Bayesian Estimations from the Two-Parameter Bathtub-Shaped Lifetime Distribution Based on Record Values

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Abstract

This article discusses Bayesian and non-Bayesian estimations problems of the unknown parameters for the two-parameter bathtub-shaped lifetime distribution based on upper record values. The ML and the Bayes estimates based on record values are derived for the two unknown parameters as well as hazard function. When the Bayesian approach is considered, under the assumption that both parameters are unknown, the Bayes estimators cannot be obtained in explicit forms. An approximation form due to Soland (Soland (1969)) is used for obtaining the Bayes estimates based on a conjugate prior for the first shape parameter and a discrete prior for the second shape parameter of this model. This is done with respect to the squared error loss and LINEX loss functions. The estimation procedure is then applied to real data set and simulation data.

Keywords: Upper record values; Maximum likelihood; Bayesian estimation; Soland's method; Two-parameter bathtub-shaped lifetime distribution.

1. Introduction

Chen (2000) proposed a new two-parameter lifetime distribution with bathtub-shaped or increasing failure rate (IFR) function. Some probability distributions have been proposed with models for bathtub-shaped failure rates, such as Hjorth (1980), Mudholkar and Srivastava (1993) and Xie and Lai (1996).The new two-parameter life time distribution with bathtub-shaped or increasing failure rate function compared with other models has some useful properties. First, it has only two parameters to model the bathtub-shaped failure rate function. Second, it holds some nice properties on the classical inferential front, where the confidence intervals for the shape parameter and the joint confidence regions for the two parameters have closed form. For more details, see Chen (2000), Wang (2002), Wu et. al. (2004, 2005) and Lee et al. (2007).

A new two- parameter bathtub-shaped lifetime distribution has a cumulative distribution function of the form (Chen 2000)

$$F(x) = 1 - e^{\lambda\left(1-e^{x^\beta}\right)}, \qquad x > 0,\ \lambda, \beta > 0 \tag{1.1}$$

and hence the probability density function (pdf) is given by

$$f(x) = \lambda\beta x^{\beta-1} e^{\left[x^\beta + \lambda\left(1-e^{x^\beta}\right)\right]}, \qquad x > 0,\ \lambda, \beta > 0. \tag{1.2}$$

The reliability $R(t)$ and hazard (failure rate) functions $H(t)$ of this distribution are given, respectively, by

$$R(t) = e^{\lambda\left(1-e^{x^{\beta}}\right)}, \qquad x > 0,\ \ \lambda, \beta > 0 \tag{1.3}$$

and

$$H(t) = \lambda\beta x^{\beta-1} e^{x^{\beta}}, \qquad x > 0,\ \ \lambda, \beta > 0. \tag{1.4}$$

The failure rate function of this distribution has a bathtub shape when $\beta < 1$ and has increasing failure rate function when $\beta \geq 1$(see, Chen (2000)).

Record data arise in several real-life problems including industrial stress testing, meteorological analysis, hydrology, seismology, athletic events, and oil and mining surveys. The formal study of record value theory probably started with the pioneering paper by Chandler (1952). After that many authors have discussed estimation problems for record values based on certain distribution. Among them are Mousa et al. (2002), Jaheen (2003), Malinowska and Szynal (2004), Soliman et al. (2006), Sultan (2008), Sultan et al. (2008), Doostparast (2009) and Habib et al. (2011).

Let $X_1, X_2, \dots X_n, \dots$ be a sequence of independent and identically distributed random variables having cumulative distribution function $F(x)$ and probability density function $f(x)$. Set $Y_n = max(min)\{X_1, X_2, \dots, X_n\}$ for $n \geq 1$. We say X_j is an upper (lower) record value of this sequence if $Y_j > (<) Y_{j-1}\ , j > 1$. Thus X_j will be called an upper (lower) record value if its value exceeds (is lower than) that of all previous observations. The notations $X_{U(n)}$and $X_{L(n)}$ are used for the n^{th} upper and lower records, respectively. For more details, see for example, Ahsanullah (1995) and Arnold et al. (1998).

This article is concerned with the Bayesian and non-Bayesian estimations based on upper record values for the two unknown parameters of the new two-parameter bathtub-shaped lifetime distribution and its hazard (failure rate) function. In Section 2, the maximum likelihood estimators are derived. In Section 3, the Bayes estimators of the parameters and hazard function are derived based on the squared error and LINEX loss functions. The estimation procedure is then applied to real data set and simulation data in Section 4. Finally, conclusions appear in Section 5.

2. Maximum Likelihood Estimation

Let $\mathrm{x} = \left\{X_{U(1)}, X_{U(2)}, \dots, X_{U(m)}\right\}$ be the first *m* upper record values from the new two-parameter bathtub-shaped lifetime distribution with pdf as given in (1.2), for simplicity of notation, we will use x_i instead of $x_{U(i)}$. The likelihood function (LF) is given by (see Ahsanullah (1995))

$$L(\,\theta \mid \underline{\mathbf{x}}\,) = f(x_m) \prod_{i=1}^{m-1} \frac{f(x_i)}{1 - F(x_i)} \tag{2.1}$$

where $\underline{\mathbf{x}} = (x_1, x_2, \dots, x_m)$.

By substituting Eqs. (1.1) and (1.2) in Eq. (2.1) we obtain

$$L(\lambda,\beta \mid \underline{\mathbf{x}}) = (\lambda\,\beta)^m exp\left\{\lambda(1-e^{x_m^\beta})\right\}\prod_{i=1}^m x_i^{\beta-1}\,e^{x_i^\beta}. \tag{2.2}$$

The natural logarithm of the likelihood function (2.2) is given by

$$l(\lambda,\beta \mid \underline{\mathbf{x}}) = m\ \ln(\lambda) + m\ \ln(\beta) + \lambda\left(1-e^{x_m^\beta}\right) + (\beta-1)\sum_{i=1}^m \ln(x_i) + \sum_{i=1}^m x_i^\beta \tag{2.3}$$

2.1 MLE with known β

Under the assumption that the parameter β is known. The maximum likelihood estimator (MLE) of λ, denoted by $\hat{\lambda}_{ML}$, can be derived from (2.3) as follows

$$\hat{\lambda}_{ML} = -\frac{m}{\left(1-e^{x_m^\beta}\right)} \tag{2.4}$$

2.2 MLE with unknown λ and β

Assuming that both parameters β and λ are unknown. The maximum likelihood estimators (MLE) of λ denoted by $\hat{\lambda}_{ML}$ can be shown to be

$$\hat{\lambda}_{ML} = -\frac{m}{\left(1-e^{x_m^{\hat{\beta}}}\right)}, \tag{2.5}$$

Where $\hat{\beta}$ is the MLE of the parameter β which, can be obtained as a solution of the following non-linear equation

$$\frac{m}{\beta} + \sum_{i=1}^m \ln(x_i)\left(1+x_i^\beta\right) + \frac{m}{\left(1-e^{x_m^\beta}\right)}\ln(x_m)\,e^{x_m^\beta}x_m^\beta = 0 \tag{2.6}$$

Using the invariance property, the corresponding MLE of the hazard rate function *H(t)* are obtained from (1.4) after replacing λ and β by their MLEs $\hat{\lambda}_{ML}$ and $\hat{\beta}_{ML}$.

3. Bayes Estimation

3.1 Known shape parameter β

Under the assumption that the parameter β is known, we consider the natural conjugate prior distribution for λ is a gamma prior density function with pdf

$$\pi(\lambda) = \frac{b^a}{\Gamma(a)}\lambda^{a-1}e^{-b\lambda}, \quad \lambda > 0,\ \ a,b > 0 \tag{3.1}$$

Combining the likelihood function (2.2) and the prior density function (3.1) and applying the Bayes theorem, we get the posterior density function of λ as follows

$$\pi(\lambda \mid \underline{x}) = \frac{\eta_{(\beta)}}{\Gamma(n)}\lambda^{n-1}e^{-\lambda\,\eta_{(\beta)}}, \tag{3.2}$$

where

$$\eta_{(\beta)} = \left(e^{x_m^\beta} + b - 1\right) \text{ and } \quad n = (m + a). \tag{3.3}$$

3.1.1 Bayes estimator based on squared error loss function

Assuming the commonly used squared error loss function, $L(\phi - \hat{\phi}_{BS})^2$, the Bayes estimator of ϕ (i.e., the value $\hat{\phi}_{BS}$ that minimizes the posterior expected loss) is the posterior mean. Then, the Bayes estimates of λ and $H(t)$ based on the square error loss function can be derived, respectively from (3.2) as follow

$$\hat{\lambda}_{BS} = E(\lambda|\underline{x}) = \int_c^d \lambda\pi(\lambda \mid \underline{x})d\lambda = \frac{n}{\eta_{(\beta)}}, \tag{3.4}$$

and

$$\widehat{H}(t)_{BS} = \frac{n(n+1)}{\eta_{(\beta)}^2}\beta\, t^{\beta-1}e^{t^\beta} \tag{3.5}$$

3.1.2 Bayes estimator based on LINEX loss function

Under the Linex loss function, the Bayes estimator $\hat{\phi}_{BL}$of a function ϕ is given by

$$\hat{\phi}_{BL} = \frac{-1}{C}\ln[E(exp\{-C\phi\}]. \tag{3.6}$$

From (3.2) and (3.6), the Bayes estimator for the parameter λ is

$$\begin{aligned}\hat{\lambda}_{BL} &= -\frac{1}{C}\ln\left[\int_0^\infty exp\{-C\lambda\}\ \pi(\lambda \mid \underline{x})d\lambda\right]\\ &= -\frac{1}{C}\ln\left[\frac{\eta_{(\beta)}}{\eta_{(\beta)} + C}\right]^n.\end{aligned} \tag{3.7}$$

Similarly, the Bayes estimator of *H(t)*, is

$$\widehat{H}(t)_{BL} = -\frac{1}{C}\ln\left[\frac{\eta_{(\beta)}}{\eta_{(\beta)} + C\beta\, t^{\beta-1}e^{t^\beta}}\right]^n \tag{3.8}$$

3.2 Unknown two parameters λ and β

Under the assumption that both the parameters λ and β are unknown, specifying a general joint prior for λ and β may leads to computational complexities. In an attempting to solve this problem and simplify the Bayesian analysis, we can use the Soland's method. Soland (1969) considered a family of joint prior distributions that places continuous distributions on the scale parameter and discrete distributions on the shape parameter to achieve the Bayesian analysis of Weibull distribution. This approximation was used for obtaining the Bayes estimates by several authors such as, Soliman et al. (2006), Sultan (2008) and Preda et al. (2010).

Suppose that the parameter β is restricted to a finite number of values $\beta_1, \beta_2, \ldots, \beta_k$ with respective prior probabilities $\ell_1, \ell_2, \ldots, \ell_k$ such that $0 \le \ell_j \le 1$ and $\sum_{j=1}^{k} \ell_j = 1$, that is $P(\beta = \beta_j) = \ell_j,\ j = 1,2,\ldots,k$. Further, suppose that a conditional prior distribution for λ given $\beta = \beta_j$ has a natural conjugate prior with distribution having a gamma $(a_j,\ b_j)$ with pdf

$$\pi(\lambda|\beta = \beta_j) = \frac{b_j^{a_j}}{\Gamma(a_j)} \lambda^{a_j - 1} e^{-\lambda\, b_j}, \qquad \lambda > 0,\ \ a_j, b_j > 0, \tag{3.9}$$

Where a_j and b_j are chosen to reflect prior beliefs on λ given that $\beta = \beta_j$.

Combining the likelihood function in (2.2) and the conditional prior in (3.9), we get the conditional posterior of $\lambda|\beta = \beta_j$ as follows

$$\pi^*(\lambda|\beta = \beta_j, \underline{\mathrm{x}}) = \frac{\lambda^{n_j - 1}}{\Gamma(n_j)}\ \eta_{j(\beta)}^{n_j}\ e^{-\lambda\, \eta_{j(\beta)}}, \tag{3.10}$$

where

$$\eta_{j(\beta)} = \left(e^{x_m^{\beta_j}} + b_j - 1\right) \text{ and } \quad n_j = (m + a_j). \tag{3.11}$$

The marginal posterior probability distribution of β_j obtained by applying the discrete version of Bayes' theorem, is given by

$$\begin{aligned} P(\beta = \beta_j, \underline{\mathrm{x}}) = P_{j(\beta)} &= A_{(\beta)} \int_0^\infty \frac{\mathrm{b_j^{a_j}}\ \ell_j\ \beta_j^m\ \nu_i\ \lambda^{n_j - 1}}{\Gamma(\mathrm{a_j})}\ \mathrm{e}^{-\lambda\, \eta_{j(\beta)}} d\lambda \\ &= \frac{\mathrm{b_j^{a_j}}\ \ell_j\ \beta_j^m\ \nu_i \Gamma(n_j)}{\eta_{j(\beta)}^{n_j}\ \Gamma(\mathrm{a_j})}, \end{aligned} \tag{3.12}$$

where $A_{(\beta)}$ is a normalized constant given by

$$(A_{(\beta)})^{-1} = \sum_{j=1}^{k} \frac{b_j^{a_j}\ \ell_j\ \beta_j^m\ \nu_i \Gamma(n_j)}{\eta_{j(\beta)}^{n_j}\ \Gamma(a_j)} \tag{3.13}$$

and

$$\nu_i = \prod_{i=1}^{m} x_i^{\beta_j - 1}\ e^{x_i^{\beta_j}}. \tag{3.14}$$

3.2.1 Estimators based on squared error loss function

The Bayes estimates of λ and β, *R(t)* and *H(t)* based on the square error loss function are derived, respectively from (3.10) and (3.12), as follow

$$\hat{\lambda}_{BS} = \int_0^\infty \sum_{j=1}^{k} P_{j(\beta)}\, \lambda\, \pi^*(\lambda|\beta = \beta_j, \underline{\mathrm{x}}) \mathrm{d}\lambda$$

$$= \sum_{j=1}^{k} \frac{P_{j(\beta)} n_j}{\eta_{j(\beta)}}, \tag{3.15}$$

$$\beta_{BS} = \sum_{j=1}^{k} P_{j(\beta)}\, \beta_j \tag{3.16}$$

and

$$\widehat{H}(t)_{BS} = \sum_{j=1}^{k} \frac{n_j\, P_{j(\beta)}\, \beta_j t^{\beta_j - 1} e^{t^{\beta_j}}}{\eta_{j(\beta)}}, \qquad t > 0. \tag{3.17}$$

3.2.2 Estimators based on LINEX loss function

Under the LINEX loss function, the Bayes estimate of a function $\varphi(\lambda, \beta)$ is given by

$$\hat{\varphi}(\lambda, \beta) = -\frac{1}{C} \ln\left(E\left(e^{-C\,\varphi(\lambda,\beta)}\right)\right). \tag{3.18}$$

From (3.10) and (3.18) the Bayes estimator for the parameter λ is

$$\hat{\lambda}_{BL} = -\frac{1}{C} \ln\left[\int_0^{\infty} \sum_{j=1}^{k} P_{j(\beta)} e^{-C\lambda}\, \pi^*\left(\lambda | \beta = \beta_j, \underline{\mathrm{x}}\right) d\lambda\right]$$

$$= -\frac{1}{C} \ln\left[\sum_{j=1}^{k} P_{j(\beta)} \left(1 + \frac{C}{\eta_{j(\beta)}}\right)^{-n_j}\right]. \tag{3.19}$$

Similarly, the Bayes estimators $\hat{\beta}_{BL}$ and $\widehat{H}(t)_{BL}$ of β and $H(t)$ based on the Linex loss function can be obtained, respectively, as follow

$$\hat{\beta}_{BL} = -\frac{1}{C} \ln\left[\sum_{j=1}^{k} P_{j(\beta)} e^{-C\beta_j}\right] \tag{3.20}$$

and

$$\widehat{H}(t)_{BL} = -\frac{1}{C}\, ln\left[\sum_{j=1}^{k} P_{j(\beta)} \left(1 + \frac{C\, \beta_j t^{\beta_j - 1} e^{t^{\beta_j}}}{\eta_{j(\beta)}}\right)^{-n_j}\right]. \tag{3.21}$$

4. Illustrations

To execute the calculation in this section, we need to determine the values of (β_j, ℓ_j) and the hyper-parameters $(a_j,\, b_j)$, $j = 1,2, \dots, k$ in the conjugate prior (3.9). But for each choice of $(a_j,\, b_j)$ it is necessary to find the prior of λ conditioned on each value of β_j and this can be difficult in practice. Alternatively, the values $(a_j,\ b_j)$ can be obtained based on the expected values of the reliability function $\left(E\left[R(t)|\beta=\beta_j\right]\right)$, as follows

$$E\left[R(t)|\beta=\beta_j\right]=\frac{b_j^{a_j}}{\Gamma(a_j)}\int_0^\infty exp\left\{\lambda\left(1-e^{t^{\beta_j}}\right)\right\}\lambda^{a_j-1}e^{-b_j\lambda}d\lambda$$

$$=\left(1-\frac{\left(1-e^{t^{\beta_j}}\right)}{b_j}\right)^{-a_j}. \qquad (4.1)$$

The values of a_j and b_j for each given β_j, $j=1,2,\dots,k$ can be obtained numerically from (4.1), when there are prior beliefs about the lifetime distribution enable one to specify two values $(R(t_1),t_1)$ and $(R(t_2),t_2)$. Or else, a nonparametric approach can be used to estimate the two values of the reliability function $(R(t_1),t_1)$ and $(R(t_2),t_2)$ (see Martz and Waller (1982)) by using

$$R\left(t_i=x_{U(i)}\right)=\frac{(m-i+0.625)}{m+0.25}\ ,\quad i=1,2,\dots,m. \qquad (4.2)$$

In order to illustrate the usefulness of the inferences discussed in the previous section, we consider the following two examples:

Example 1

We consider the real data of the amount of annual rainfall (in inches) recorded at the Los Angeles Civic Center for the last 100 years, from 1910 to 2009. During this period, we observe the following six upper record values

12.63, 16.18, 23.65, 32.76, 33.44, 37.96

Based on these six upper record values, the hyper-parameters a_j and b_j and the values of β_j are obtained by the following steps:

1. By using the nonparametric approach of the reliability function, we set $t_1=16.18$ and $t_2=32.76$ in (4.2), we obtain $R(t_1)=0.74$ and $R(t_2)=0.42$.
2. Based on these six upper record values, the MLE of the parameter β from (2.6), is $\beta_{ML}=0.4691$. Therefore, we suppose that β_j takes ten values around $\hat{\beta}_{ML}$ 0.43 (0.01) 0.52, each has probability 0.1.
3. The values of the hyper-parameters a_j and b_j for each given β_j are obtained numerically from (4.1), using the Newton-Raphson method.

Table 1 shows the values of the hyper-parameters and the posterior probabilities derived for each β_j. Table 2 contains the MLEs $(.)_{ML}$ and the Bayes estimates $((.)_{BS},(.)_{BL})$ of $\lambda,\ \beta$ and $H(t)$ which are computed from data in Table 1.

Table 1: Prior information, Hyper-parameters of the gamma and the posterior probabilities

j	ℓ_j	β_j	a_j	b_j	$P_{j(\beta)}$
1	0.1	0.43	2.10541	171.65912	0.88263967
2	0.1	0.44	1.47403	128.29103	0.02490062
3	0.1	0.45	1.13088	105.24769	0.00565856
4	0.1	0.46	0.91508	91.22700	0.00321935
5	0.1	0.47	0.76672	82.02387	0.00299023
6	0.1	0.48	0.65837	75.71850	0.00373855
7	0.1	0.49	0.57570	71.31145	0.00569596
8	0.1	0.50	0.51050	68.23388	0.00999762
9	0.1	0.51	0.45772	66.14102	0.01954016
10	0.1	0.52	0.41409	64.81373	0.04161927

Table 2: Estimates of λ, β and $H(t)$ with $(t = 7)$

	$(.)_{ML}$	$(.)_{BS}$	$(.)_{BL}$		
			C= -1	C= -2	C=1
λ	0.02447	0.02649	0.02654	0.0266	0.02643
β	0.46909	0.43712	0.4374	0.4376	0.4369
$H(t)$	0.04934	0.03915	0.03925	0.03935	0.03904

Example 2

Let us consider the first seven upper record values simulated from a new two-parameter lifetime distribution (1.2) with $\beta = 0.8$ and $\lambda = 3$, they are as follows:

0.025, 0.054, 0.258, 0.868, 0.888, 1.091, 1.192

Based on these seven upper record values, the maximum likelihood and Bayes estimates of λ, β and $H(t)$ are obtained by the following steps:

1. We approximate the prior for β over the interval (0.675, 0.9) by the discrete prior with β taking the 10 values 0.675 (0.025) 0.9, each with probability 0.1.

2. By using the nonparametric procedure in (4.2), we assume that the reliability function for times $t_1 = 0.054$ and $t_2 = 0.888$ are, respectively, $R(t_1) = 0.776$ and $R(t_2) = 0.362$.

3. Substituting the two values of $R(t_1)$, $R(t_2)$ obtained in step 2 into equation (4.1). The values of the hyper-parameters a_j and b_j for each given β_j, $j = 1,2,\ldots,10$ are obtained numerically by using the Newton-Raphson method. The values of the hyper-parameters and the posterior probabilities for each β_j are displayed in Table 3.

4. Based on the entries of Table 3 the MLEs $(.)_{ML}$ and the Bayes estimates $((.)_{BS}, (.)_{BL})$ of λ, β and $H(t)$are computed and the results are presented in Table 4.

Table 3: **Prior information, Hyper-parameters of the gamma and the posterior probabilities**

j	ℓ_j	β_j	a_j	b_j	$P_{j(\beta)}$
1	0.1	0.675	0.495	0.224	0.1546
2	0.1	0.700	0.464	0.19	0.1372
3	0.1	0.725	0.437	0.163	0.1228
4	0.1	0.750	0.413	0.14	0.1104
5	0.1	0.775	0.392	0.121	0.0997
6	0.1	0.800	0.374	0.105	0.0904
7	0.1	0.825	0.357	0.091	0.0819
8	0.1	0.850	0.342	0.079	0.0742
9	0.1	0.875	0.328	0.069	0.0674
10	0.1	0.900	0.316	0.061	0.0615

Table 4: **Estimates of λ, β and $H(t)$ with $(t = 1.2)$**

	True value	$(.)_{ML}$	$(.)_{BS}$	$(.)_{BL}$		
				C= 0.5	C=1	C= 2
λ	3	3.422	3.25	2.938	2.694	2.333
β	0.8	0.613	0.767	0.765	0.764	0.762
$H(t)$	7.36	5.984	7.56	6.019	5.114	4.038

5. Conclusion

In this article, we present the maximum likelihood and Bayes estimates of the two unknown parameters and hazard function for the new two-parameter lifetime model based on record values. Bayes estimators, under squared error loss and LINEX loss functions, are derived in approximate forms by using Soland's method. The comparisons between different estimators are made based on simulation study and a real record values set. It has been noticed from Tables 4 that, the Bayes estimates based on squared error loss and LINEX loss functions are perform better than the maximum likelihood estimates. The Bayes estimates of the parameters that are obtained based on the LINEX loss function tend to the corresponding estimates which are obtained based on squared error loss when C tends to zero.

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