂) = reify (find uf i) h, reify (find_compress uf i) h in
   r₁ = r₂ ∧ fst (reify (find uf j) h₁) = fst (reify (find uf j) h₂))

7 RELATED WORK

Relational verification is a large research area, here we only discuss the works that we found to be closest related. Many of the techniques we discuss are specialized to relational reasoning and sometimes even to a specific application domain (e.g., IFC, cryptography, differential privacy, program equivalence, etc). In contrast, in this work we argue that modern proof assistants and verification systems already have all the ingredients needed for effectively verifying relational properties in a generic way.

**Static IFC tools** Many IFC type systems and other static analyses were proposed for showing noninterference automatically (Assaf et al. 2017; Hammer and Snelting 2009; Hedin and Sabelfeld 2012; Sabelfeld and Myers 2003a; Volpano et al. 1996). While these approaches are most often sound, they are inherently incomplete, rejecting many programs that are in fact secure. Other verification techniques for IFC are explicitly targeted at better completeness (Amtoft and Banerjee 2004; Amtoft et al. 2012; Banerjee et al. 2016; Barthe et al. 2014; Darvas et al. 2005; Nanevski et al. 2013; Rabe 2016; Scheben and Schmitt 2011), but give up a part of the automation in the process. As we illustrate in subsection 5.2, our method allows mixing automatic typechecking with semantic arguments where typechecking would be too restrictive. While this kind of hybrid IFC approach aimed at obtaining the best of both worlds was also proposed recently by Küsters et al. (2015), here we show that it can be easily achieved in a unified (non-relational) verification framework.

**Relational program logics and type systems** Many program logics for reasoning about relational properties have been proposed including Benton’s (2004) Relational Hoare Logic, Yang’s (2007) Relational Separation Logic, and Barthe et al.’s (2009) Probabilistic Relational Hoare Logic (pRHL, cf. §4). These logics target a wide variety of application domains such as, access control (Nanevski et al. 2013), cryptography (Barthe et al. 2009, 2012, 2013a; Petcher and Morrisett 2015), differential privacy (Barthe et al. 2013b), mechanism design (Barthe et al. 2015), cost analysis (Çiček et al. 2017), program approximations (Carbin et al. 2012). Many of these logics are embedded in proof assistants for interactive verification and some of them also provide SMT-based automation based on relational weakest precondition calculi.

Barthe et al.’s (2014) RF⋆, is worth pointing out for its connection to F⋆. The authors extend a prior, value-dependent version of F⋆ (Swamy et al. 2013) with a probabilistic semantics and a type system that combines pRHL with refinement types. Like many other relational Hoare logics, RF⋆ provided only incomplete rules for relational verification, but aimed at capturing many relational properties by intrinsic typing only.

In this paper we advocate a different approach based on modeling effectful computations using monads and proving relational properties on their monadic representations, making the most of existing support for full dependent types, reasoning about pure programs and SMT-based automation in F⋆. We believe this approach provides a versatile generic alternative to building specialized program logics for each application domain.

**Product program constructions** Product program constructions and self-composition are techniques aimed at reducing the verification of k-safety properties (Clarkson and Schneider 2010) to the verification of traditional (unary) safety properties of a product program that emulates the behavior of multiple input programs. Multiple such constructions have been proposed (Barthe et al. 2016) targeted for instance at secure IFC (Barthe et al. 2011; Naumann 2006; Terauchi and Aiken 2005; Yasuoka and Terauchi 2014), program equivalence for compiler validation (Zaks and Pnueli 2008), equivalence checking and computing semantic differences (Lahiri et al. 2012), program approximation (He et al. 2016). Sousa and Dillig’s (2016) recent Descartes tool for k-safety properties also creates k copies of the program, but uses lockstep reasoning to improve performance by more tightly coupling the key invariants across the program copies. Recently Antonopoulos et al. (2017) propose a tool called Blazer...
that obtains better scalability by decomposing the problem instead of using self-composition for k-safety problems satisfying an additional decomposition property called $\psi$-quotient partitionability.

**Other program equivalence techniques** Beyond the ones already mentioned above, many other techniques targeted at program equivalence have been proposed; we briefly review several recent works: Benton et al. (2009) do manual proofs of correctness of compiler optimizations using partial equivalence relations. Kundu et al. (2009) do automatic translation validation of compiler optimizations by checking equivalence of partially specified programs that can represent multiple concrete programs. Godlin and Strichman (2010) propose proof rules for proving the equivalence of recursive procedures. Lucanu and Rusu (2015) and (Ștefan Ciobăcă et al. 2016) generalize this to a set of co-inductive equivalence proof rules that are language-independent. Automatically checking the equivalence of processes in a process calculus is an important building block for security protocol analysis (Blanchet et al. 2008; Chadha et al. 2016).

**Semantic techniques** Many semantic techniques have been proposed for reasoning about relational properties such as observational equivalence, including techniques based on binary logical relations (Ahmed et al. 2009; Benton et al. 2009, 2013, 2014; Dreyer et al. 2010, 2011, 2012; Mitchell 1986), bisimulations (Koutavas and Wand 2006; Sangiorgi et al. 2011; Sumii 2009) and combinations thereof (Hur et al. 2012, 2014). While these very powerful techniques are often not directly automated, they can be used to provide semantic correctness proofs for relational program logics (Dreyer et al. 2010, 2011) and other verification tools.

**Proof assistants** We are not the first to propose the use of monads in proof assistants, and in fact that is already very popular in the interactive theorem proving community. Also, once one uses monads in a proof assistant one very naturally starts doing relational verification; even something as simple as the monad laws are relational properties about program equivalence, although in that community it is not very common to single out these class of properties as “relational.” Finally, the idea of using a proof assistant as a powerful unified verification framework for all kinds of properties should not surprise anyone. Still the precise instantiation of these ideas in F$^\star$ might have some merits over other uses of monads in a proof assistant: the efficient implementation of effects enabled by abstraction and reification; the proofs that heavily use SMT-based automation; and the extra convenience of writing effectful code in direct syntax with returns, binds, and lifts automatically inserted.

8 **FUTURE WORK**

While we found F$^\star$ to be a versatile tool for relational verification, we also contemplated about features that would make it even better suited to relational proofs of effectful programs.

**Tactics** F$^\star$’s combination of SMT solving and dependent typechecking with higher-order unification and normalization provides good automation, but the addition of a tactic language would provide still more control and the possibility of user-defined decision procedures.

**Extrinsic termination reasoning** Aside from its use in relational reasoning, extrinsic proofs of reified terms allow programmers to defer proof obligations, rather than insisting on proofs (while anticipating all uses) at the time of definition. While convenient, extrinsic proofs in F$^\star$ only apply to programs that are intrinsically proved terminating. Building on our use of McBride’s (2015) approach in §6.1, we aim to define divergence as a reifiable effect, placing it on par with other effects in F$^\star$. We could then reason about the partial correctness of a program declared in this effect or to prove its termination after its definition. Going back to the while interpreter from subsection 5.1, we could forget about the decreasing metric and use either Bove and Capretta’s (2005) termination witnesses or step-indexing, proving, for example, noninterference of arbitrary reachable states of a non-terminating program.

**Observational purity** Another desirable feature would be to hide the effect of a term if it is proven observationally pure, e.g., in subsection 6.1 this would provide the ability to replace the original pure code by its equivalent memoized variant. Since we are able to prove that the memoized code has the same extensional behaviour as
the pure code up to some private data that we could abstract over, we would like to implement a mechanism to encapsulate observationally pure code, building on techniques already available in F* to abstract local state (Swamy et al. 2013). We hope that this mechanism could also be applied to programs proven total extrinsically.

**Finer grained effects** Generalizing effect definitions in F* to encompass a wider class of effects would allow relational reasoning about programs with non-determinism and probabilities. Another interesting extension would be to encode more invariants in the effect type and weakest precondition calculus through indexed effects. One particularly interesting application would be to have separation logic-flavored effects where the computation type carries frame information. This development may need a theory of effects over indexed types.

9 CONCLUSION
This paper advocates verifying relational properties of effectful programs using generic tools that are not specific to relational reasoning: monadic effects, reification, dependent types, non-relational weakest preconditions, and SMT-based automation. Our experiments in F* verifying relational properties about a variety of examples show the wide applicability of this approach. One of the strong points is the great flexibility in modelling effects and expressing relational properties about code using these effects. The other strong point is the good balance between interactive control and SMT-based automation. Thanks to this, the effort required from the F* programmer seems on par with non-relational reasoning in F* and with specialized relational program logics.

While in this paper we used F*, we hope that our approach to relational reasoning could also be applied to other proof assistants (e.g., Coq, Lean, Agda, Idris, etc), for which the automation would likely come in quite different styles.

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